Closure Frames and Web Spaces

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Abstract. Those topological spaces for which not only the lattice of open sets but also that of closed sets is a frame are described by means of closure operators and of neighborhood systems. This class of so-called web spaces includes, for example, all strongly locally connected spaces, and every space occurs as a dense subspace of a web space. On the other hand, a very weak separation axiom forces web spaces to be discrete.

We provide a universal construction principle producing all T_0 web spaces and show that those T_0 closure systems which are frames arise precisely as the join-ideal systems associated with so-called locally approximating standard extensions of partially ordered sets. This extends classical results about ideal systems and Scott topologies of posets and lattices.

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0. Introduction

It is well-known that every topology is a *frame* (*locale*, $\bigvee$-*distributive lattice*), that is, a complete lattice in which binary meets distribute over arbitrary joins. Not much attention seems to have been paid, however, to topologies that hap-

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pen to be *dual frames* (*coframes*) as well. One reason for the disregard of such spaces may be that they have almost no separation properties. In fact, for T_0-topologies in which every nonempty open set contains a nonempty closed set (a postulate much weaker than the T_1-axiom), the coframe property already implies discreteness. On the other hand, it turns out that the class of spaces with coframe topologies includes many classes that are of interest for general topology (e.g. strongly locally connected and, in particular, locally supercompact spaces). Moreover, many posets equipped with their Scott topologies are web spaces, among them all meet-continuous lattices. We shall give a precise characterization of all posets whose Scott topology is a coframe.

From the topological point of view, it is important to have handy descriptions of the spaces in question by means of their closure operator (which is rather easy and known since two decades [10]) and by suitable systems of neighborhoods (which is slightly more involved).

In Section 1, we study *closure frames*, i.e., general (not necessarily topological) closure systems that are frames. Note that abstractly *every* frame is isomorphic to a closure frame (viz. that of its principal ideals). For an arbitrary system $\mathcal{M}$ of subsets of a closure space (X, Γ) (with closure operator Γ), we put together several necessary and sufficient conditions ensuring that the closure system of $\mathcal{M}$-*ideals* (i.e. subsets Y such that $M \subseteq Y$ entails $\Gamma M \subseteq Y$ for all $M \in \mathcal{M} \cup \{\{x\} : x \in X\}$) becomes a frame. Much of the material in Section 1 is already contained in [10] and [17], but we have included the basic definitions and facts to make the paper selfcontained and easier to understand.

In Section 2, we present a construction of all closure frames as certain ideal systems (in the sense of Section 1) from suitable systems $\mathcal{M}$ of subsets of partially ordered sets, regarded as closure spaces with respect to their cut operator. This construction generalizes (and is motivated by) the two facts that a lattice is distributive iff its ideal lattice is a frame [6, 22], and meet-continuous iff its lattice of Scott-closed sets is a frame [21].

In Section 3, we show that the spaces with coframe topologies are precisely the so-called *web spaces*, having local bases of web neighborhoods. These are slightly more general than the strongly connected neighborhoods studied in R.-E. Hoffmann's work on spaces *"admitting a dual"* [23].

The coframe property for topologies is highly non-hereditary. This will follow from the fact, to be shown in the last section, that *every* space may be represented as a dense subspace of a web space (and, moreover, of a space with a least base).

The (dual) frame property that concerns us here is closely related to several other relevant properties of topological spaces that are stable under lattice isomorphism of topologies (with the spaces not necessarily homeomorphic). Such *lattice-invariant* properties and their mutual dependencies will be studied in a forthcoming paper. In particular, the interplay between distributivity properties and low separation axioms will be investigated in more detail.

1. Closure Frames

The *downsets* (*lower sets, lower ends, decreasing sets, initial segments,...*) of a preordered (= quasiordered) set $(X, \leq)$ are the unions of *principal ideals*

$$\downarrow y = \{x \in X : x \leq y\} \quad (y \in X).$$

They form a (topological) closure system $\mathcal{A} = \mathcal{A}(\leq)$, the *Alexandroff completion.* The complements of downsets are the *upsets* (*upper sets*,etc.), which constitute the *Alexandroff topology* $\alpha(\leq)$ [1]. The associated closure operator (the *downset operator*) sends each subset Y to

$$\downarrow Y = \{x \in X : x \leq y \text{ for some } y \in Y\} = \bigcup\{\downarrow y : y \in Y\}.$$

Specific downsets are the *(lower) cuts*, that is, the intersections of principal ideals. They form a closure system $\mathcal{N} = \mathcal{N}(\leq)$, known as the *normal completion, Dedekind-MacNeille completion* or *completion by cuts.* The associated closure operator (the *cut operator*) assigns to each subset Y the *cut*

$$\Delta Y = \bigcap\{\downarrow z : z \in X,\ Y \subseteq \downarrow z\}.$$

Note that whenever Y has a join (supremum), denoted by $\bigvee Y$, then $x \in \Delta Y$ is tantamount to $x \leq \bigvee Y$.

Recall the one-to-one correspondence between closure systems and closure operators Γ, characterized by the equivalence $Y \subseteq \Gamma Z \Leftrightarrow \Gamma Y \subseteq \Gamma Z$. (As in the specific cases of downset operators and cut operators, we omit functional parentheses whenever convenient). In accordance with the corresponding topological definition, the T_0-*axiom* for closure spaces requires that

$$\Gamma\{x\} = \Gamma\{y\} \Leftrightarrow x = y.$$

But observe that, in contrast to the situation of *topological spaces*, closure operators do not preserve finite unions in general. If Γ is a closure operator then its range $\mathcal{C} = \mathcal{C}(\Gamma)$ is the associated closure system; the pair (X, Γ) or, alternatively, $(X, \mathcal{C})$ is a *closure space.* Every closure space carries a natural preorder, usually referred to as the *specialization (pre)order* of the space, defined by

$$x \leq y \Leftrightarrow x \in \Gamma\{y\} = \bigcap\{Y \in \mathcal{C} : y \in Y\} \Leftrightarrow \Gamma\{x\} \subseteq \Gamma\{y\}.$$

In the sequel, all order-theoretical statements refer to the specialization pre-order, unless another order relation is explicitly mentioned. Thus, for example, every closed set is a downset, being a union of point closures $\downarrow x = \Gamma\{x\}$, and every open set is an upset, being a union of *(neighborhood) cores*, i.e., principal dual ideals

$$\uparrow x = \bigcap\{U : x \in U,\ X \setminus U \in \mathcal{C}\}.$$

If a closure system, regarded as a complete lattice, is a frame (that is, if binary meets distribute over arbitrary joins), we speak of a *closure frame*.

Below, we shall use the shorthand notation

$$Y_x = \downarrow x \cap \downarrow Y \ \text{ for } x \in X \ \text{ and } \ Y \subseteq X.$$

The following notions (introduced in [10]) will be central for our investigations: an element x of a closure space (X, Γ) is said to Γ-*distribute* if

$$(\Gamma Y)_x \subseteq \Gamma(Y_x) \ \text{ for all } \ Y \subseteq X,$$

and to be Γ-*near* if

$$x \in \Gamma Y \ \text{ implies } \ x \in \Gamma(Y_x) \ \text{ or, equivalently, } \downarrow x = \Gamma Z \ \text{ for some } \ Z \subseteq \downarrow Y.$$

The condition $\downarrow x = \Gamma Z$ implies $x = \bigvee Z$ (in the T_0 case), but not conversely. Also note that one may replace inclusion with equality in the definition of Γ-distributivity, because the reverse inclusion $(\Gamma Y)_x \supseteq \Gamma(Y_x)$ is always valid. Similarly, $x \in \Gamma(Y_x)$ is tantamount to $\downarrow x = \Gamma(Y_x)$. Every Γ-distributing element is Γ-near and, on the other hand, an element Γ-distributes whenever each element below it is Γ-near.

The name "Γ-*distributing elements*" comes from the observation that in case Γ is the cut operator of a complete lattice (thus $\Gamma Y = \downarrow \bigvee Y$), such elements x are characterized by the identity

$$x \wedge \bigvee Y = \bigvee (x \wedge Y) \ \ (Y \subseteq X).$$

Γ-distributing elements might also be referred to as Γ-*continuous* ones. This is justified by the following

Lemma 1.1 *Let a closure space (X, Γ) be a meet-semilattice in its specialization order. Then an element x is Γ-continuous iff the unary meet operation $y \longmapsto x \wedge y$ is a continuous selfmap of the closure space (X, Γ).*

Proof. The continuity condition

$$x \wedge \Gamma Y \subseteq \Gamma(x \wedge Y) \ \ (Y \subseteq X)$$

is equivalent to

$(\Gamma Y)_x = \downarrow x \cap \Gamma Y = \downarrow(x \wedge \Gamma Y) \subseteq \Gamma(x \wedge Y) = \Gamma \downarrow (x \wedge Y) = \Gamma(\downarrow x \cap \downarrow Y) = \Gamma(Y_x)$. □

The following result, characterizing closure frames by means of their closure operator, has been established in [10] (see also [17], where more distributive laws for closure systems were investigated):

Proposition 1.2 *For a closure space (X, Γ) the following are equivalent:*
(a) *The closure system associated with Γ is a frame.*
(b) *Γ induces a homomorphism from the frame of downsets onto $\mathcal{C}(\Gamma)$.*
(c) *Γ preserves binary (resp. finitary) intersections of downsets.*
(d) *Each element of X is Γ-distributing.*
(e) *Each element of X is Γ-near.*

Recall that a meet-semilattice with a topology is said to be *semitopological* if all the unary operations $y \longmapsto x \wedge y$ are continuous. Thus, we obtain

Corollary 1.3 *A topological (closure) space is a semitopological meet-semilattice in its specialization order iff its lattice of closed sets is a frame in which the space is embedded as a meet-semilattice.*

Sometimes, it will be convenient to extend the previous definitions to so-called *semiclosure operators*, by which we mean functions Γ on the power set $\mathcal{P}X$ that are extensive ($Y \subseteq \Gamma Y$), preserve inclusion ($Y \subseteq Z \Rightarrow \Gamma Y \subseteq \Gamma Z$) and satisfy

$$\Gamma Y = \Gamma(\bigcup\{\Gamma\{y\} : y \in Y\}) \ (Y \subseteq X).$$

Using the downset operator $\downarrow$ associated with the natural preorder, determined by

$$x \leq y \ \Leftrightarrow \ \Gamma\{x\} \subseteq \Gamma\{y\},$$

we may characterize semiclosure operators as inclusion preserving maps Γ satisfying the two identities

$$\Gamma\{y\} = \downarrow y \ \text{ and } \ \Gamma \downarrow Y = \Gamma Y.$$

Clearly, the natural preorder $\leq$ is a partial order iff Γ fulfils the T_0-axiom, this time requiring that distinct points have distinct *semiclosures*. Note also that a closure operator is just an idempotent semiclosure operator.

For any semiclosure operator (and, more generally, for any *preclosure operator*, i.e., for any extensive and inclusion preserving function) $\Gamma : \mathcal{P}X \longrightarrow \mathcal{P}X$, the fixed points form a closure system

$$\mathcal{C}(\Gamma) = \{Y \subseteq X : \Gamma Y = Y\},$$

and the corresponding closure operator Γ^{∞}, given by

$$\Gamma^{\infty} Y = \bigcap \{Z \in \mathcal{C}(\Gamma) : Y \subseteq Z\},$$

may be obtained by transfinite iteration of Γ: setting

$\Gamma^{0} = \mathrm{id}_{\mathcal{P}X}$, $\Gamma^{\kappa+1} = \Gamma \circ \Gamma^{\kappa}$ and $\Gamma^{\lambda} Y = \bigcup \{\Gamma^{\kappa} Y : \kappa < \lambda\}$ for limit ordinals λ,

one finally arrives at an ordinal κ with $\Gamma^{\kappa} = \Gamma^{\infty}$. Clearly, $\mathcal{C}(\Gamma) = \mathcal{C}(\Gamma^{\kappa})$ for $\kappa > 0$.

In contrast to arbitrary preclosure operators, our semiclosure operators Γ have the advantage that passing to the closure operators Γ^{∞} does not change the specialization preorder, because $\Gamma\{y\} = \Gamma\Gamma\{y\} = \Gamma^{\infty}\{y\}$.

By transfinite induction, one proves the implications

$$\begin{gathered}(\Gamma Y)_x = \Gamma(Y_x) \;\Rightarrow\; (\Gamma^{\kappa} Y)_x = \Gamma^{\kappa}(Y_x) \;\Rightarrow \\ (\Gamma Y)_x \subseteq \Gamma^{\infty}(Y_x) \;\Leftrightarrow\; (\Gamma^{\kappa} Y)_x \subseteq \Gamma^{\infty}(Y_x) \;\Leftrightarrow\; (\Gamma^{\infty} Y)_x = \Gamma^{\infty}(Y_x) \quad (\kappa > 0).\end{gathered}$$

Hence, every Γ-distributing element Γ^{∞}-distributes, too (but not conversely). Similarly,

$$x \text{ is } \Gamma\text{-near} \quad \Rightarrow \quad (x \in \Gamma Y \text{ implies } x \in \Gamma^{\infty}(Y_x)) \quad \Leftrightarrow \quad x \text{ is } \Gamma^{\infty}\text{-near}.$$

In order to include a wider range of distributive laws, consider now an arbitrary system $\mathcal{M}$ of subsets of X. Of particular relevance for our studies is the collection

$$\mathcal{M}^{\wedge} = \{\downarrow M : M \in \mathcal{M} \cup \{\{x\} : x \in X\}\}$$

of all $\mathcal{M}$-*downsets* (with respect to a given preorder), and the meet-semilattice $\mathcal{M}^{\cap}$ of all finite intersections of $\mathcal{M}$-downsets. By definition, $\emptyset^{\wedge}$ is the set of all principal ideals (point closures), and $(\mathcal{P}X)^{\wedge}$ is the Alexandroff completion $\mathcal{A}$.

For any closure operator $\Gamma : \mathcal{P}X \longrightarrow \mathcal{P}X$ and the associated natural preorder $\leq$, the map

$$\Gamma_{\mathcal{M}} : \mathcal{P}X \longrightarrow \mathcal{P}X, \; Y \longmapsto \bigcup \{\Gamma M : M \in \mathcal{M}^{\wedge}, M \subseteq \downarrow Y\}$$

is a semiclosure operator (but need not be a closure operator), and $\Gamma_{\mathcal{M}}$ has the same natural preorder as Γ. Indeed,

$$\downarrow y = \Gamma_{\mathcal{M}}\{y\} = \Gamma\{y\},\ \downarrow Y \subseteq \Gamma_{\mathcal{M}} Y \subseteq \Gamma Y, \text{ and}$$

$$\Gamma_{\mathcal{M}} M = \Gamma_{\mathcal{M}}^{\infty} M = \Gamma M \text{ for all } M \in \mathcal{M} \cup \mathcal{M}^{\wedge}.$$

The members of the closure system

$$\mathcal{M}^{\Gamma} = \mathcal{C}(\Gamma_{\mathcal{M}})$$

are just those downsets Y which satisfy $\Gamma M \subseteq Y$ for all $M \in \mathcal{M}$ with $M \subseteq Y$. They are referred to as the $\mathcal{M}$-*ideals* of the closure space (X, Γ). Note that $\Gamma_{\mathcal{P}X}$ coincides with Γ, while $\Gamma_{\emptyset}$ is the downset operator $\downarrow$. Thus,

$$\mathcal{C}(\Gamma) = (\mathcal{P}X)^{\Gamma} \subseteq \mathcal{M}^{\Gamma} \subseteq \emptyset^{\Gamma} = \mathcal{A}.$$

An element x with

$$(\Gamma M)_x = \Gamma(M_x) \text{ for all } M \in \mathcal{M}$$

is called Γ-$\mathcal{M}$-*distributing*, and an element x with

$$x \in \Gamma(M_x) \text{ for all } M \in \mathcal{M} \text{ and } x \in \Gamma M$$

is called Γ-$\mathcal{M}$-*near*. Now, we are in a position to summarize the main relationships between various distribution and nearness properties. For proofs, see [10].

Theorem 1.4 *Given a closure operator $\Gamma : \mathcal{P}X \longrightarrow \mathcal{P}X$ and a subset $\mathcal{M}$ of $\mathcal{P}X$, consider the following statements:*

(a1) *The system of all $\mathcal{M}$-ideals is a closure frame with closure operator $\Gamma_{\mathcal{M}}$.*

(b1) *$\Gamma_{\mathcal{M}}$ induces a homomorphism from the frame of downsets to that of $\mathcal{M}$-ideals.*

(c1) *$\Gamma_{\mathcal{M}}$ preserves finite intersections of downsets.*

(d1) *Each element of X is $\Gamma_{\mathcal{M}}$-distributing.*

(e1) *Each element of X is $\Gamma_{\mathcal{M}}$-near.*

(a2) *The system of all $\mathcal{M}$-ideals is a closure frame (with closure operator $\Gamma_{\mathcal{M}}^{\infty}$).*

(b2) *$\Gamma_{\mathcal{M}}^{\infty}$ induces a homomorphism from the frame of downsets to that of $\mathcal{M}$-ideals.*

(c2) *$\Gamma_{\mathcal{M}}^{\infty}$ preserves finite intersections of downsets.*

(d2) *Each element of X is $\Gamma_{\mathcal{M}}^{\infty}$-distributing.*

(e2) *Each element of X is $\Gamma_{\mathcal{M}}^{\infty}$-near.*

(a3) *The system of all $\mathcal{M}^{\cap}$-ideals is a closure frame.*

(b3) *Γ preserves binary intersections of $\mathcal{M}$-downsets.*

(c3) Γ *preserves finite intersections of $\mathcal{M}$-downsets.*

(d3) *Each element of X is Γ-$\mathcal{M}$-distributing.*

(e3) *Each element of X is Γ-$\mathcal{M}$-near.*

The following implications and equivalences hold in general:

$$\begin{array}{ccccccccc}
(a1) & \Leftrightarrow & (b1) & \Rightarrow & (c1) & \Rightarrow & (d1) & \Leftrightarrow & (e1) \\
 & & & & & & \Downarrow & & \\
(a2) & \Leftrightarrow & (b2) & \Leftrightarrow & (c2) & \Leftrightarrow & (d2) & \Leftrightarrow & (e2) \\
 & & & & & & \Downarrow & & \\
(a3) & \Leftrightarrow & (b3) & \Leftrightarrow & (c3) & \Leftrightarrow & (d3) & \Leftrightarrow & (e3)
\end{array}$$

For idempotent $\Gamma_{\mathcal{M}}$, the first 10 conditions are equivalent. If $\mathcal{M}^{\cap}$ is equal to $\mathcal{M}^{\wedge}$ (or at least $M_x \in \mathcal{M}^{\wedge}$ for all $M \in \mathcal{M}$) then the last 12 conditions are equivalent.

An example of a system $\mathcal{M}$ for which $\Gamma_{\mathcal{M}}$ is *always* idempotent (provided Γ is a closure operator) is that of all finite subsets; more generally, the system of all subsets with less than κ elements for a regular cardinal κ.

We call a closure space (X, Γ) *finitely $\mathcal{M}$-distributive* if

$$M \in \mathcal{M} \text{ and } x \in \Gamma M \;\Rightarrow\; x \in \Gamma(M_x),$$

in other words, if the equivalent conditions (a3) - (e3) are satisfied. If, moreover,

$$M \in \mathcal{M} \text{ and } x \in \Gamma M \;\Rightarrow\; M_x \in \mathcal{M}^{\wedge} \text{ and } x \in \Gamma(M_x),$$

we say the system $\mathcal{M}$ is *locally approximating* for (X, Γ), or (X, Γ) is *locally $\mathcal{M}$-approximable*. Clearly, this implies the conditions (d1) and (e1). Thus, from Theorem 1.4 we infer the following facts:

Corollary 1.5 *For $\mathcal{M}^{\Gamma}$, the closure system of all $\mathcal{M}$-ideals, to be a frame it is necessary that (X, Γ) be finitely $\mathcal{M}$-distributive, and sufficient that $\mathcal{M}$ be locally approximating.*

Unfortunately, in general neither of these two conditions is both necessary and sufficient (although they coincide in the frequent instance of $\mathcal{M}^{\cap} = \mathcal{M}^{\wedge}$). Thus, the question naturally arises whether we can find suitable systems $\mathcal{M}'$ (depending on $\mathcal{M}$ and Γ) such that the implication

$$M \in \mathcal{M} \text{ and } x \in \Gamma M \;\Rightarrow\; M_x \in \mathcal{M}' \text{ and } x \in \Gamma(M_x)$$

is necessary and sufficient for the frame property of $\mathcal{M}^{\Gamma}$. The answer is in the affirmative but requires some more definitions.

Let us consider the following systems:

$$
\begin{array}{ll}
\mathcal{M}^e = \{Y \subseteq X : \Gamma_{\mathcal{M}}^{\infty} Y \in \mathcal{C}(\Gamma)\} & \textit{(extended } \mathcal{M}\textit{-sets)}, \\
\mathcal{M}^f = \{Y \subseteq X : \exists \kappa \, (\Gamma_{\mathcal{M}}^{\kappa} Y \in \mathcal{M}^{\wedge})\} & \textit{(final } \mathcal{M}\textit{-sets)}, \\
\mathcal{M}^g = \{Y \subseteq X : \Gamma_{\mathcal{M}}^{\infty} Y \in \mathcal{M}^{\wedge}\} & \textit{(generating } \mathcal{M}\textit{-sets)}, \\
\mathcal{M}^h = \{Y \subseteq X : \Gamma_{\mathcal{M}}^{\infty} Y \in \emptyset^{\wedge}\} & \textit{(hulling } \mathcal{M}\textit{-sets)}.
\end{array}
$$

Note that $Y \in \mathcal{M}^e$ means $\Gamma_{\mathcal{M}}^{\infty} Y = \Gamma Y$, while $Y \in \mathcal{M}^h$ means $\Gamma_{\mathcal{M}}^{\infty} Y = \downarrow x$ for some $x \in X$. Clearly,

$$\mathcal{M} \subseteq \mathcal{M}^e, \;\; \mathcal{M}^{\wedge} \subseteq \mathcal{M}^f \text{ and } \mathcal{M}^h \subseteq \mathcal{M}^g \subseteq \mathcal{M}^f \subseteq \mathcal{M}^e.$$

Occasionally one may wish to avoid the (possibly complicated) operators $\Gamma_{\mathcal{M}}$ and $\Gamma_{\mathcal{M}}^{\infty}$. Then it may be opportune to generalize the previously introduced "nearness" concepts once more: given two systems $\mathcal{M}, \mathcal{M}' \subseteq \mathcal{P}X$, we say an element $x \in X$ is *Γ-$\mathcal{M}$-$\mathcal{M}'$-near* if for each $M \in \mathcal{M}$ with $x \in \Gamma M$, there is an $M' \in \mathcal{M}'$ such that $M' \subseteq \downarrow M$ and $\downarrow x = \Gamma M'$. Or, given a second semiclosure operator Γ', we say an element x is *Γ-$\mathcal{M}$-Γ'-near* if for all $M \in \mathcal{M}$, $x \in \Gamma M$ entails $x \in \Gamma'(M_x)$.

The following lemma is rather technical but helpful for concrete computations:

Lemma 1.6 *Let (X, Γ) be a closure space and $\mathcal{M}, \mathcal{M}' \subseteq \mathcal{P}X$ with $\mathcal{M}^h \subseteq \mathcal{M}' \subseteq \mathcal{M}^e$. Then:*

(1) $\Gamma_{\mathcal{M}}$*-near* $=$ Γ-$\mathcal{M}$-$\mathcal{M}^{\wedge}$*-near* $=$ Γ-$\mathcal{M}$-$\Gamma_{\mathcal{M}}$*-near.*

(2) $\Gamma_{\mathcal{M}}^{\infty}$*-near* $=$ Γ-$\mathcal{M}$-$\mathcal{M}'$*-near* $=$ Γ-$\mathcal{M}$-$\Gamma_{\mathcal{M}}^{\infty}$*-near.*

(3) Γ-$\mathcal{M}$*-near* $=$ Γ-$\mathcal{M}$-$\mathcal{M}^{\cap}$*-near* $=$ Γ-$\mathcal{M}$-Γ*-near.*

Proof. M always designates a member of $\mathcal{M}$.

(1) If x is $\Gamma_{\mathcal{M}}$-near and $x \in \Gamma M = \Gamma_{\mathcal{M}} M$ then $x \in \Gamma_{\mathcal{M}}(M_x)$, i.e., $x \in \Gamma M'$ for some $M' \in \mathcal{M}^{\wedge}$ with $M' \subseteq \downarrow M_x$, whence $\downarrow x \subseteq \Gamma M' \subseteq \Gamma(M_x) \subseteq \Gamma(\downarrow x) = \downarrow x$. Thus, x is Γ-$\mathcal{M}$-$\mathcal{M}^{\wedge}$-near. This, in turn, together with the hypothesis $x \in \Gamma M$ implies $\downarrow x = \Gamma M'$ for some $M' \in \mathcal{M}^{\wedge}$ with $M' \subseteq \downarrow M$, hence $M' \subseteq M_x$ and so $x \in \Gamma_{\mathcal{M}}(M_x)$, which means that x is Γ-$\mathcal{M}$-$\Gamma_{\mathcal{M}}$-near. On the other hand, $x \in \Gamma_{\mathcal{M}} Y$ entails $x \in \Gamma M'$ for some $M' \in \mathcal{M}^{\wedge}$ with $M' \subseteq \downarrow Y$, and if x is

Γ-$\mathcal{M}$-$\Gamma_{\mathcal{M}}$-near then it follows that $x \in \Gamma_{\mathcal{M}}(M'_x) \subseteq \Gamma_{\mathcal{M}}(Y_x)$. In other words, x is $\Gamma_{\mathcal{M}}$-near.

(2) If x is $\Gamma_{\mathcal{M}}^{\infty}$-near and $x \in \Gamma M = \Gamma_{\mathcal{M}}^{\infty} M$ then $x \in \Gamma_{\mathcal{M}}^{\infty}(M_x)$. This gives the inclusions $\downarrow x \subseteq \Gamma_{\mathcal{M}}^{\infty}(M_x) \subseteq \Gamma(M_x) \subseteq \Gamma(\downarrow x) = \downarrow x$, so that $M_x \in \mathcal{M}^h \subseteq \mathcal{M}'$. Thus, x is Γ-$\mathcal{M}$-$\mathcal{M}'$-near. On the other hand, if that property is assumed then $x \in \Gamma M$ implies $\downarrow x = \Gamma M'$ for some $M' \in \mathcal{M}' \subseteq \mathcal{M}^e$ with $M' \subseteq M_x$, hence $x \in \downarrow x = \Gamma M' = \Gamma_{\mathcal{M}}^{\infty}(M') \subseteq \Gamma_{\mathcal{M}}^{\infty}(M_x)$, and we conclude that x is Γ-$\mathcal{M}$-$\Gamma_{\mathcal{M}}^{\infty}$-near. Now assume x has the latter property. Again, $x \in \Gamma_{\mathcal{M}} Y$ entails $x \in \Gamma M'$ for some $M' \in \mathcal{M}^{\wedge}$ with $M' \subseteq Y$. But then either $M' = \downarrow M$ for some $M \in \mathcal{M}$, so that $x \in \Gamma M$ and therefore $x \in \Gamma_{\mathcal{M}}^{\infty}(M_x) \subseteq \Gamma_{\mathcal{M}}^{\infty}(Y_x)$, or $M' = \downarrow y$ for some $y \in X$, and then $x \in \downarrow x \cap \Gamma M' = \downarrow x \cap \downarrow y = M'_x \subseteq Y_x \subseteq \Gamma_{\mathcal{M}}^{\infty}(Y_x)$. In any case, $x \in \Gamma_{\mathcal{M}} Y$ entails $x \in \Gamma_{\mathcal{M}}^{\infty}(Y_x)$, and by an earlier remark, this ensures $\Gamma_{\mathcal{M}}^{\infty}$-nearness of x.

(3) This is clear from the definitions (note that $M_x = \downarrow x \cap \downarrow M \in \mathcal{M}^{\cap}$). $\square$

Now, Theorem 1.4 together with Lemma 1.6 yields:

Theorem 1.7 *Let (X, Γ) be a closure space and $\mathcal{M}, \mathcal{M}'$ subsets of $\mathcal{P}X$ such that $\mathcal{M}^h \subseteq \mathcal{M}' \subseteq \mathcal{M}^e$. Then the following statements are equivalent:*

(a) *The closure system $\mathcal{M}^{\Gamma}$ of all $\mathcal{M}$-ideals is a frame.*

(b) *$\mathcal{M}^e$ is locally approximating.*

(c) *$\mathcal{M}^f$ is locally approximating.*

(d) *$M \in \mathcal{M}$ and $x \in \Gamma M$ imply that $M_x \in \mathcal{M}'$ and $x \in \Gamma(M_x)$.*

If (X, Γ) is a meet-semilattice in its specialization order then these conditions are also equivalent to

(e) *The unary meet operations $y \longmapsto x \wedge y$ are continuous as selfmaps of the*

$\mathcal{M}$-ideal closure space $(X, \Gamma_{\mathcal{M}}^{\infty})$.

Proof. (a)$\Rightarrow$(b) : By 1.4 (a2)$\Rightarrow$(e2), each $x \in X$ is $\Gamma_{\mathcal{M}}^{\infty}$-near. Hence, $x \in \Gamma M = \Gamma_{\mathcal{M}}^{\infty} M$ implies $\downarrow x = \Gamma_{\mathcal{M}}^{\infty}(M_x) \subseteq \Gamma(M_x) \subseteq \downarrow x$ and therefore $M_x \in \mathcal{M}^h \subseteq \mathcal{M}^e$.

(b)$\Rightarrow$(c) : For $M \in \mathcal{M}^f \subseteq \mathcal{M}^e$ and $x \in \Gamma M$, it follows that $M_x \in \mathcal{M}^e$ and $\downarrow x = \Gamma(M_x) = \Gamma_{\mathcal{M}}^{\infty}(M_x)$, whence $M_x \in \mathcal{M}^h \subseteq \mathcal{M}^f$.

(c)$\Rightarrow$(d) is clear since $\mathcal{M}$ is contained in $\mathcal{M}^f$, and $\downarrow M \in (\mathcal{M}^f)^{\wedge}$ is tantamount to $M \in \mathcal{M}^f$.

(d) $\Rightarrow$ (a) is an immediate consequence of Lemma 1.6 (2) and the implication (e2) $\Rightarrow$ (a2) in Theorem 1.4.

For the equivalence of (a) and (e), use 1.1. $\square$

The next proposition shows that our definition of finitely $\mathcal{M}$-distributive closure spaces actually generalizes known distributive laws for complete lattices.

Proposition 1.8 *Let $L = (X, \leq)$ be a complete lattice and $\mathcal{M} \subseteq \mathcal{P}X$. Then the following statements are equivalent:*

(a) *L is an $\mathcal{M}$-frame:*
$x \wedge \bigvee M = \bigvee(x \wedge M)$ $(= \bigvee\{x \wedge y : y \in M\})$ *for all* $x \in X$ *and* $M \in \mathcal{M}$.

(b) *L is finitely $\mathcal{M}$-distributive:*
$\bigvee \bigcap \mathcal{M}' = \bigwedge\{\bigvee M : M \in \mathcal{M}'\}$ *for all finite $\mathcal{M}' \subseteq \mathcal{M}^\wedge$.*

(c) *The cut closure space (X, Δ) is finitely $\mathcal{M}$-distributive.*

Similarly, $(X, \leq)$ satisfies the conditions
$x \wedge \bigvee M = \bigvee(x \wedge M)$ *and* $\downarrow(x \wedge M) \in \mathcal{M}^\wedge$ *for all* $x \in X$ *and* $M \in \mathcal{M}$
iff $\mathcal{M}$ is locally approximating for the cut closure space (X, Δ).

These facts are easy consequences of the equalities

$M_x = \downarrow(x \wedge M)$, $\Delta M = \downarrow \bigvee M$, $\Delta(M_x) = \downarrow \bigvee(x \wedge M)$ and $(\Delta M)_x = \downarrow (x \wedge \bigvee M)$.

On account of 1.8, a complete lattice with cut closure space (X, Δ) is

a frame iff (X, Δ) is finitely $\mathcal{A}$-distributive,
distributive iff (X, Δ) is finitely $\mathcal{F}$-distributive,
meet-continuous [21] iff (X, Δ) is finitely $\mathcal{D}$-distributive,

and in all three cases, the respective system is, moreover, locally approximating.

Proposition 1.8 may be generalized from complete lattices (and their cut spaces) to arbitrary closure spaces, as follows:

Proposition 1.9 *Let (X, Γ) be a closure space, $\mathcal{M}$ a subset of $\mathcal{P}X$, and $\mathcal{M}'$ a subset of $\mathcal{P}\,\mathcal{C}(\Gamma)$ such that $\mathcal{M}^\wedge = \{\bigcup \mathcal{Y} : \mathcal{Y} \in \mathcal{M}'\}$.*

Then (X, Γ) is finitely $\mathcal{M}$-distributive iff the closure system $\mathcal{C}(\Gamma)$ is finitely $\mathcal{M}'$-distributive.

Note. For instance, $\mathcal{M}' = \{\{\downarrow y : y \in M\} : M \in \mathcal{M}^\wedge\}$ has the property required.

Proof. If (X, Γ) is finitely $\mathcal{M}$-distributive then, by (e3), $M \in \mathcal{M}$ and $x \in \Gamma M$ imply $x \in \Gamma(M_x)$. Let $\mathcal{Y} \in \mathcal{M}'$, $Z \in \mathcal{C}(\Gamma)$ and $x \in Z \wedge \bigvee \mathcal{Y} = Z \cap \Gamma(\bigcup \mathcal{Y})$. Since $M = \bigcup \mathcal{Y} \in \mathcal{M}^\wedge$, we get

$x \in \Gamma(M_x) = \Gamma(\downarrow x \cap \bigcup \mathcal{Y}) \subseteq \Gamma(Z \cap \bigcup \mathcal{Y}) = \Gamma(\bigcup\{Z \cap Y : Y \in \mathcal{Y}\}) = \bigvee(Z \wedge \mathcal{Y})$.

Conversely, suppose that the closure system $\mathcal{C}(\Gamma)$ is finitely $\mathcal{M}'$-distributive. From $M \in \mathcal{M}$ and $x \in \Gamma M$, we infer $x \in \Gamma \downarrow M$ with $\downarrow M = \bigcup \mathcal{Y}$ for some $\mathcal{Y} \in \mathcal{M}'$, hence $x \in \downarrow x \cap \Gamma(\bigcup \mathcal{Y}) = \downarrow x \wedge \bigvee \mathcal{Y} = \bigvee(\downarrow x \wedge \mathcal{Y}) = \Gamma(\downarrow x \cap \bigcup \mathcal{Y}) = \Gamma(M_x)$. $\square$

For a similar result involving "global subset selections", see [14], and for "infinitary" analogues, [4, 9, 16].

2. Closure Frames from Locally Approximating Standard Extensions

As in [11] and [12], a *standard extension* of a preordered set is a system of downsets that contains all principal ideals. A *standard completion* is then a standard extension that is closed under arbitrary intersections, in other words, a closure system whose point closures are precisely the principal ideals. Every (T_0) closure system $\mathcal{C}$ is the standard completion of a unique preordered (partially ordered) set $(X, \leq)$, namely the underlying set equipped with the specialization preorder.

For any poset $(X, \leq)$, the standard completions form a set of representatives for the join-dense completions of $(X, \leq)$ [2, 28], and a similar remark holds for the standard extensions [12, 19]. Clearly, the largest standard completion of a preordered set is the Alexandroff completion $\mathcal{A}$, and the smallest standard completion is the Dedekind-MacNeille completion $\mathcal{N}$. Other famous standard completions are the *Frink ideal completion* [20], consisting of all directed unions of cuts, and the *Scott completion* (usually defined for posets only), whose members are the *Scott-closed sets*, i.e. downsets closed under joins of directed subsets. Their complements form the *Scott topology* [21], denoted by $\sigma(X, \leq)$ or $\sigma(\leq)$.

As explained in a series of papers by the first author [9 - 15], the above completions are special instances of the following general construction. Let $\mathcal{M}$ be any subset of the power set $\mathcal{P}X$. Then

$$\mathcal{M}^{\Delta} = \{A \in \mathcal{A} : M \in \mathcal{M} \text{ and } M \subseteq A \quad \text{imply} \quad \Delta M \subseteq A\} = \mathcal{C}(\Delta_{\mathcal{M}})$$

is the closure system of all $\mathcal{M}$-*ideals* for the cut operator Δ, while

$$\mathcal{M}^{\vee} = \{A \in \mathcal{A} : M \in \mathcal{M}, M \subseteq A \text{ and } x = \bigvee M \quad \text{imply} \quad x \in A\}$$

is the closure system of all $\mathcal{M}$-*join ideals*. Of course, the two systems $\mathcal{M}^{\Delta}$ and $\mathcal{M}^{\vee}$ are intimately related: indeed, we have $\mathcal{M}^{\vee} = {\mathcal{M}_{\vee}}^{\Delta}$ where $\mathcal{M}_{\vee}$ consists of all members of $\mathcal{M}$ possessing a join; in particular, if $(X, \leq)$ is $\mathcal{M}$-*complete* (i.e., all sets in $\mathcal{M}$ have joins) then $\mathcal{M}^{\vee} = \mathcal{M}^{\Delta}$. Denoting by $\mathcal{F}$ and $\mathcal{D}$ the collections of all finite and of all directed subsets of X, respectively, we see that $\mathcal{F}^{\Delta}$ is the Frink completion and $\mathcal{D}^{\vee}$ is the Scott completion, while $\mathcal{F}^{\vee}$ is a modified join-ideal completion and $\mathcal{D}^{\Delta}$ is a modified Scott completion (cf. [8, 15]).

Now, let us call a standard extension $\mathcal{M}$ of a poset $P = (X, \leq)$ *locally approximating* if

$$M \in \mathcal{M} \text{ and } x \leq \bigvee M \quad \text{imply} \quad M_x \in \mathcal{M}^{\wedge} \text{ and } x = \bigvee M_x.$$

(This is in agreement with our earlier definition of locally approximating systems for closure spaces.) If, in addition, P is $\mathcal{M}$-complete, $\mathcal{M}$ is said to be *strictly locally approximating*. Furthermore, we say $\mathcal{M}$ is *locally prime* if $\emptyset \notin \mathcal{M}$ and

$$Y \cup Z \in \mathcal{M}_{\vee}x \text{ implies } Y \in \mathcal{M}_{\vee}x \text{ or } Z \in \mathcal{M}_{\vee}x,$$

where

$$\mathcal{M}_{\vee}x = \{M \in \mathcal{M} : \bigvee M = x\}.$$

Now, we are in a position to present the promised construction of all T_0 closure frames as join-ideal systems. (The case of arbitrary closure systems is easily reduced to the T_0 case, by the usual antisymmetrization process for preorders and the T_0-reflection for spaces, respectively.)

Theorem 2.1 *The T_0 closure frames with a given specialization order $\leq$ on X are precisely the join-ideal systems of (strictly) locally approximating standard extensions for $(X, \leq)$. In the same vein, the topological T_0 closure frames come from locally approximating and prime standard extensions.*

Proof. From 1.5, we infer that for any locally approximating standard extension $\mathcal{M}$ of a poset $P = (X, \leq)$, the join-ideal system

$$\mathcal{M}^{\vee} = {\mathcal{M}_{\vee}}^{\Delta}$$

is in fact a closure frame, and since its specialization is the original order, it is T_0. If, in addition, $\mathcal{M}$ is locally prime then, for $Y, Z \in \mathcal{M}^{\vee}$, we have that $Y \cup Z$ is a downset, because Y and Z are such. In order to check that $Y \cup Z$ is in $\mathcal{M}^{\vee}$, consider an $M \in \mathcal{M}$ with $M \subseteq Y \cup Z$ and put $x = \bigvee M$. Then $x = \bigvee((M \cap Y) \cup (M \cap Z))$ and $(M \cap Y) \cup (M \cap Z) = M \in \mathcal{M}$ imply $M \cap Y \in \mathcal{M}_{\vee}x$ or $M \cap Z \in \mathcal{M}_{\vee}x$, hence $x = \bigvee(M \cap Y) \in Y$ or $x = \bigvee(M \cap Z) \in Z$; in any case $x \in Y \cup Z$. Clearly, $\emptyset \in \mathcal{M}^{\vee}$ since $\emptyset \notin \mathcal{M}$. In all, this shows that $\mathcal{M}^{\vee}$ is a topological closure system.

Conversely, let $\mathcal{C}$ be an arbitrary T_0 closure frame with closure operator Γ and specialization order $\leq$. Then

$$\mathcal{M} = \{M = \downarrow M : \Gamma M \in \emptyset^{\wedge}\}$$

is a standard extension of $(X, \leq)$. For $M \in \mathcal{M}$, choose $y \in X$ with $\Gamma M = \downarrow y$. Then we have $y = \bigvee M$, ensuring that $(X, \leq)$ is $\mathcal{M}$-complete. Furthermore, if $x \leq \bigvee M = y$ then $x \in \Gamma M$, and by 1.2, we conclude that $\downarrow x = \Gamma(M_x)$, whence $M_x \in \mathcal{M}$. As before, it follows that $x = \bigvee M_x$, and consequently, $\mathcal{M}$ is (strictly) locally approximating. If $\mathcal{C}$ happens to be topological, i.e., Γ preserves finite unions, then $\mathcal{M}$ is locally prime. In fact, suppose $Y \cup Z \in \mathcal{M}_{\vee}x$. Thus $\bigvee(Y \cup Z) = x$ and $\downarrow x = \Gamma(Y \cup Z) = \Gamma Y \cup \Gamma Z$. Then $x \in \Gamma Y \subseteq \downarrow x$ or $x \in \Gamma Z \subseteq \downarrow x$, i.e. $Y \in \mathcal{M}_{\vee}x$ or $Z \in \mathcal{M}_{\vee}x$. Furthermore, $\emptyset \notin \mathcal{M}$ since $\Gamma\emptyset = \emptyset \neq \downarrow x$.

It remains to prove that $\mathcal{C} = \mathcal{M}^{\vee}$. Each $Y \in \mathcal{C}$ is a downset with respect to specialization. For $M \in \mathcal{M}$ with $M \subseteq Y$ and $y = \bigvee M$, we get $\Gamma M = \downarrow y$, whence $y \in \Gamma M \subseteq Y$, and so $Y \in \mathcal{M}^{\vee}$. Conversely, assume $Y \in \mathcal{M}^{\vee}$. Again by 1.2, $x \in \Gamma Y$ implies $\downarrow x = \Gamma(Y_x)$, and as Y is a downset, $M = Y_x \in \mathcal{M}$. Now, $M \subseteq Y$ entails $x = \bigvee M \in Y$. Thus, $Y = \Gamma Y \in \mathcal{C}$. □

Next, we are going to apply the results of Section 1 to the cut operator Δ of a poset $(X, \leq)$ and to the system $\mathcal{M}_{\vee}$ of all members of $\mathcal{M}$ possessing a join.

For these specific data, Corollary 1.5 further yields

Proposition 2.2 *For $\mathcal{M}^{\vee}$, the system of all $\mathcal{M}$-join ideals of a poset, to be a frame it is necessary that*

$$M \in \mathcal{M}_{\vee} \text{ and } x \leq \bigvee M \quad \Rightarrow \quad x = \bigvee M_x,$$

and sufficient that

$$M \in \mathcal{M}_{\vee} \text{ and } x \leq \bigvee M \quad \Rightarrow \quad x = \bigvee M_x \text{ and } M_x \in \mathcal{M}^{\wedge}.$$

A careful analysis shows that neither of these two criteria is both necessary and sufficient for the frame property of $\mathcal{M}^{\vee}$ - not even in the “classical” case of directed sets, where the $\mathcal{M}$-join ideals are the Scott-closed sets. The discussion of some relevant constructions and examples in that particular framework is deferred to a separate note.

It will be convenient to write $Y^{(\kappa)}$ for $\Delta_{\mathcal{M}_\vee}^{\kappa} Y$ whenever κ designates an ordinal or is the symbol ∞. We have

$$\mathcal{M}_\vee^f = \{Y \subseteq X : Y^{(\kappa)} \in \mathcal{M}_\vee^\wedge \text{ for some } \kappa\} \ =$$

$$\mathcal{M}_\vee^g = \{Y \subseteq X : Y^{(\infty)} \in \mathcal{M}_\vee^\wedge\} \ =$$

$$\mathcal{M}_\vee^h = \{Y \subseteq X : Y^{(\infty)} = \downarrow x \text{ for some } x \in X\}.$$

(The equality follows from the fact that $Y^{(\kappa)} \in \mathcal{M}_\vee^\wedge$ implies $Y^{(\kappa+1)} = \downarrow \bigvee Y^{(\kappa)}$.) Applying 1.7 to the present situation, we arrive at

Theorem 2.3 *Let $(X, \leq)$ be a poset and $\mathcal{M}$ an arbitrary subset of $\mathcal{P}X$.*

In order that the closure system $\mathcal{M}^\vee$ of all $\mathcal{M}$-join ideals be a frame, it is necessary and sufficient that

$$M \in \mathcal{M}_\vee \text{ and } x \leq \bigvee M \quad \Rightarrow \quad x = \bigvee M_x \text{ and } {M_x}^{(\kappa)} \in \mathcal{M}^\wedge \text{ for some } \kappa.$$

The last requirement may be omitted if $\Delta_{\mathcal{M}_\vee}$ is idempotent or if $\mathcal{M}^\cap = \mathcal{M}^\wedge$.

Similar results hold, of course, for the closure system $\mathcal{M}^\Delta$ instead of $\mathcal{M}^\vee$.

For specific choices of $\mathcal{M}$, viz. $\mathcal{A}, \mathcal{F}$ and $\mathcal{D}$, Theorem 2.3 amounts to

Corollary 2.4 *Let $P = (X, \leq)$ be an arbitrary poset.*

(1) *The closure system $\mathcal{A}^\vee$ of all join-closed downsets is a frame iff for all $A \subseteq X$ possessing a join, $x \leq \bigvee A$ implies $x = \bigvee A_x$.*

(2) *The closure system $\mathcal{F}^\vee$ of all $\vee$-ideals (i.e., downsets closed under finite joins) is a frame iff for all finite $F \subseteq X$ possessing a join, $x \leq \bigvee F$ implies $x = \bigvee F_x$.*

(3) *The Scott topology $\sigma(X, \leq)$ is a coframe iff for all directed $D \subseteq X$ possessing a join, $x \leq \bigvee D$ implies that $\downarrow x$ is the Scott closure of D_x (or, equivalently, $x = \bigvee D_x$ and some power of $\Delta_{\mathcal{D}_\vee}$ makes D_x directed).*

The latter somewhat tedious restriction is automatically fulfilled whenever the directed downsets are closed under binary intersections, in particular, when $(X, \leq)$ is a $\vee$- or $\wedge$-semilattice. Actually, in these cases, D_x is already directed.

Corollary 2.5 *Let $(X, \leq)$ be a $\wedge$-semilattice. Then:*

(1) *$\mathcal{A}^\vee$ is a frame iff $x \wedge \bigvee A = \bigvee (x \wedge A)$ for all subsets (downsets) A having a join.*

(2) *$\mathcal{F}^\vee$ is a frame iff $x \wedge \bigvee F = \bigvee (x \wedge F)$ for all finite subsets F having a join.*

(3) *$\mathcal{D}^\vee$ is a frame iff $x \wedge \bigvee D = \bigvee (x \wedge D)$ for all directed D having a join.*

That the closure system of all Scott-closed sets is *topological* can be checked directly but also follows from Theorem 2.1 and the fact that the directed downsets always form a locally prime standard extension. Other locally prime standard extensions are obtained by taking the downsets generated by arbitrary chains, by countable chains, etc., providing interesting examples for domain theory.

For a linearly ordered set, every standard extension not containing the empty set is locally prime (because all nonempty subsets are directed).

Here are some classes of strictly locally approximating standard extensions:

(1) The Alexandroff completion is strictly locally approximating iff the underlying poset is a frame. For any such frame, $\mathcal{A}^\vee$ is the system of all principal ideals, hence isomorphic to the given frame.

(2) The system of all downsets generated by at most two (or a finite number of) elements is strictly locally approximating iff the underlying poset is a distributive $\vee$-semilattice with 0 (cf. [22]). The ideals of any such semilattice form a frame.

(3) The system of all directed downsets is strictly locally approximating iff the underlying poset is strictly locally continuous in the sense that each directed subset D has a join, and $x \leq \bigvee D$ implies that D_x is directed, too, with join x. For any such poset, the Scott-closed sets form a frame.

3. Web Spaces and D-Spaces

Let us turn now to a neighborhood description of topological closure frames. As is the custom in topology, we denote, from now on, the closure of a subset Y by $\overline{Y}$ rather than ΓY. Recall that the principal ideal $\downarrow x$ coincides with the point closure $\overline{\{x\}}$. By 1.2, the topology of a space X is a coframe iff

$$\downarrow x \cap \overline{Y} = \overline{\downarrow x \cap \downarrow Y}$$

for all points x and all subsets Y of X, or, equivalently, iff

$$x \in \overline{Y} \;\Rightarrow\; x \in \overline{\downarrow x \cap \downarrow Y}$$

for all $x \in X$ and $Y \subseteq X$. It is not evident *a priori* how to characterize conveniently such spaces by means of neighborhood systems. Here is a solution:

Proposition 3.1 *The topology of a space X is a coframe iff each point x has a base of (open) neighborhoods W such that for each $y \in W$, the set $W \cap \downarrow x \cap \downarrow y$ is nonempty.*

Neighborhoods of the above type (where the "center" of a neighborhood is connected with every other point in the neighborhood via a common lower bound) will be referred to as *web neighborhoods*, and spaces with the described property as *web spaces* (which are, perhaps, widespread over the INTERNET). Thus, Proposition 3.1 may be rephrased as follows:

The web spaces are those topological spaces whose lattice of closed sets is a frame.

Proof. Let X be a web space and W a web neighborhood of some $x \in \overline{Y}$. Then W intersects Y. take a $y \in W \cap Y$. Thus, there is a $z \in W \cap \downarrow x \cap \downarrow y$, whence $W \cap \downarrow x \cap \downarrow Y \neq \emptyset$, and it follows that $x \in \overline{\downarrow x \cap \downarrow Y}$.

Conversely, assume that the closed sets form a frame. That is, $x \in \overline{Y}$ implies $x \in \overline{\downarrow x \cap \downarrow Y}$. Let U be any open neighborhood of x. Then there is an open neighborhood $V \subseteq U$ of x with $U \cap \downarrow x \cap \downarrow y \neq \emptyset$ for all $y \in V$. Indeed, for $Y = \{y \in X : U \cap \downarrow x \cap \downarrow y = \emptyset\}$, we obtain $U \cap \downarrow x \cap \downarrow Y = \emptyset$ and hence $x \notin \overline{\downarrow x \cap \downarrow Y}$. But then $x \notin \overline{Y}$, and we find an open $V \subseteq U$, disjoint from Y and containing x; by contraposition, $y \in V$ implies $U \cap \downarrow x \cap \downarrow y \neq \emptyset$.

Note that V need not yet be a web neighborhood of x. In order to construct a web one, apply the preceding reasoning to each $z \in U$ and let $W(z)$ be the largest open neighborhood of z contained in U such that $U \cap \downarrow z \cap \downarrow y \neq \emptyset$ for all $y \in W(z)$. We claim that $W(x)$ is a web neighborhood of x. Given $y \in W(x)$, choose a $z \in U \cap \downarrow x \cap \downarrow y$. Sitting above z, the element x must belong to the upset $W(z)$, and for each $y \in W(z)$, we have $U \cap \downarrow z \cap \downarrow y \neq \emptyset$, *a fortiori* $U \cap \downarrow x \cap \downarrow y \neq \emptyset$. Thus, $z \in W(z) \subseteq W(x)$ by the maximality of $W(x)$. Consequently, we obtain $z \in W(x) \cap \downarrow x \cap \downarrow y \neq \emptyset$, as desired. □

The proof also shows that, in the definition of web spaces, it actually does not matter whether the web neighborhoods are required to be open or not.

Closely related to web spaces are the *D-spaces* or *strongly locally connected spaces*, that is, spaces having a base of strongly connected open sets. Here *"strongly connected"* [27] (*alias "ultraconnected"* [29]) is tantamount to *"$\vee$-prime in the lattice of open sets"* and to *"down-directed with respect to specialization"* (whereas *"irreducible"* means *"$\wedge$-prime in the lattice of open sets"*, respectively, "(up-)directed with respect to specialization"). As observed by R.-E. Hoffmann [23], the strongly locally connected spaces are precisely those spaces which "have a dual", meaning that their topology is *dually* isomorphic to another topology. Therefore, we have chosen the name "D-space", pointing not only to a base of down-directed sets, but also to the duality property. Since down-directed neighborhoods are web neighborhoods, all D-spaces are web spaces. While the latter are characterized by the dual frame property of their topologies, the former are precisely those spaces whose topologies are dual *spatial* frames.

The subsequent more restricted classes of topological spaces have been studied in [13] and elsewhere. (Recall that the *core* of a point is the intersection of its neighborhoods.)

A-spaces	=	Alexandroff spaces	=	spaces in which all cores are open
B-spaces	=	monotope spaces	=	spaces having a smallest base [24]
C-spaces	=	core spaces	=	spaces having local bases of cores

The implication chain

$$A\text{-spaces} \;\Rightarrow\; B\text{-spaces} \;\Rightarrow\; C\text{-spaces} \;\Rightarrow\; D\text{-spaces} \;\Rightarrow\; \text{web spaces}$$

shows that we have quite a large store of web spaces.

4. Subspaces of Web Spaces

In this final section, we shall see that the web property is highly non-hereditary. In fact, we show that *every* topological space can be densely embedded in a web space - indeed, in a B-space. Moreover, we shall see that every T_1-space arises as the dense subspace of all maximal elements in a T_0-B-space with the pleasant additional property that

(M) every non-singleton core is open and contains a singleton core.

Using the specialization order, this may be expressed by saying that every non-maximal element has a least neighborhood and is dominated by a maximal one. Spaces with that property will be referred to as B_1-spaces. Such spaces are of interest in domain theory and other parts of theoretical computer sciences. For example, the tree of all finite or infinite words over finite alphabet becomes a B_1-space when endowed with the Scott topology.

Theorem 4.1 (0) *Each topological (T_0-)space is a dense subspace of a (T_0-)B-space.*

(1) *The T_1-spaces are the dense subspaces of all maximal elements in B_1-spaces.*

Proof. (0) Let X be a T_0-space, $\mathcal{X}_0$ the collection of all complements of cores, i.e., all sets of the form $X \setminus \uparrow x \quad (x \in X)$. By $\mathcal{X}_1$ we denote the collection of all closed proper subsets of X. Define a topology on the union $\mathcal{X} = \mathcal{X}_0 \cup \mathcal{X}_1$ by declaring as basic open the sets

$$[A) = \{C \in \mathcal{X} : A \subseteq C\}.$$

These sets actually form a base because for $C \in [A) \cap [B)$ we obtain $C \in [A \cup B) = [A) \cap [B)$ and $A \cup B$ is a proper closed set, being contained in $C \neq X$.

The specialization order of the space $\mathcal{X}$ is just the set inclusion. Indeed, for $C, D \in \mathcal{X}$ we have (observing that $C \neq X$):

$$\begin{aligned} C \subseteq D \;&\Rightarrow\; \forall A \in \mathcal{X}_1 (A \subseteq C \;\Rightarrow\; A \subseteq D) \;\Rightarrow\; \forall x \in X (\downarrow x \subseteq C \;\Rightarrow\; \downarrow x \subseteq D) \\ &\Rightarrow\; C \subseteq D. \end{aligned}$$

As X is T_0, the map

$$\varphi : X \longrightarrow \mathcal{X},\; x \longmapsto X \setminus \{x\}$$

is one-to-one (with range $\mathcal{X}_0$). We have

$$A \not\subseteq X \setminus \uparrow x \;\Leftrightarrow\; x \in A.$$

(For $\Rightarrow$ observe that if $A \not\subseteq X \setminus \uparrow x$, there is a $y \in \uparrow x \cap A$, and so $x \in \overline{\{y\}} \subseteq \overline{A} = A$.) Consequently, $\varphi^{-1}[[A)] = \{x : A \subseteq X \setminus \uparrow x\} = X \setminus A$ is open and φ is continuous; further, each closed proper subset A of X is equal to $\varphi^{-1}[\mathcal{X} \setminus [A)]$ so that φ is an embedding.

The general case is now easily obtained by "blowing up" points to the blocks of closure-equivalent points.

(1) In the T_1 case, $\uparrow x = \{x\}$ so that $\mathcal{X}_0$ is precisely the set of all maximal elements in $\mathcal{X}$. Hence, the non-maximal elements of $\mathcal{X}$ are in $\mathcal{X}_1$ and therefore they have open cores $[A)$. □

Note. With some more effort one can show that every metric space is densely embedded as the maximal layer of a B_1-space equipped with the Scott topology. This is closely related to recent work on so-called *domain models* for specific classes of metric spaces (cf. [7, 18, 26]).

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