Table of Contents

Preface

The Seventh Prague Midsummer Combinatorial Workshop was held from August 7 to 11, 2000 at Malostranské náměstí building of Charles University which is depicted on the cover of this publication. The workshop was organized by the Department of Applied Mathematics (KAM) of Charles University jointly with the DIMATIA centre. Only a small but distinguished group of mathematicians was invited and we were particularly happy to have Shalom Eliahou and Imre Bárány among us.

As it already became a tradition, the workshop benefited from participation of young researchers and PhD students, and as in the last year, 5 selected undergraduate students from the USA and 3 students from Charles University took part in the workshop, together with student guides Jana Maxová and Laura Ciobanu, in the framework of the joint DIMATIA-DIMACS DREU program.

The workshop followed an informal daily routine with morning and early afternoon discussions and presentations. This report reflects some of the presentations during the workshop. Perhaps you can digest from these proceedings some of the atmosphere at the workshop and you can also see that the fruitful exchange of ideas led directly to some new results and papers.

This volume was edited by Helena Nyklová. Most of the problems described here were supplied by the authors in electronic form; in a few cases, slight typographical changes were necessary. We apologize for any possible inaccuracies which might have occurred in the editing process.

The conference photos were taken in front of the Nebozízek restaurant where we had a conference dinner. You may notice that the leader of the Prague combinatorial group is missing in the photo. The participants of the workshop remember that Jarik spent most of the time of the workshop in hospital. I am happy to report that he got out from the physicians' hands soon after the workshop, regained all his strength immediately, and is approaching his semiround birthday - March 2001 - in the hectic and creative way of life so typical for him.

This summer workshop was partly supported by a Charles University grant GAUK 158, a Czech grant GAČR 201/99/0242 and Kontakt 337. The new Institute for Theoretical Computer Science (supported by the Ministry of Education of the Czech Republic as project LN00A056) has been introduced during the workshop.

Based on our past experience and being encouraged by several participants, we hope to organize the Eighth Prague Combinatorial Workshop in the summer of 2001. We hope to meet you all there!

Jan Kratochvíl

About H-structure of Graphs

Tomáš Chudlarský

For given graphs H and G we define the *H-graph's structure of graph G* (or briefly *H-structure of G*) as hypergraph $\mathcal{S}_H(G) = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of vertices of the graph G and a subset A of $\mathcal{V}$ forms an edge of $\mathcal{E}$ iff the subgraph of G induced by A is isomorphic to H.

Reed [4] has proved that P_4-structure has important role in the perfection of a graph — so called semi strong perfect graph theorem says: *A graph is perfect iff each graph with isomorphic P_4-structure is perfect.*

In this context Chvátal asked how it is difficult to recognize P_3-structure and P_4-structure. Both problems are solvable in polynomial time, the first one has been proved by Hayward [2] and the second one by Hayward, Hougardy and Reed [3]. We can investigate this problem for arbitrarily long path:

Recognizing P_n-structure (an integer n is a fixed parameter)
Instance: A hypergraph $\mathcal{H}$.
Question: Is there such graph G that $\mathcal{S}_{P_n}(G) = \mathcal{H}$?

Problem 1. *Is there integer n such that problem of recognizing P_n-structure is NP-complete?*

Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph, a k-coloring of $\mathcal{H}$ is such a function c from $\mathcal{V}$ to $\{1, \ldots, k\}$, that each edge $e \in \mathcal{E}$ is not monochromatic. The chromatic number of a hypergraph $\mathcal{H}$ (we denote it by $\chi(\mathcal{H})$) is minimum k such that k-coloring of $\mathcal{H}$ exists. We study the computational complexity of the following problem:

Bicolorability of H-structure (a graph H is fixed parameter)
Instance: A graph G.
Question: Is $\mathcal{S}_H(G)$ 2-colorable?

This problem obviously belongs to the class NP. Kratochvíl, Skrekovski and Chudlarský have proved following theorem.

Theorem 2. *For all graphs H with at least three vertices the problem of bicolorability of H-structure is NP-complete.*

Note that for H with at most two vertices this problem can be trivially solved in a polynomial time (fact that $\mathcal{S}_G(K_2) = G$ and $\mathcal{S}_G(\overline{K_2}) = \overline{G}$ reduces the problem to decide if G, or its complement is bipartite; if H is a single vertex then the answer is "no").

If we restrict this problem to planar input (a graph G), we can easily show that for each non-bipartite graph H problem of bicolorability of $\mathcal{S}_H(G)$ is trivial.

Observation 3. *Let G be planar graph and let H be non-bipartite graph. Then $\chi\big(\mathcal{S}_H(G)\big) \leq 2$.*

Proof: Since G is planar, from the four color theorem we know that G is colorable by colors from the set $\{1,2,3,4\}$. This coloring (of G) gives us bicoloring (of $\mathcal{S}_H(G)$): we color vertices colored by 1 or 2 by color a and we color other vertices by color b. Since H is not two colorable each copy of H (in G) contains at least three colors from $\{1,2,3,4\}$ and thus it has to use both new colors — a and b. $\diamondsuit$

It would be nice to prove this proposition without the four color theorem (4CT). However it is not clear even if we take the triangle as a graph H.

Problem 4. *Prove (without using 4CT) that vertices of each planar graph G can be bicolored such that there is no monochromatic triangle.*

Problem 5. *Prove (without using 4CT) observation 3.*

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Modular Hadamard Matrices

Shalom Eliahou (with Michel Kervaire)

A square matrix H of size n, with entries ± 1, is a Hadamard *matrix* if $HH^\top = nI$, where $H^\top$ is the transpose of H and I the identity matrix of size n. We may weaken this orthogonality condition by requiring only a congruence $HH^\top \equiv nI$ modulo some fixed integer m; in this case, we say that H is an *m-modular* Hadamard *matrix*. This notion was introduced in 1972 by Marrero and Butson [MB1, MB2].

If a Hadamard matrix of size $n \geq 3$ exists, then n must be divisible by 4. Conversely, Hadamard conjectured in 1893 that every positive integer n divisible by 4 is the size of some Hadamard matrix. The smallest open cases are currently n = 428, 668, 716.

It is of interest to consider the m-modular version of Hadamard's conjecture, as it might be easier to tackle, at least for small m. In fact, the classical Hadamard conjecture is equivalent to the set of its m_i-modular versions for any fixed infinite sequence of moduli m_i. This follows from the plain fact that any m-modular Hadamard matrix of size $n < m$ is necessarily a true Hadamard matrix. One may thus expect that the difficulty of solving the m-modular version of the conjecture will increase with m (at least for the ordering given by divisibility).

The case $m = 6$ of the modular version of the conjecture has been settled by Marrero and Butson [MB1, MB2]. They construct 6-modular Hadamard matrices of every size n divisible by 2 (not only by 4). Note that the condition $n \equiv 0 \bmod 4$ on the size $n \geq 3$ of an m-modular Hadamard matrix remains necessary only if m itself is divisible by 4, as easily seen. Their solution for the case $m = 6$ actually yields 12-modular Hadamard matrices of every size n divisible by 4, thereby settling in fact the case $m = 12$ of the conjecture. To wit: let J denote the constant all one square matrix of appropriate size, and let $K = J - 2I$. A 12-modular Hadamard matrix of size n is given by J, K or $\begin{pmatrix} K & K \\ K & -K \end{pmatrix}$ depending on whether $n \equiv 0, 4$ or $8 \bmod 12$ respectively.

There seems to have been no further progress in the modular context until recently. In [EK], we settled the case $m = 32$ of the conjecture. The solution involves modular Golay pairs and quadruples, modular Williamson matrices, and the Goethals-Seidel array.

Among natural moduli m to try are the prime numbers and the powers of 2. Currently, the m-modular version of Hadamard's conjecture is open for all primes $m \geq 5$, and for all $m = 2^t$ with $t \geq 6$. Note that a solution for $m = 512 = 2^9$ necessarily entails a true Hadamard matrix of size 428.

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The Edge Precoloring Extension Problem on Bipartite Graphs

Jiří Fiala

We have showed that the following problem is *NP* complete: Let G be a cubic bipartite graph and f be a precoloring of a subset of edges of G using at most three colors. Can f be extended to a proper edge 3-coloring of the entire graph G? This result provides a natural counterpart to classical Holyer's result [3] on edge 3-colorability of cubic graphs and a strengthening of results on precoloring extension of perfect graphs given by Kratochvíl and Sebö [4].

As far as we know this brings the first complexity results on the edge precoloring extension problem. There are several variants of the problem that are still open, e.g. the edge precoloring extension on planar bipartite graphs or perfect graphs.

Our complexity study was motivated by properties of locally injective homomorphism on small graphs (see [2], the graph $P(1, 3, 5)$). The same concept also states the problem of determining the computational complexity of the edge-coloring choosability, i.e. the problem asking whether a graph allows a proper edge coloring s.t. every edge is colored by a color belonging to a prescribed set of feasible colors and these sets are specific for each edge.

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Path Systems

Frank Göring

Three new and short proofs of Menger's classical graph theorem concerning the number of disjoint AB-paths will be given. One of them is an algorithmical proof which works even in the case that the considered graph is infinite (but has a finite separator). Furthermore, a theorem obtained from Menger's Theorem which has many forms of Menger's Theorem and some classical results of transversal theory as special cases will be proved.

Erdős's conjecture, that there is always an AB-separator and a system of disjoint AB-paths such that every path of the system contains exactly one vertex of the separator, remains open in the case that the separator is infinite.

Additive Latin Transversals

Gyula Károlyi

A *transversal* of an $n \times n$ matrix is a collection of n cells, no two of which are in the same row or column. A transversal of a matrix is a *latin transversal* if no two of its cells contain the same element. A conjecture of Snevily [3] asserts that, for any odd n, every $k \times k$ submatrix of the Cayley addition table of $\mathbf{Z}/n\mathbf{Z}$ contains a latin transversal. Putting it differently, for any two subsets A and B with $|A| = |B| = k$ of a cyclic group G of odd order $n \geq k$, there exist numberings $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ of the elements of A and B respectively such that the k sums $a_i + b_i,\ 1 \leq i \leq k$, are pairwise different. In fact, it is also conjectured for arbitrary Abelian groups G of odd order. The statement does not hold for cyclic groups of even order as shown, for example, by taking $A = B = G$, whereas for this choice it clearly holds when $|G|$ is odd (just take $a_i = b_i, i = 1, \ldots, n$). For arbitrary groups of even order take $A = B = \{0, g\}$, with g an involution, to get a counterexample.

The following results are obtained together with Oriol Serra and Balázs Szegedy [2]. First we verify Snevily's conjecture for arbitrary cyclic groups of odd order.

Theorem 1 *Let G be a cyclic group of odd order. Let $A = \{a_1, a_2, \ldots, a_k\}$ and B be subsets of G, each of cardinality k. Then there is a numbering $b_1, \ldots, b_k$ of the elements of B such that the sums $a_1 + b_1, \ldots, a_k + b_k$ are pairwise different.*

Alon [1] proved the conjecture in the particular case when $n = p$ is a prime number. Actually he proved a stronger result which can be considered as a special case of the following result when $\alpha = 1$.

Theorem 2 *Let p be a prime number, α a positive integer and $G = \mathbf{Z}_{p^\alpha}$ or $G = (\mathbf{Z}_p)^\alpha$. Let $(a_1, \ldots, a_k)$, $k < p$, be a sequence of not necessarily distinct elements in G. Then, for any subset $B \subset G$ of cardinality k there is a numbering $b_1, \ldots, b_k$ of the elements of B such that the sums $a_1 + b_1, \ldots, a_k + b_k$ are pairwise different.*

Note that the above theorem is not true with $k = p$ (see [1]). We prove these results with the adaptation of the so-called polynomial method in various finite and infinite fields, which reduces the problems to the study of permanents of Vandermonde matrices. Theorem 2 also relies on the following intermediate result. Write $q = p^{\alpha}$.

Lemma *If $\epsilon_1, \dots, \epsilon_t$ are q^{th} roots of unity such that $\sum_{i=1}^{t} \epsilon_i = 0$, then t is divisible by p.*

We also mention two problems.

Problem 1 (Snevily [3]) *Can Theorem 1 be extended to arbitrary groups of odd order?*

Let G be any Abelian group and $A = (a_1, a_2, \dots, a_k)$, $k \le |G|$ be any sequence of group elements. A is said to be a *bad* sequence if there is a subset $B \subset G$ of cardinality k such that, for any numbering $b_1, \dots, b_k$ of the elements of B, there are $1 \le i < j \le k$ such that $a_i + b_i = a_j + b_j$.

Problem 2 *Is it true, that if $A = (a_1, a_2, \dots, a_k)$ is a bad sequence in G, then there exists a subgroup $H \le G$ with $|H| = k$, a bad sequence $A' = (a'_1, a'_2, \dots, a'_k)$ in H, and an element $c \in G$ such that $a_i = a'_i + c$ for every $1 \le i \le k$?*

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Large Empty Convex Polygons

Gyula Károlyi

We say that a set of points in the plane is in *general position* if no three of them are collinear. $\mathcal{X}$ will denote a set of points in the plane in general position. Let $\mathrm{vert}(\mathcal{X})$ denote the vertex set of the convex hull of $\mathcal{X}$. A polygon is said to be *empty*, if it contains no elements of $\mathcal{X}$ in its interior. If every triple in $\mathrm{vert}(\mathcal{X})$ determines an empty triangle, then $\mathcal{X} = \mathrm{vert}(\mathcal{X})$ is in *convex position* or, in short, *convex*.

According to a well known theorem of Erdős and Szekeres [ES1, ES2], for any integer $n \ge 3$, there exists $E(n) = O(4^n)$ with the property that every set $\mathcal{X}$ of at least $E(n)$

points in general position in the plane has n elements in convex position. (In this case, we say that $\mathcal{X}$ *determines* a convex n-gon.) For a long time it appeared to be only a technicality that none of the existing proofs yielded the stronger result that every sufficiently large point set contains the vertex set of an *empty* convex n-gon. Harborth [Ha] showed that every 10-element point set determines an empty convex *pentagon*, and that this does not remain true for all 9-element sets. Finally, in 1983 Horton [Ho] surprised most experts by a simple recursive construction of arbitrarily large finite point sets determining no empty convex *heptagons*. The corresponding problem for *hexagons* is still open.

Bialostocki, Dierker, and Voxman [BDV] proposed the following elegant "modular" version of the original problem.

Conjecture *For any $n \geq 3$ and $p \geq 2$, there exists an integer $B(n,p)$ such that every set of $B(n,p)$ points in general position in the plane determines a convex n-gon such that the number of points in its interior is $0 \bmod p$.*

Bialostocki et al. verified this conjecture for every $n \geq p+2$. The original upper bound on $B(n,p)$ was later improved by Caro [C], but his proof also relied heavily on the assumption $n \geq p+2$. In a joint work with János Pach and Géza Tóth [KPT] we somewhat relaxed this condition.

Theorem 1 *For any $n \geq 5p/6 + O(1)$, there exists an integer $B(n,p)$ such that every set of $B(n,p)$ points in general position in the plane determines a convex n-gon such that the number of points in its interior is $0 \bmod p$.*

If every triple in $\mathrm{vert}(\mathcal{X})$ determines a triangle with *at most one* point in its interior, then $\mathcal{X}$ is said to be *almost convex*.

Our proof of Theorem 1 is based on the following

Theorem 2 *For any $n \geq 3$, there exists an integer $K(n)$ such that every almost convex set of at least $K(n)$ points in general position in the plane determines an empty convex n-gon. Moreover, we have $K(n) = \Omega(2^{n/2})$.*

In this talk I presented the proof of Theorem 2 and a possible extension of the result was addressed. Shortly after the workshop Theorem 2 was indeed extended by Pavel Valtr from almost convex sets to sets where each triangle determined by the set contains at most k points in its interior, where k is an arbitrary, but fixed, positive integer.

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A Problem on the Ordinary Generating Function of Bell Numbers

Martin Klazar

Bell numbers B_n count partitions of an n-element set, say $\{1, 2, \ldots, n\}$, into disjoint nonempty subsets. For example, $B_3 = 5$ and the partitions are $123, 1/23, 12/3, 13/2$, and $1/2/3$. Let

$$B = \sum_{n=0}^{\infty} B_n x^n = 1 + x + 2x^2 + 5x^3 + 15x^4 + 52x^5 + \cdots$$

be the ordinary generating function of the sequence $(B_n)_{n \geq 0}$. We pose the following problem.

Does B satisfy an algebraic differential equation (ADE) ??

Notice that B has zero radius of convergence and hence is a purely formal power series. In other words, we ask if B satisfies a nontrivial equation of the form

$$P(x, B, B', B'', \ldots, B^{(k)}) = 0$$

where P is a polynomial (with complex coefficients) in its $k + 2$ variables.

We give few comments. It is more common to work with the exponential generating function of $(B_n)_{n\geq 0}$

$$B_e = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!} = \mathrm{e}^{\mathrm{e}^x - 1}.$$

The reader will easily find an ADE satisfied by B_e. It is not difficult to prove that B_e and B satisfy no linear differential equation (LDE), that is, ADE with P linear in the variables $B, B', \ldots, B^{(k)}$. We outline the argument. If B_e satisfied a LDE, e^x would be an algebraic function which is not the case. Thus B_e satisfies no LDE. We have $B_e = B * \mathrm{e}^x$ where $*$ is the Hadamard product (coefficientwise multiplication, also called "student product"). Clearly, e^x satisfies a LDE. It is well known ([3], Theorem 6.4.12) that the class of power series solutions of LDEs is closed to Hadamard product. Thus B satisfies no LDE for otherwise B_e would satisfy one.

This argument does not work for solutions of ADEs. Examples in [2] show that this class is not closed to Hadamard product, even if one of the factors is e^x or $\sum_{n=0}^{\infty} n!x^n$. It seems plausible that B satisfies no ADE. On the other hand, to mention a classical and surprising result, in [1] it was proved that the power series

$$\sum_{n=0}^{\infty} x^{n^2}$$

satisfies an ADE.

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Equivalence of Fleischer's and Thomassen's Conjectures

Martin Kochol

We show that conjectures of Thomassen (every 4-connected line graph is hamiltonian) and Fleischner (every cyclically 4-edge-connected cubic graph has either an edge-3-coloring or a dominating cycle) are equivalent.

On Hamiltonian Cycles in Strong Products of Graphs

Daniel Král' (with Jana Maxová, Pavel Podbrdský, and Robert Šámal)

Abstract

We prove that the strong product of $G_1, \ldots, G_n$ contains a hamiltonian cycle for $\Delta \leq n$ where all G_i are connected graphs of maximum degree at most Δ; in particular $G^{\Delta(G)}$ contains a hamiltonian cycle. For large Δ we prove the same statement for $n \approx c\Delta$ for any $c > \ln 25/12 + 1/60$.

1 Introduction

Let the vertex set of the graph G be denoted by $V(G)$ and let its edge set be denoted by $E(G)$. The strong product of two graphs G and H is the graph $G \times H$ with the vertex set $V(G) \times V(H)$. Its two distinct vertices $[u_1, v_1]$ and $[u_2, v_2]$ are joined by an edge iff $u_1 = u_2 \vee u_1u_2 \in E(G)$ and $v_1 = v_2 \vee v_1v_2 \in E(H)$. A cycle containing all the vertices of the graph is called a hamiltonian cycle; each vertex is contained in such a cycle exactly once. We write G^k for the strong product of k copies of G and we call graphs containing a hamiltonian cycle hamiltonian graphs for the brevity.

Zaks asked in [3] whether there exists $k(G)$ for any connected graph G with at least two vertices such that $G^{k(G)}$ is hamiltonian. Bermond, Germa and Heydemann proved in [1] the existence of the number $k(G)$ and they proved that if G^k is hamiltonian, then also G^h is hamiltonian for all $k \leq h$ using some results of Rosehfeld and Barnette contained in [2]; let the smallest possible $k(G)$ be denoted as $h(G)$. Bermond et al. did not give any upper bound on $h(G)$ in terms of maximum degree of the graph G, but they conjectured that $h(G) \leq \Delta(G)$ holds for all connected graphs G with at least two vertices where $\Delta(G)$ is the maximum degree of G. We can prove this conjecture, but we focus here our attention to the problem of finding the smallest possible value of $h(G)$. Let $h_{\max}(\Delta)$ be $\max\{h(G)|\Delta(G) \leq \Delta\}$; we prove that $h_{\max}(\Delta) < \Delta$ for large value of Δ. On the other hand Zaks in [3] proved the following inequality (S_n is $K_{1,n}$, see the next section):

$$h(S_n) \geq \left\lceil \frac{\ln 2}{\ln\left(1+\frac{1}{n}\right)} \right\rceil$$

Thus it is impossible to prove sublinear upper bounds for $h_{\max}(\Delta)$.

2 The Result

All the graphs considered in this paper contains at least two vertices. We use instead of the strong product of graphs only the product of graphs for shortness; we do not use other definitions of the graph product in the paper. We write C_n for the cycle containing n vertices and S_n for the star of order n (i.e. $S_n = K_{1,n}$). We write for the set of all the positive integer numbers less or equal to k by $\leq k$; we use $\geq k$, $< k$ and $> k$ in the similar manner. Let S_I denote the set of all the stars S such that $i \in I$ where I is a set of integer numbers; C_I is used in the similar manner. If A and B are sets of graphs, then we write $A \times B$ for $\{G \times H | G \in A \wedge H \in B\}$ and A^2 for $A \times A$, A^3 for $A \times A \times A$ etc.

The proof of the result uses notation of covering — first, we cover the product of graphs by suitable special subgraphs (stars and cycles) and then we find a hamiltonian cycle by "conecting" these subgraphs together. We mean by a covering of the graph G by the set of graphs I a set J of subgraphs of G such that each vertex of G is contained exactly in one of the graphs of J and each graph of J is isomorphic to some of the graph of I. The following almost trivial lemma is a starting point of the proof:

Lemma 1. *Each connected graph G of maximum degree at most Δ can be covered by $S_{\leq\Delta}$.*

The next lemma states that the product of stars can be covered by stars of substantially smaller order:

Lemma 2. *For all $1 \leq k$, $2 \leq l$, any graph of $S^k_{\leq kl}$ can be covered by $S_{\leq l}$.*

It is possible to prove a better bound for the order of stars covering the product of two stars:

Lemma 3. *Any graph of $S_{\leq 2n} \times S_{\leq 2n+1}$ can be covered by $S_{\leq n}$.*

The proof of the result for stars is easier than for general graphs, thus we prove it for stars first and later we extend it to general graphs:

Lemma 4. *Any graph of $S^D_{\leq D}$ is hamiltonian for $D \geq 2$.*

It is possible to prove with a lot of technical effort that $h_{\max}(\Delta) \leq \Delta$ using Lemma 4. We cover the product by cycles, we find some special edges connecting these cycles, join all these cycles together and thus we find a hamiltonian cycle.

We focus our attention to coverings by cycles now; the proof of the following lemma needs to consider several special cases (some of them are contained in Lemma 4):

Lemma 5. *Any graph of $S_{\leq 8} \times S_{\leq 8} \times S_{\leq 8} \times S_{\leq 9} \times S_{\leq 9} \times S_{\leq 9} \times S_{\leq 15}$ can be covered by $C_{\geq 120}$.*

The following corollary follows from Lemma 1, Lemma 2 and Lemma 5 after some more or less straightforward work:

Corollary 1. *The product of any at least $\lfloor \frac{19}{24}\Delta \rfloor + \lceil \log_2 \Delta \rceil + 1$ connected graphs of maximum degree at most Δ can be covered by $C_{\geq \Delta}$ for $32 < \Delta$.*

It is possible to improve the previous corollary using Lemma 3:

Corollary 2. *For each $c > \ln \frac{25}{12} + \frac{1}{60}$ there exists c' such that the product of any at least $\lfloor c\Delta \rfloor + c'$ connected graphs of maximum degree at most Δ can be covered by $C_{\geq 120}$ for $32 < \Delta$.*

We are now ready to prove the main theorem:

Theorem 1. *The product of any at least $\lfloor \frac{19}{24}\Delta \rfloor + \lceil \log_2 \Delta \rceil + 3$ connected graphs of maximum degree at most Δ is hamiltonian for $32 < \Delta$.*

Proof: We can suppose without lost of generality that one of the graphs in the product has maximum degree exactly Δ — call this graph H. Let G be any other of the graphs and let I be the product of the rest graphs. I can be covered by $C_{\geq \Delta}$ due to Corollary 1 — let F_C be the cycle edges of this covering. I contains a spanning tree of degree at most $\Delta + 1$; let us call this spanning tree T. Let F_T be the minimal set of edges of T such that the subgraph of I induced by the edges of $F_C \cup F_T$ is connected. It is possible to colour the edges of F_T by colours $\{1, \ldots, \Delta + 1\}$ in such way that no vertex of I is adjacent to two edges of the same colour.

The product of H and each cycle of the covering of I is hamiltonian; this gives us a covering of $H \times I$ by $C_{\geq 2\Delta}$. Let E_C be the set of edges of cycles in this covering. Let $v_1, \ldots, v_{d+1}$ be any distinct vertices of H. Let E_T be the set of all the edges $[v_i, x][v_i, y]$ such that $xy \in F_T$ and xy is coloured by colour i. Clearly, the subgraph of $H \times I$ induced by edges of $E_T \cup E_C$ is connected and its only cycles are those of the covering of $H \times I$. Each vertex of $H \times I$ is adjacent to at most one edge of E_T.

We are now ready to prove that $G \times H \times I$ is hamiltonian. We assume that G is a tree; let u be any of its leaves and let v be its (only) neighbour. First, consider a product of a cycle of length at least 2Δ and G. We cover this product by the copies of the cycle corresponding to each vertex except for u and v; the product of the cycle

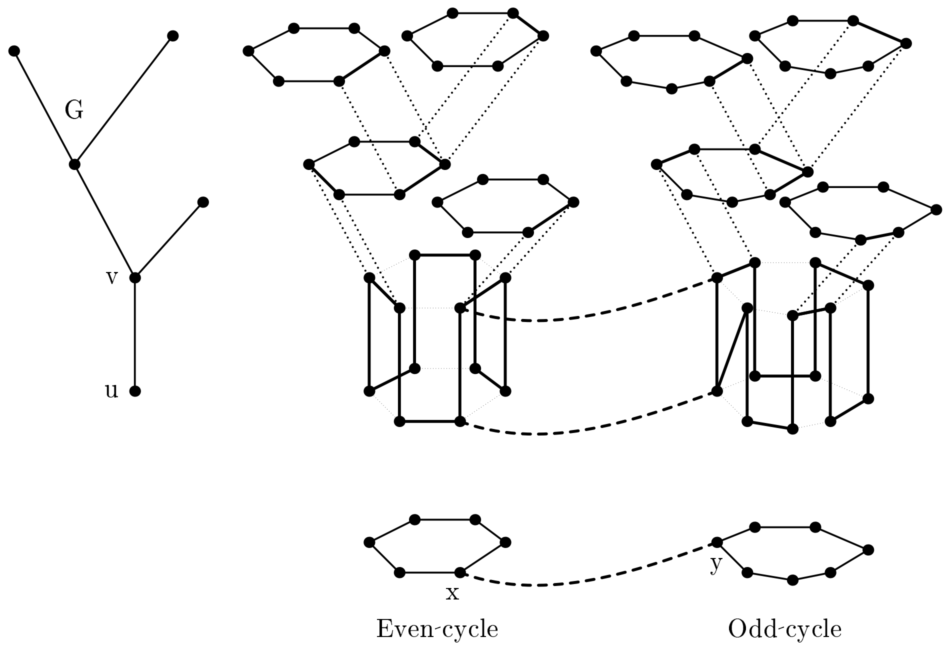

Figure 1: Covering the product of cycles and G by comb graphs and cycles and their concatenation

and the edge uv we cover by a comb graph as shown in Figure 1. We can concatenate all the copies of one cycle together, since there are at least Δ edges with the same G-coordinate in each copy of the cycle (see Figure 1). Let xy be an edge of E_T. The edges $[u,x][v,x]$ and $[u,y][v,y]$ are in different cycles covering $G \times H \times I$ and we can join these two different cycles — see Figure 1. Each vertex of $H \times I$ is adjacent to at most one edge of E_T and it is possible to join all the cycles simultaneously. Since the subgraph of $H \times I$ induced by $E_C \cup E_T$ is connected and its only cycles are those of the covering of $H \times I$, we obtain the hamiltonian cycle of $G \times H \times I$. $\square$

If we use Corollary 2 instead of Corollary 1 in the proof of Theorem 1, we obtain the following theorem:

Theorem 2. *For each $c > \ln \frac{25}{12} + \frac{1}{60}$ there exists c' such that the product of any at least $\lfloor c\Delta \rfloor + \lceil \log_2 \Delta \rceil + c'$ connected graphs of maximum degree at most Δ is hamiltonian for* $32 < \Delta$.

Note that if we use Lemma 4 instead with a trivial lower bound on the length of hamiltonian cycles in the product of stars, we get:

Theorem 3. *The product of any at least $\Delta + 2$ connected graphs of maximum degree at most Δ is hamiltonian.*

3 Conclusion

The question of precise determining of $h_{\max}(\Delta)$ remains open; it seems that stars are in some sense the worst graphs and that the lower bound given by Zaks could be tight. The more interesting task is to describe the linear behaviour of $h_{\max}$. The lower bound proved by Zaks and our upper bound gives:

$$\liminf_{\Delta \to \infty} \frac{h_{\max}(\Delta)}{\Delta} \geq \ln 2 \approx 0.6931$$

$$\limsup_{\Delta \to \infty} \frac{h_{\max}(\Delta)}{\Delta} \leq \ln \frac{25}{12} + \frac{1}{60} \approx 0.7506$$

We conjecture that:

$$\lim_{\Delta \to \infty} \frac{h_{\max}(\Delta)}{\Delta} = \ln 2$$

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The Matching Polynomials

Bodo Laß

Let G be a simple graph with n vertices. An r-matching in a graph G is a set of r edges, no two of which have a vertex in common. The number of r-matchings in G will be denoted by $p(G, r)$. We set $p(G, 0) = 1$ and define the matchings polynomial of G by (see [G], chapter 1)

$$\mu(G, x) := \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \cdot p(G, r) \cdot x^{n-2r}$$

and the signless matchings polynomial of G by

$$\overline{\mu}(G, x) := \sum_{r=0}^{\lfloor n/2 \rfloor} p(G, r) \cdot x^{n-2r}.$$

Thus the matchings polynomials count the matchings in a graph. In particular, the number of perfect matchings in G is equal to $\overline{\mu}(G, 0)$.

The complement $\overline{G}$ of G is the graph with the same vertex set of G, with two distinct vertices adjacent in $\overline{G}$ if and only if they are not adjacent in G.
It is well-known that the matchings polynomials of $\overline{G}$ are determined by the matchings polynomials of G (see the beautiful books [G], chapter 1, or [L], 5.18). However, I could not find the following formulae in the literature:

Duality theorem for the matchings polynomials ($\frac{d^2}{dx^2}$):

$$\begin{aligned} \overline{\mu}(\overline{G}, x) &= \exp\left[\tfrac{d^2}{dx^2}/2\right] \cdot \mu(G, x), \\ \mu(\overline{G}, x) &= \exp\left[-\tfrac{d^2}{dx^2}/2\right] \cdot \overline{\mu}(G, x). \end{aligned}$$

Duality theorem for the matchings polynomials ($\frac{d}{dx}$):

$$\begin{aligned}\overline{\mu}(\overline{G},x) &= e^{-x^2/2}\cdot\mu(G,\tfrac{d}{dx})\cdot e^{x^2/2},\\ \mu(\overline{G},x) &= e^{x^2/2}\cdot\overline{\mu}(G,-\tfrac{d}{dx})\cdot e^{-x^2/2}.\end{aligned}$$

Putting $y=0$ in the next theorem yields the main result of [G], chapter 1:

Duality theorem for the matchings polynomials (int):

$$\overline{\mu}(\overline{G},y)=\tfrac{1}{\sqrt{2\pi}}\cdot \mathrm{int}_{-\infty}^{\infty} e^{-(x-y)^2/2}\cdot\mu(G,x)\cdot dx.$$

Let $e^{-x^2/2}\mu(G,x)$ be the matchings function of G. Then we have ($i=\sqrt{-1}$):

Duality theorem for the matchings polynomial (C):

$$e^{-y^2/2}\mu(\overline{G},y)=\tfrac{(-i)^n}{\sqrt{2\pi}}\cdot \mathrm{int}_{-\infty}^{\infty} e^{xyi}\cdot e^{-x^2/2}\mu(G,x)\cdot dx.$$

Duality theorem for the matchings polynomial (R):

$$\begin{aligned}(-1)^{n/2}\cdot e^{-y^2/2}\mu(\overline{G},y) &= \sqrt{\tfrac{2}{\pi}}\cdot \mathrm{int}_0^{\infty}\cos(xy)\cdot e^{-x^2/2}\mu(G,x)\cdot dx, \qquad n \quad even,\\ (-1)^{(n-1)/2}\cdot e^{-y^2/2}\mu(\overline{G},y) &= \sqrt{\tfrac{2}{\pi}}\cdot \mathrm{int}_0^{\infty}\sin(xy)\cdot e^{-x^2/2}\mu(G,x)\cdot dx, \qquad n \quad odd.\end{aligned}$$

Thus the matchings functions of $\overline{G}$ and G are, up to an eventual multiplication by -1, real Fourier transforms of one another.

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Partitions by k-fans

Jiří Matoušek (with Imre Bárány)

A k-fan is a point x in the plane with k semilines (the *rays*) emanating from it. The rays divide the plane into k open *sectors*. We also admit k-fans with x at the infinity; then we have k parallel lines, and both of the unbounded regions together form one sector.

Is the following true? For every two be absolutely continuous probability measures μ_1 and μ_2 in the plane, there exists a 4-fan such that each sector has μ_1-measure $\frac{1}{4}$ and μ_2-measure $\frac{1}{4}$ too. (It is not essential that we work with probability measures; the analogous question for simultaneous partitioning of two point sets is essentially equivalent.)

This seems to be the most appealing case of a more general question, namely in what ratios can be two measures partitioned by 3-fans or 4-fans. At present we have no counterexample to the following general statement: for every choice of μ_1 and μ_2 and of positive reals $\alpha_1, \ldots, \alpha_4$ summing up to 1, there exists a 4-fan whose ith sector has μ_1-measure α_i and μ_2-measure α_i, $i = 1, 2, 3, 4$. We can prove this statement e.g. for $\alpha_1 = \frac{2}{5}$ and $\alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{5}$. The proof and other related results can be found in our paper *Simultaneous partitions of measures by k-fans*, to appear in Discr. Comput. Geom., preprint at
`http://www.ms.mff.cuni.cz/~matousek/preprints.html.`

Lipschitz Subsets of Point Sets In R^3

Jiří Matoušek

For a point $p \in R^d$, let $x_i(p)$ denote the ith coordinate. Call a finite set $S \subset R^d$ *C-Lipschitz in the ith coordinate*, where $C > 0$ is a real parameter, if $|x_i(p) - x_i(q)| \leq C \cdot \max_{j \neq i} |x_j(p) - x_j(q)|$ for every $p, q \in S$. In R^3, we can imagine a C-Lipschitz set in the z-coordinate as points on a terrain which is nowhere very steep (but, in the definition, there is no terrain; we only look at the points themselves).

Is it true that for some constants $c > 0$ and C, every n-point set $P \subset R^3$ contains a subset S of at least $cn^{2/3}$ points that is C-Lipschitz in one of the coordinates?

In the plane, it is easy to show, using Dilworth's theorem or the Erdős–Szekeres lemma, that every n-point set contains a subset of $\sqrt{n}$ points that is 1-Lipschitz in one of the coordinates. This was used in [J. Matoušek: On Lipschitz mappings onto a square In: The Mathematics of Paul Erdős II (R. Graham, J. Nešetřil ed.), Springer-Verlag 1997, pages 303–309] to give a simple alternative proof of a result of Preiss, namely that every set of positive Lebesgue measure in the plane can be mapped onto a square by a Lipschitz mapping. This problem was raised by Laczkovich, and for dimension 3 and higher it remains wide open. Answering the above question about C-Lipschitz subsets affirmatively doesn't seem to imply directly a positive answer to the 3-dimensional Laczkovich problem, but it might mean a significant step to a solution. Moreover, David Preiss said that a positive answer to the above question would imply a solution to another interesting problem in geometric measure theory.

Recently, the author of this note managed to prove that every n-point set in R^3 has a subset of size $n^{1/2}\varphi(n)$ that is 2-Lipschitz in one of the coordinates, where $\varphi(n)$ is a function tending (very slowly) to the infinity as $n \to \infty$. The proof uses a difficult result of Ruzsa and Szemerédi on triple systems with at most 3 triples on any 6 points. It is just a tiny improvement of the $\sqrt{n}$ bound (trivially obtained from the planar result) and a very long way to the desired $n^{2/3}$.

The Complexity of Minimizing Certain Cost Metrics for k-source Spanning Trees

Andrzej Proskurowski (with H. S. Connamacher and A. M. Farley)

We discuss recent results concerning k-**Source Spanning Tree Problems**:

Instance: A graph $G = (V, E)$ with a length function, $l : E \to \Re$,
k sources $S = \{s_1, \ldots, s_k\} \subseteq V$, a positive integer K.

Question: Is there a spanning tree T of G such that $cost(T) \leq K$?

Each of the problems investigated has a specific cost metric and the number of sources, $k > 1$. All the metrics are combinations of distances between sources and vertices in the tree, and the operations combining the distances are *max* and *sum*. Combining the operations in all possible ways (varying the order of operations and their ranges), we obtain six cost functions. Table 1 lists the corresponding decision problems and their complexity status.

	Problem	Cost Metric	Complexity	Reference
1	k-SPST	$cost_1(T) = \sum_{s \in S} \sum_{v \in V} d_T(s,v)$	$\mathcal{NP}$-complete	[2]
2	k-MVST	$cost_2(T) = \max_{v \in V} \sum_{s \in S} d_T(s,v)$	$\mathcal{NP}$-complete	[1]
3	k-MSST	$cost_3(T) = \max_{s \in S} \sum_{v \in V} d_T(s,v)$	$\mathcal{NP}$-complete	[1]
4	k-SVET	$cost_4(T) = \sum_{v \in V} \max_{s \in S} d_T(s,v)$	$\mathcal{P}$	[1]
5	k-SSET	$cost_5(T) = \sum_{s \in S} \max_{v \in V} d_T(s,v)$	$\mathcal{P}$	[1]
6	k-MEST	$cost_6(T) = \max_{s \in S, v \in V} d_T(s,v)$	$\mathcal{P}$	[6]

Table 1: Multi-source Spanning Tree Problems and Their Complexity Status

The first of these problems, k-Source Shortest Paths Spanning Tree (k-SPST), is an instance of a more general Optimum Communication Spanning Tree (cf. [ND7] in [3]) as defined in [4]. If every vertex is a source, this problem becomes the Shortest Total Path Length Spanning Tree (cf. [ND3] in [3]). Both these problems are $\mathcal{NP}$-hard ([5]), and the k-SPST problem is $\mathcal{NP}$-complete even with two sources and uniform edge weights ([2]). An efficient solution exists for the last problem in Table 1, k-Source Maximum Eccentricity Spanning Tree (k-MEST), and [6] presents an $\mathcal{O}$ $(|V|^3 + |E||V| \log |V|)$ algorithm. The remaining four metrics were introduced as open problems in [6]; the complexity status of each of them is completely characterized in [1]: two are $\mathcal{NP}$-hard, namely k-MVST and k-MSST, while the other two problems, k-SVET and k-SSET, are tractable.

We use the following, standard definitions. A *graph* is a pair $G = (V, E)$ where V is the set of *vertices* and $E \subseteq V \times V$ is the set of *edges*. There is a *length function* defined on the edges, $l : E \to \Re$. We also make use of points where a *point* may be either a vertex of G or a location along an edge of G. The set S of *sources* in a graph is a nonempty subset of the vertices. A *spanning tree* T of G is a connected acyclic subgraph of G using all vertices V. The *distance function*, $d : V \times V \to \Re$, on nodes u and v is the minimum of the sum of the length of each edge on a path from u to v, over all such paths. Depending on the set of edges considered, we distinguish between the (spanning) *tree distance* $d_T(u, v)$ in which the u, v-path is unique and the *graph distance* $d_G(u, v)$ which is defined over all possible such paths in G.

A generalization of the above set of problems could consider different cost measures defined as combination of operations $\bigoplus$ and $\bigotimes$ acting on distances $d_T(v, s)$, over all $v \in V$ and $s \in S$. The operations $\bigoplus$ and $\bigotimes$ extend ring operations $(\mathcal{R}, \oplus, \otimes)$ to sets of arguments, in the natural way. The question arises whether there are other reasonable cost-defining operations $\oplus, \otimes$ with interesting behavior of the complexity of the corresponding decision problems. We note that tractability of the last three problems in Table 1 follows from the fact that there is a point p on the graph (not necessarily a vertex) such a solution spanning tree is the shortest-path single-source p spanning tree of G.

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Shannon Capacity of Graphs

Miklós Ruszinkó

The *zero error capacity* of a discrete noisy channel C was invented by Shannon [6]. A channel consists of a finite set X of possible input letters and for each $x \in X$ a subset Y_x of a (not necessarily finite) output set Y. Here Y_x is the set of possible outputs of the channel on input x. Clearly, if the decoder receives an output $y \in Y_{x_1} \cap Y_{x_2}$ where $x_1 \neq x_2 \in X$ then the decoder cannot be certain of the input letter, i.e., it will make an error in decoding with a certain probability. On the other hand, if for $x \neq x' \in X$ we have $Y_x \cap Y_{x'} = \emptyset$, the decoder will be able to determine the exact input letter: it is the unique $x \in X$ for which the output y is contained in Y_x.

In order to determine the maximum number of letters that can be transmitted through the channel without the possibility of an error, Shannon associated a (characteristic) graph $G = G(C)$ to the channel C as follows. The vertices $V(G)$ of the graph G are labeled by the possible input letters ($|V(G)| = |X|$), and two vertices x_1, x_2 are adjacent iff $Y_{x_1} \cap Y_{x_2} \neq \emptyset$. Clearly, the labels of an independent set can be transmitted without an error. Therefore, the number of letters that can be transmitted by C without an error is exactly the *independence number* $\alpha(G)$.

If the sender transmitted k letters, say, $x_1, \ldots, x_k$, i.e., the channel has been used k times, then the output of the channel will also contain k symbols $y_1, \ldots, y_k$, $y_i \in Y_{x_i}$. This situation can be considered as a single use of the channel C^k, which has input set X^k, output set Y^k and the set of possible outputs $Y_{x_1} \times \cdots \times Y_{x_k}$ on input $x_1, \ldots, x_k$. The k^{th} power G^k of a graph G is defined as follows. The vertex set of G^k is $V(G^k) = V(G)^k$, and two vertices $(x_1, x_2, \ldots, x_k)$ and $(x'_1, x'_2, \ldots, x'_k)$ are adjacent iff for all $1 \leq i \leq k$ either $x_i = x'_i$ or x_i and x'_i are adjacent in G. It is easy to see that the number of sequences of length k that can be transmitted without an error is the independence number of the k^{th} power of $G(C)$.

The Shannon capacity of C is defined as

$$\Theta(C) = \sup_k \left(\alpha\big(G(C)^k\big)\right)^{1/k} = \lim_{k\to\infty} \left(\alpha\big(G(C)^k\big)\right)^{1/k}.$$

Note that the capacity gives a measure of the optimal perfomance of the channel when transmitting long sequences. This limit, by super-multiplicativity exists and – since $(\alpha(G))^k \leq \alpha(G^k)$ for an arbitrary graph G – it is always at least $\alpha(G)$. It is worth of mentioning, that Shannon originally [6] defined the capacity as $\log\Theta$ (we use the definition and notation of Lovász [5]). Also notice, that Θ depends on the graph $G(C)$ only, and every graph is the characteristic graph of some channel. Therefore, we consider the Shannon capacity of graphs:

$$\Theta(G) = \sup_k \left(\alpha(G^k)\right)^{1/k} = \lim_{k\to\infty} \left(\alpha(G^k)\right)^{1/k}.$$

Since Shannon's invention of the capacity [6] in 1956, it has been one of the central topics in both information theory and extremal graph theory.

We investigated [2] the Shannon capacity of large odd cycles C_{2n+1}. It follows from a result of Hales [3] that for an infinite subsequence n_k, $k = 1, 2, \ldots$, of positive integers the difference $n_k + 1/2 - \Theta(C_{2n_k+1})$ tends to zero as k tends to infinity (note $\alpha(C_{2n+1}) = n$), i.e.,

Theorem 0.1 (Hales).

$$\liminf_{n\to\infty}(n + 1/2 - \Theta(C_{2n+1})) = 0.$$

Hales also showed that the $\limsup$ of this difference is at most $1/4$. Modifying Hales linear algebraic construction, we showed [2] the difference cannot be larger then $1/6$ as n tends to infinity

Theorem 0.2 (Bohman, Ruszinkó, Thoma).

$$\limsup_{n\to\infty}(n + 1/2 - \Theta(C_{2n+1})) \leq 1/6.$$

We conjectured [2] that the limit as n tends to infinity of the difference $n + 1/2 - \Theta(C_{2n+1})$ exists and is equal to 0. Recently, Bohman [1] found a proof of this conjecure. Although, it would be great to have a transparent construction implying the above statement which I would like to pose as as open problem.

The other problem is to identify the Shannon capacity of $G_{n,1/2}$. This is known to be between $\log n$ and $\sqrt{n}$. The lower bound comes from the independence number of $G_{n,1/2}$ and the upper bound follows from the computation of the Lovász' θ function for $G_{n,1/2}$ [4].

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Simple Colorings of Oriented Graphs

Petra Smolíková

For an integer k, a k–coloring of $\vec{G}$ is an assignement c of colors from the set $\{1, 2, \dots k\}$ to the vertices of $\vec{G}$ such that the following rules are accomplished:

(1) for any $(u, v) \in E(\vec{G})$, $c(u) \neq c(v)$;

(2) if $(u, v), (x, y) \in E(\vec{G})$ and $c(u) = c(y)$ then $c(v) \neq c(x)$.

The oriented chromatic number $\chi_o(\vec{G})$ of $\vec{G}$ is the minimum k for which there exists a k–coloring of $\vec{G}$.

We define a notion of *simple k–coloring* of an oriented graph $\vec{G}$ as an assignement c of colors from the set $\{1, 2, \dots k\}$ to the vertices of $\vec{G}$ in such a way that the two following conditions are satisfied:

(1) there are $u, v \in V(\vec{G})$ such that $c(u) \neq c(v)$;

(2) if $(u, v), (x, y) \in E(\vec{G})$ and $c(u) = c(y)$ then $c(v) \neq c(x)$ or $c(u) = c(v)$.

We define *the simple chromatic number* of $\vec{G}$ as the minimum k for which there exists a simple k–coloring of $\vec{G}$. We show that this number is close to the oriented chromatic number for some families $\mathcal{F}$ of oriented graphs in the following sense. Let $\chi_S(\mathcal{F})$ be the maximum of $\chi_S(\vec{G})$, over all $\vec{G} \in \mathcal{F}$. We show that for some families $\mathcal{F}$ of oriented graphs, $\chi_S(\mathcal{F}) = \chi_o(\mathcal{F})$. If this is true, we call such a family $\mathcal{F}$ *maximal*. Our main theorems can be stated as follows.

Theorem 1. *The family of bipartite graphs is maximal.*

Theorem 2. *For any $k > 2$, the family of partial k–trees is maximal.*

Theorem 3. *For any $a > 3$, the family of graphs of acyclic chromatic number a is maximal.*

Theorem 4. *The family $\mathcal{P}$ of planar graphs is maximal.*

It was proved in [1] that every planar graph has oriented chromatic number at most 80. Could Theorem 4 be used to find better upper bound for $\chi_o(\mathcal{P})$?

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The Upper Chromatic Number of Circular Mixed Hypergraphs

Heinz-Jürgen Voss (with Vitaly Voloshin)

A *mixed hypergraph* is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where X is the *vertex set*, $|X| = n$, and each of $\mathcal{C}$, $\mathcal{D}$ is a family of subsets of X, the $\mathcal{C}$*-edges* and $\mathcal{D}$*-edges*, respectively. A *proper k-coloring* of a mixed hypergraph is a mapping from the vertex set X to a set of k *colors* so that each $\mathcal{C}$-edge has two vertices with a common color and each $\mathcal{D}$-edge has two vertices with distinct colors. A mixed hypergraph is *k-colorable (uncolorable; uniquely colorable)* if it has a proper coloring with at most k colors (admits no colorings; admits precisely one coloring apart from permutation of colors). A *strict k-coloring* is a proper k-coloring when all k colors are used. The minimum number of colors in a strict coloring of $\mathcal{H}$ is called the *lower chromatic number* $\chi(\mathcal{H})$; the maximum number is called the *upper chromatic number* $\bar{\chi}(\mathcal{H})$.

A mixed hypergraph is *reduced* if no edge contains another edge of the same type as a subset, and moreover, the size of each $\mathcal{C}$-edge is at least 3, and the size of each $\mathcal{D}$-edge is at least 2. Here only reduced mixed hypergraphs are considered.

A mixed hypergraph $\mathcal{H}$ is called *circular* if there exists a host cycle L on the vertex set X such that every $\mathcal{C}$-edge and every $\mathcal{D}$-edge induces a connected subgraph of the cycle L. If $u \in X$ then u^+ immediately succeeds u in the direction of the host cycle L. If the host graph is a path or a tree then an *interval mixed hypergraph* or a *mixed hypertree* is obtained, investigated in [1], or in [6, 2, 7, 5, 4] and [3], respectively.

In the class of all circular mixed hypergraphs of order n we have characterized both the class of all *uncolorable* circular mixed hypergraphs by a fixed subhypergraph $\mathcal{F}_n$ and the class of all *uniquely colorable* circular mixed hypergraphs (see [8]).

In a circular mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ the subfamily Σ of $\mathcal{C}$-edges is a *sieve*, if for any $C, C' \in \Sigma$, $C \neq C'$, the intersection $C \cap C'$ induces a K_1 or K_2 of $(X, \mathcal{D})$. The maximum cardinality of a sieve of $\mathcal{H}$ is the sieve-number $s(\mathcal{H})$.

The following bounds for $\bar{\chi}(\mathcal{H})$ have been derived in [9].

Theorem 1. *Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a colorable circular mixed hypergraph with n vertices and sieve number s. Then $n - s - 1 \leq \bar{\chi}(\mathcal{H}) \leq n - s + 2$. Each of the possible four values for $\bar{\chi}(\mathcal{H})$ in this inequality is attained at some circular mixed hypergraph.*

By Theorem 1 we have four classes of circular mixed hypergraphs.

The class with upper chromatic number $\bar{\chi}(\mathcal{H}) = n - s + 2$ is the class of all circular mixed hypergraphs $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ with an even number n of vertices, where $\mathcal{C}$ consists of all possible $\mathcal{C}$-edges of size 3 and $\mathcal{D}$ consists of all possible $\mathcal{D}$-edges of size 2, i.e., $(X, \mathcal{D})$ is a cycle. $\mathcal{H}$ is called a *complete circular mixed hypergraph*.

The class of all circular mixed hypergraphs with upper chromatic number $\bar{\chi}(\mathcal{H}) = n - s + 1$ and only with $\mathcal{C}$-edges C of size $|C| \geq 5$ is characterized by the following theorem.

Theorem 2. *Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a circular mixed hypergraph with n vertices, sieve number s, and $|C| \geq 5$ for all $C \in \mathcal{C}$. Then*

$$\bar{\chi}(\mathcal{H}) = n - s + 1$$

if and only if $\mathcal{C} = \Sigma \cup \mathcal{C}'$, $\Sigma \cap \mathcal{C}' = \emptyset$; where

(i) $\Sigma = \{C_1, C_2, \ldots C_s\}$ is a maximum sieve in this cyclic order on the host cycle L with the property that $C_i \cap C_{i+1} \neq \emptyset$ and $C_i \cap C_{i+2} = \emptyset$ for all $1 \leq i \leq s$ (indices mod s), and

(ii) each $\mathcal{C}$-edge $C \in \mathcal{C}'$ has the property: there exist two $\mathcal{C}$-edges $C_i, C_{i+1} \in \Sigma$ with two common vertices, say u and u^+, such that either $u^+ \notin C$ and $C_i \setminus \{u^+\} \subset C$ or $u \notin C$ and $C_{i+1} \setminus \{u\} \subset C$, and there is no other $\mathcal{C}$-edge of $\mathcal{C}'$ containing precisely one of the vertices u and u^+.

Such a simple (only slightly more complicated) characterization can also be given for the class of all circular mixed hypergraphs with upper chromatic number $\bar{\chi}(\mathcal{H}) = n - s + 1$ in which each $\mathcal{C}$-edge C has a size $|C| \geq 4$. If $\mathcal{C}$-edges of size 3 occur a characterization is much more complicated. But in a forthcomming paper we are able to give a polynomial characterization of the class of all circular mixed hypergraphs with upper chromatic number $\bar{\chi}(\mathcal{H}) = n - s + 1$.

Here we present the open problem related with the two remaining classes.

Problem 1. *What is the complexity of the problem: decide whether or not a circular mixed hypergraph has the upper chromatic number $\bar{\chi}(\mathcal{H}) = n - s$ or $\bar{\chi}(\mathcal{H}) = n - s - 1$.*

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A Continuos Analogue of the Upper Bound Theorem

Uli Wagner (with Emo Welzl)

Abstract

This note is a brief synopsis of the findings in [4]. It aims at indicating what the Upper Bound Theorem for convex polytopes on one hand and a new extension of an old theorem from geometric probability on the other hand have got to do with each other.

1 An Extension of Wendel's Theorem

Let μ be a probability distribution on $\mathbb{R}^d$; we will assume that μ is *absolutely continuous* (a.c.), i.e. that it has a density function. Now, if we pick n independent random points $P_1, \ldots, P_n$ according to μ, what is the probability $\tilde{s}_{n-d-1}(\mu, \mathbf{0})$ (the funny subscript will be clarified later) that the origin $\mathbf{0}$ is contained in the convex hull of the P_i's?

James Wendel [6] showed that if the distribution μ is centrally symmetric about $\mathbf{0}$, then[1]

$$\tilde{s}_{n-d-1}(\mu, \mathbf{0}) = 1 - \frac{\sum_{i=0}^{d-1} \binom{n-1}{i}}{2^{n-1}}. \tag{1}$$

Theorem 1 (Wagner & Welzl, [4]). *For general μ, the probability $\tilde{s}_{n-d-1}(\mu, \mathbf{0})$ is at most the number given in (1). Moreover, this value is attained if and only if every hyperplane through $\mathbf{0}$ equipartitions[2] μ.*

We will see how and why this can be considered a continuous analogue of

2 The Upper Bound Theorem

The Upper Bound Theorem (UBT) for convex polytopes [2] gives tight upper bounds for the possible numbers of k-dimensional faces, $-1 \leq k \leq r-1$ of an r-dimensional convex polytope with n vertices. It is best formulated (and proved) in terms of *h-vectors* of simplicial polytopes[3].

The h-vector $h(P)$ of an r-dimensional simplicial polytope P has integer entries $h_0, \ldots, h_r$, each h_j being a certain linear combination of the face numbers f_k; the face numbers can be recovered from the h-vector, in fact, each f_k can be expressed as a *nonnegative* linear combination of the h_j's, cf. [7]. The UBT gives tight upper bounds (in terms of the number f_0 of vertices and the dimension r) for the h_j's, which translate to bounds for the face numbers.

The connection to Theorem 1 is formed by the following chain of links:

3 n Points and a Line

Fix a set S of n points in general position and a directed line ℓ in $\mathbb{R}^d$; the line should be disjoint from the convex hull of any $d-1$ points from S. For integer j, an oriented simplex $\sigma = [p_1, \ldots, p_d]$ spanned by an ordered d-tuple of points from S is called a *j-facet* of S if $|S \cap \mathcal{H}^+(\sigma)| = j$, where $\mathcal{H}^+(\sigma) := \{q \in \mathbb{R}^d \mid \det[\begin{smallmatrix} 1 & 1 & \cdots & \cdots & 1 \\ q & p_1 & & & p_d \end{smallmatrix}] > 0\}$ is the *positive side* of σ; the *negative side* $\mathcal{H}^-(\sigma)$ is defined analogously. Such an oriented simplex σ is said to be *entered* by ℓ if ℓ intersects the relative interior of σ in a single point and is directed from the positive to the negative side. Let us denote by $h_j(S, \ell)$ the number of j-facets of S which are entered by ℓ. Why this notation, and

[1] Note that the numerator on the right hand side is exactly the number of topes (or regions) in an arrangement of $n-1$ affine hyperplanes in dimension $d-1$.

[2] By absolute continuity, the hyperplane itself has μ-measure zero.

[3] A r-dimensional polytope is *simplicial* if all its proper faces are simplices, i.e. if any $r+1$ of its vertices are affinely independent. The face numbers are maximized by these polytopes, see [1].

why should you care? The reason is that an $(n-d)$-dimensional integer vector h is the h-vector $h(S,\ell) = (h_0(S,\ell), \ldots, h_{n-d}(S,\ell))$ of a point set S of size n and a directed line ℓ in $\mathbb{R}^d$ if and only if $h = h(P)$ for some simplicial polytope P on $f_0 = n$ vertices in dimension $r = n - d$ [5]. That is, up to a shift in dimension, h-vectors of point sets and lines and those of simplicial polytopes are the same thing. Thus, the UBT can be equivalently rephrased as

$$h_j(S,\ell) \leq \binom{j+d-1}{d-1} \tag{2}$$

for $S, \ell \subset \mathbb{R}^d$ and integer $j \geq 0$. It is this form of h-vectors and of the UBT that can be translated to

4 The Continuous Set-Up: h-Functions

We keep the directed line ℓ, but let us replace the finite set S by an a.c. probability distribution μ on $\mathbb{R}^d$, and let us replace counting by measuring. More precisely, for $0 \leq y \leq 1$, let $H_{\mu,\ell}(y)$ denote the probability that the oriented simplex $\sigma = [P_1, \ldots, P_d]$ spanned[4] by d independent μ-random points is entered by ℓ and that $\mu(\mathcal{H}^+(\sigma)) \leq y$. This defines a "distribution" function $H_{\mu,\ell}$ from the unit interval $[0,1]$ into itself, which can be shown to be continuously differentiable [4]. This allows us to define the *h-function* of μ and ℓ to be the derivative

$$h(y) = h_{\mu,\ell}(y) := \frac{\mathrm{d}H(y)}{\mathrm{d}y}. \tag{3}$$

(Intuitively speaking, $h_{\mu,\ell}(y)$ measures the "probability" that a "y-facet" of μ is entered by ℓ.) Thus we see that h-functions are, in a sense, the natural extension of h-vectors to the continuous realm. What is more, they also enjoy many key properties (reflecting those) of their discrete twins, for instance

$$h_{\mu,\ell}(y) = h_{\mu,\ell}(1-y)\,, \tag{4}$$

in analogy to the Dehn-Sommerville Relations[5]. The advertised continuous analogue of the UBT is this:[6]

Theorem 2.

$$h_{\mu,\ell}(y) \leq \frac{d}{2} \min\{y^{d-1}, (1-y)^{d-1}\}\,. \tag{5}$$

[4]By absolute continuity, the random points are almost surely in general position. For this and other technical details, cf. [4].

[5]Which state that $h_j(S,\ell) = h_{n-d-j}(S,\ell)$, see [5].

[6]Here, equality holds if and only if every hyperplane containing ℓ equipartitions μ.

Currently, we are investigating continuous analogues of the *Generalized Lower Bound Theorem* (GLBT) and the *g-Theorem*.[7]

Of course, we haven't told you any proofs, and moreover, the question remains unanswered what all this has to do with Theorem 1. In fact, these issues are intertwined; the missing link is provided by

5 Random Liftings and Moments

Let us modify the set-up of Theorem 1: We identify $\mathbb{R}^d$ with the subspace $\mathbb{R}^d \times \{0\} \subset \mathbb{R}^{d+1}$. Now, take your favorite a.c. probability distribution on the real line, for instance the uniform distribution on some bounded interval, and pick, for each μ-random point P_i, a $(d+1)^{\text{st}}$ coordinate $X_i \in \mathbb{R}$ according to that distribution.[8] Then the lifted points $\tilde{P}_i := (P_i, X_i)$ are i.i.d. according to some a.c. probability measure $\tilde{\mu}$ on $\mathbb{R}^{d+1}$, and if we take $\tilde{\ell}$ to be the x_{d+1}-axis, then[9]

$$\begin{aligned} \tilde{s}_k(\mu, \mathbf{0}) &= \Pr[\,\ell \text{ intersects } \operatorname{conv}\{\tilde{P}_1, \ldots \tilde{P}_{k+(d+1)}\}\,] && (6) \\ &= 2\binom{d+k+1}{k} \mathrm{int}_0^1 y^k \tilde{h}(y) \mathrm{d}y && (7) \end{aligned}$$

where $\tilde{h} = h_{\tilde{\mu},\tilde{\ell}}$ is the h-function of $\tilde{\mu}$ and $\tilde{\ell}$, see [4]. Thus, (5) immediately implies Theorem 1.

On the other hand, this expresses the *moments* $M_k(\tilde{h}) = \mathrm{int}_0^1 y^k \tilde{h}(y)\,\mathrm{d}y$ in terms of a quantity which is invariant under permutations of the $\tilde{P}_i$'s and stems from a lower dimension, and $\tilde{h}$ is uniquely determined by the sequence $(M_k)_{k \in \mathbb{N}_0}$ of its moments, see [4]. This quite directly yields the continuous Dehn-Sommerville Equation (4), and provides a stepstone for an inductive proof of Theorem 2.

6 Discussion

We have analyzed the probability of the convex hull of n random points (i.i.d. $\sim \mu$) to contain the origin. For that we borrowed the seminal concept of h -vectors from polytope theory, and introduced their continuous counterparts, h-functions.

Here is a challenging related open question. For μ on $\mathbb{R}^2$ given, consider

$$a(\mu) = \mathrm{Prob}\,[\,P_1 \in \operatorname{conv}\{P_2, P_3, P_4\}\,]$$

[7] A note on how to derive the continuous GLBT from the discrete one can be found at `http://www.inf.ethz.ch/~uli`.

[8] In such a way that the X_j's and P_i's are mutually independent.

[9] Observe that this does not depend on the distribution of the $(d+1)^{\text{st}}$ coordinate.

for points P_1, P_2, P_3, P_4 i.i.d. $\sim \mu$. What is a tight upper bound for this probability over all (a.c.) probability distributions μ on $\mathbb{R}^2$? The probability $a(\mu)$ determines

$$b(\mu) = \mathrm{Prob}\,[\,\mathrm{conv}\{P_1, P_2\} \cap \mathrm{conv}\{P_3, P_4\} \neq \emptyset\,]$$

by the relation $4a(\mu) + 3b(\mu) = 1$. Hence, it is closely connected to the problem of determining a good bound for the crossing number of straight-line embeddings of complete graphs, cf. [3].

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List of participants

Robert Babilon
Charles University, Prague
Czech Republic
babilon@kam.ms.mff.cuni.cz

Imre Bárány
Alfré Rényi Institute of Mathematics, Budapest
Hungary
barany@renyi.hu

Laura Ciobanu
Rutgers University, New Jersey
USA
ciobanu@math.rutgers.edu

Ricardo Collado
University of Puerto Rico
USA
a931313@goliath.cnnet.clu.edu

Sidney Cox
USA
sidney_cox@yahoo.com

Shalom Eliahou
L'Université du Littoral Cote d'Opale
France
Shalom.Eliahou@lmpa.univ-littoral.fr

Jiří Fiala
Charles University, Prague
Czech Republic
fiala@kam.ms.mff.cuni.cz

Frank Göring
Technische Universität Ilmenau
Germany
Frank.Goering@mathematik.tu-ilmenau.de

Jean Griffin
Rutgers University, New Jersey
USA
Jeangrif@eden.rutgers.edu

Jochen Harant
Technische Universität Ilmenau
Germany
harant@mathematik.tu-ilmenau.de

Peter Horak
Kuwait University, Safat
Kuwait
horak@math-1.sci.kuniv.edu.kw

Jan Kára
Charles University, Prague
Czech Republic
jack@atrey.karlin.mff.cuni.cz

Gyula Károlyi
Eötvös University, Budapest
Hungary
karolyi@cs.elte.hu

Martin Klazar
Charles University, Prague
Czech Republic
klazar@kam.ms.mff.cuni.cz

Martin Kochol
MÚ SAV, Bratislava
Slovakia
kochol@savba.sk

Jan Kratochvíl
Charles University, Prague
Czech Republic
honza@kam.ms.mff.cuni.cz

Bodo Laß
RWTH Aachen
Germany
lass@math2.rwth-aachen.de

Stephen Marotta
Drew University, Madison
USA
smarotta@drew.edu

Jiří Matoušek
Charles University, Prague
Czech Republic
matousek@kam.ms.mff.cuni.cz

Jana Maxová
Charles University, Prague
Czech Republic
jana@kam.ms.mff.cuni.cz

Jaroslav Nešetřil
Charles University, Prague
Czech Republic
nesetril@kam.ms.mff.cuni.cz

Helena Nyklová
Charles University, Prague
Czech Republic
nyklova@kam.ms.mff.cuni.cz

Ondřej Pangrác
Charles University, Prague
Czech Republic
pangrac@kam.mff.cuni.cz

Pavel Podbrdský
Charles University, Prague
Czech Republic
podbrdsky@kma14.karlin.mff.cuni.cz

Andrzej Proskurowski
University of Oregon, Eugene
Oregon, USA
andrzej@cs.uoregon.edu

Miklós Ruszinkó
SZTAKI, Budapest
Hungary
ruszinko@helly.maca.sztaki.hu

Robert Šámal
Charles University, Prague
Czech Republic
samal@atrey.karlin.mff.cuni.cz

Jakub Šimek
Charles University, Prague
Czech Republic
simek@kam.ms.mff.cuni.cz

Petra Smolíková
Charles University, Prague
Czech Republic
petra@kam.ms.mff.cuni.cz

William Steiger
Rutgers University, New Jersey
USA
steiger@cs.rutgers.edu

Eberhard Triesch
RWTH Aachen
Germany
triesch@math2.rwth-aachen.de

Pavel Valtr
Charles University, Prague
Czech Republic
valtr@kam.ms.mff.cuni.cz

Jan Vondrák
Charles University, Prague
Czech Republic
vondrak@kam.ms.mff.cuni.cz

Heinz-Jürgen Voss
Technische Universität Dresden
Germany
voss@zpr.uni-koeln.de

Uli Wagner
ETH Zentrum, Zürich
Switzerland
uli@inf.ethz.ch

Ryan Williams
Cornell University, Ithaca
USA
rrw9@cornell.edu