Point configurations in d-space without large subsets in convex position*

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Abstract

In this paper we give a lower bound for the Erdős-Szekeres theorem in higher dimensions. Namely, in two different ways we construct for every $n > d \geq 2$, a configuration of n points in general position in $\mathbb{R}^d$ containing at most $c_d(\log n)^{d-1}$ points in convex position. (Points in $\mathbb{R}^d$ are in convex position if none of them lies in the convex hull of the others.)

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1 Introduction

A set of points in d-dimensional Euclidean space $\mathbb{R}^d$ is said to be in general position if any $\leq d+1$ of the points are affinely independent. In their classical paper written in 1935, Erdős and Szekeres [2] proved that, for any $n \geq 3$, there is a smallest integer $f(n)$ such that any set of at least $f(n)$ points, in general position in the plane $\mathbb{R}^2$, contains the vertex set of a convex n-gon. In fact, they proved the following quantitative result.

Theorem 1 (Erdős and Szekeres [2, 3])

$$2^{n-2} + 1 \leq f(n) \leq \binom{2n-4}{n-2} + 1.$$

Various extensions of this result, and its relation to Ramsey theory are explored e.g. in [7] and [9]. The lower bound is conjectured to be sharp [2, 3]. The best upper bound so far is $f(n) \leq \binom{2n-5}{n-2} + 2$, see [8].

Much less is known about the situation in higher dimensions. We say that a set of points in $\mathbb{R}^d$ is *in convex position* if none of them lies in the convex hull of the others. Let, for $n > d \geq 2$, $f_d(n)$ denote the smallest integer such that any set of at least $f_d(n)$ points, in general position in $\mathbb{R}^d$, contains n points in convex position. Thus, $f(n) = f_2(n)$. For $d \geq 3$, the only known values of $f_d(n)$ are $f_d(n) = 2n - d - 1$ for $d+1 \leq n \leq \lfloor 3d/2 \rfloor + 1$ (see [1] for the upper bound and [7] for the lower bound), and $f_3(6) = 9$ [1].

The study of $f_d(n)$ was initiated by Grünbaum in [4] who also established its existence for every $n > d$ via Ramsey's theorem. A more effective general upper bound $f_d(n) \leq f(n)$ follows from a simple projective argument (see [10]) and is slightly improved to $f_d(n) \leq f(n-d+2) + d - 2 \leq \binom{2n-2d-1}{n-d} + d$ in [6]. The aim of the present paper is to obtain the following general lower bound.

Theorem 2 *For every $d \geq 2$, there is a constant $c = c_d > 1$ such that*

$$f_d(n) = \Omega(c^{n^{1/(d-1)}}) .$$

Equivalently, for every $N > d \geq 2$, there exists a configuration of N points in general position in Euclidean d-space which does not contain more than $c'_d(\log N)^{d-1}$ points in convex position.

We present two proofs for this result based on two different constructions. After we introduce some notation in Section 2, the first proof is presented in Section 3. The second proof is based on the notion of so-called d-Horton sets which generalize Horton's construction of planar point sets that do not contain empty convex 7-gons, see [5, 10]. This notion is explained in Section 4 and is used in Section 5 for the second proof of Theorem 2.

2 Preliminaries

Fix the dimension $d \geq 2$. Identify, for every $1 \leq e \leq d$, $\mathbb{R}^e$ with the unique e-dimensional subspace of $\mathbb{R}^d$ spanned by the first e coordinate axes. This way $\mathbb{R}^f$ is identified with a subspace of $\mathbb{R}^e$, for every $f \leq e \leq d$.

For any $e \leq d$, denote by π_e the orthogonal projection from $\mathbb{R}^d$ onto $\mathbb{R}^e$. Thus, $\pi_e((a_1, \ldots, a_d)) = (a_1, \ldots, a_e)$. We will also use the same symbol to denote the restriction of π_e to any $\mathbb{R}^f$, $e \leq f \leq d$. If it is not a cause for ambiguity we will denote the projection from $\mathbb{R}^e$ to $\mathbb{R}^{e-1}$ simply by π.

We say that a set P of points in $\mathbb{R}^e$ is in *strongly general position* if it is in general position and, for $f = 1, \ldots, e-1$, any $f+1$ points of P determine an f-dimensional affine subspace which is not parallel to the $(e-f)$-dimensional subspace of $\mathbb{R}^e$ spanned by the last $e-f$ coordinate axes of $\mathbb{R}^e$. In this case $|\pi_f(P)| = |P|$ and $\pi_f(P)$ is in strongly general position in $\mathbb{R}^f$ for every $1 \leq f \leq e$.

The maximum size of any subset of P in convex position will be denoted by $\mathrm{mc}(P)$.

3 Recursive construction

We will need the following general construction. Suppose that $X = \{x_1, \ldots, x_t\}$ is in strongly general position in $\mathbb{R}^e$. Let $\alpha > 0$. Choose, for every $x \in X$ a vector $v(x) = (v_1, \ldots, v_e)$ such that $0 < v_1 < v_2 < \ldots < v_e < \alpha$ and $v_f < \alpha v_{f+1}$ for every $1 \le f < e$. If $y_i \in \{x_i - v(x_i), x_i, x_i + v(x_i)\}$ for $1 \le i \le t$, then the sequence $(y_1, \ldots, y_t)$ has the same order type (combinatorial structure) as $(x_1, \ldots, x_t)$, assuming that $\alpha > 0$ is small enough. If, moreover, the set $X' = \{x \pm v(x) \mid x \in X\}$ of size $2|X|$ is in strongly general position, then X' is called an *α-double of X*.

It is clear that such an α-double of X can be constructed for any $\alpha > 0$. Note that if X' is an α-double of X then $\pi_f(X')$ is an α-double of $\pi_f(X)$ for any $1 \le f \le e$. The key observation is compressed in the following lemma.

Lemma 3 *Let $X \subset \mathbb{R}^e$ be in strongly general position. If $\alpha > 0$ is small enough, then for any α-double X' of X*

$$\mathrm{mc}(X') \le \mathrm{mc}(X) + \mathrm{mc}(\pi(X)) \ .$$

Proof. Suppose that $C \subseteq X'$ is in convex position. Consider first $C_1 = \{x \in X \mid x - v(x) \in C \text{ or } x + v(x) \in C\}$. By the definition of α-double, C_1 is also in convex position. Thus, $|C_1| \le \mathrm{mc}(X)$. Next, consider $C_2 = \{x \in X \mid x - v(x) \in C \text{ and } x + v(x) \in C\}$. If α is small enough, then the vectors $v(x), x \in X$, are almost parallel to the e^{th} coordinate axis, and therefore (also due to the strongly general position) $\pi(C_2)$ is in convex position. Thus, $|C_2| = |\pi(C_2)| \le \mathrm{mc}(\pi(X))$. Since $|C| = |C_1| + |C_2|$, the result follows. □

Proof of Theorem 2. Fix any one-point set X_0 in $\mathbb{R}^d$. Suppose that, for some integer $i \ge 0$, a set X_i of points in strongly general position in $\mathbb{R}^d$ has already been defined. Choose a very small $\alpha_i > 0$ and consider an α_i-double X_i' of X_i; then $\pi_e(X_i')$ is an α_i-double of $\pi_e(X_i)$ for every $1 \le e \le d$. Applying Lemma 3 to the sets $\pi_e(X_i)$ for $d \ge e \ge 2$, we obtain that, if α_i is small enough, then $\mathrm{mc}(\pi_e(X_i')) \le \mathrm{mc}(\pi_e(X_i)) + \mathrm{mc}(\pi_{e-1}(X_i))$, for $2 \le e \le d$. Choose such a small α_i, and set $X_{i+1} = X_i'$.

This way an infinite sequence $X_0, X_1, X_2, \ldots$ of sets, in strongly general position in $\mathbb{R}^d$ is constructed such that $|X_i| = 2^i$. Theorem 2 follows immediately from the following lemma.

Lemma 4 $\mathrm{mc}(\pi_e(X_i)) \leq 2i^{e-1}$ *for every* $1 \leq e \leq d$ *and* $i \geq 1$.

Proof. The statement is clearly valid if $e = 1$ or $i = 1$. For double induction let $e \geq 2, i \geq 1$ and suppose that the assertion has already been proved for the pairs e, i and $e-1, i$. Then, according to the construction of X_{i+1},

$$\begin{aligned}
\mathrm{mc}(\pi_e(X_{i+1})) &\leq \mathrm{mc}(\pi_e(X_i)) + \mathrm{mc}(\pi_{e-1}(X_i)) \\
&\leq 2i^{e-1} + 2i^{e-2} \\
&\leq 2(i+1)^{e-1},
\end{aligned}$$

as stated. □

A more careful calculation in fact yields that $\mathrm{mc}(\pi_e(X_i)) \leq \frac{2}{(e-1)!} i^{e-1} + O(i^{e-2})$. Thus, for large n and N, Theorem 2 is valid with $c_d \approx 2^{0.37d}$ and $c'_d \approx \frac{2}{(d-1)!}$, respectively.

4 d-Horton sets

Before we define d-Horton sets, we need to define some other notions.

We say that a point a lies *below* a hyperplane h, if $a + (0, 0, \ldots, 0, c)$ lies on h for a unique $c > 0$. Similarly, a lies *above* h, if $a + (0, 0, \ldots, 0, c)$ lies on h for a unique $c < 0$.

Let A, B be two finite sets of points in strongly general position in $\mathbb{R}^d$. We say that A lies *deep below* B and B lies *high above* A if there are two sets $A' \supseteq A$, $B' \supseteq B$ in strongly general position, each of size at least d, such that the following holds: Any point of A' lies below any hyperplane determined by d points of B' and any point of B' lies above any hyperplane determined by d points of A'.

We denote the $(d-1)$th prime number by p_d (thus, $p_2 = 2$, $p_3 = 3, \ldots$). Let $H = \{h_0, h_1, ..., h_k\}$ be a set of points in strongly general position in $\mathbb{R}^d, d \geq 2$, ordered according to the first coordinate (i.e., if $i < j$ then h_i has smaller first coordinate than h_j). (Note that the definition of strongly general position implies that any two points of H differ in the first coordinate.) We define p_d sets $H_z, z = 0, \ldots, p_d - 1$, forming a partition of H as follows:

$$H_z := \{h_i \in H : i \equiv z \,(\mathrm{mod}\ p_d)\}, \qquad z = 0, \ldots, p_d - 1.$$

We now define so-called d-Horton sets introduced in [10]. A finite set of points in strongly general position in $\mathbb{R}^d, d \geq 1$, is said to be a *d-Horton set* if either $d = 1$ or $|H| \leq 1$ or if it satisfies the following three recursive conditions:

(a) $\pi(H)$ is $(d-1)$-Horton,

(b) each of the sets $H_z, z = 0, 1, ..., p_d - 1$, is d-Horton,

(c) any index set $I, I \subseteq \{0, 1, ..., p_d - 1\}, |I| \geq 2$, can be partitioned into nonempty sets J and $I - J$ in such a way that the set $\bigcup_{z \in J} H_z$ lies deep below the set $\bigcup_{z \in (I-J)} H_z$.

5 Construction using d-Horton sets

A recursive construction of d-Horton sets of arbitrary size was given in [10]. Thus, it suffices to prove the following theorem:

Theorem 5 *No d-Horton set of size $n \geq 2$ contains a subset in convex position of size $\geq c''_d \log^{d-1} n$.*

The proof of Theorem 5 relies on the following lemma.

Lemma 6 *Let A_1, A_2, A_3 be three finite sets of points in strongly general position in $\mathbb{R}^d$ such that A_i lies deep below A_j for all $1 \leq i < j \leq 3$. If C is a set in convex position intersecting both A_1 and A_3, then $\pi(C \cap A_2) \subset \mathbb{R}^{d-1}$ is in convex position.*

Proof. Suppose that the set $\pi(C \cap A_2)$ is not in convex position. Then there are points $t, t_1, \ldots, t_k \in C \cap A_2$ such that $\pi(t)$ lies in the convex hull of $\pi(t_1), \ldots, \pi(t_k)$. Consequently, t lies in the convex hull of $k+2$ points $t_1, \ldots, t_k, a, b \in C$, where a is any point in $C \cap A_1$ and b is any point in $C \cap A_3$ — a contradiction. $\square$

Proof of Theorem 5. We proceed by induction on d. The statement is trivially true for $d = 1$, since no three points in $\mathbb{R}^1$ are in convex position. Now, let $d > 1$, let H be a d-Horton set of size $n \geq 2$, and let $C \subseteq H$ be in convex position.

We inductively choose d-Horton sets $H = H^{(0)} \supseteq H^{(1)} \supseteq H^{(2)} \supseteq \ldots$ so that, for each $s \geq 0$, $H^{(s+1)}$ is one of the p_d sets $H_i^{(s)}$ intersecting C, and all other sets $H_i^{(s)}$ intersecting C lie high above it. (The existence of such a set $H_i^{(s)}$ follows from condition (c) in the definition of d-Horton sets.) Similarly, we inductively choose d-Horton sets $H = G^{(0)} \supseteq G^{(1)} \supseteq G^{(2)} \supseteq \ldots$ so that, for each $s \geq 0$, $G^{(s+1)}$ is one of the sets $G_i^{(s)}$ intersecting C, and all other sets $G_i^{(s)}$ intersecting C lie deep below it.

If possible, we choose different sets $H^{(1)}$ and $G^{(1)}$. We may then assume that $H^{(1)} \neq G^{(1)}$, since otherwise we could consider the smaller set $H^{(1)} \supseteq C$ instead of H.

We have $|H^{(s+1)}| \in \{\lfloor |H^{(s)}|/p_d \rfloor, \lceil |H^{(s)}|/p_d \rceil\}$. It follows that $|H^{(w)}| = 1$, where $w = \lceil \log_{p_d} n \rceil$. Similarly, $|G^{(w)}| = 1$.

We consider the decomposition of H into sets $H \setminus (H^{(1)} \cup G^{(1)}), H^{(1)} \setminus H^{(2)}, \ldots, H^{(w-1)} \setminus H^{(w)}, H^{(w)}, G^{(1)} \setminus G^{(2)}, \ldots, G^{(w-1)} \setminus G^{(w)}, G^{(w)}$. We will show that each of these $2w+1$ sets contains at most $(p_d - 1)c''_{d-1} \log^{d-1} n$ points of C. Then the size of C is at most $(2w+1)(p_d - 1)c''_{d-1} \log^{d-1} n < c''_d \log^d n$, where $c''_d = 10(p_d - 1)c''_{d-1}$ (say), and the theorem follows.

For each $s = 1, \ldots, w-1$, the set $H^{(s)} \setminus H^{(s+1)}$ is a disjoint union of the $p_d - 1$ sets $H_i^{(s)}$ different from $H^{(s+1)}$. If we intersect any of these sets with C and make the π-projection, the resulting set is a subset of $\pi(H)$ in convex position by Lemma 6 (applied on $A_1 := H^{(s+1)}$, $A_2 := H_i^{(s)}$, $A_3 := G^{(1)}$ — here we use that $H^{(1)} \neq G^{(1)}$). Since $\pi(H)$ is a $(d-1)$-Horton set of size n, it follows from the inductive hypothesis that $H^{(s)} \setminus H^{(s+1)}$ contains at most $(p_d - 1)c''_{d-1} \log^{d-1} n$ points of C. Analogously, the same estimate holds for each of the sets $G^{(s)} \setminus G^{(s+1)}$ and also for the set

$H \setminus (H^{(1)} \cup G^{(1)})$. It certainly also holds for the one-point sets $H^{(w)}$ and $G^{(w)}$. $\square$

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