On covers of graphs

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Abstract

We consider two problems related to the Cycle Double Cover (CDC) conjecture for graphs: the Oriented Perfect Path Double Cover (OPPDC) and the Oriented Faithful Cycle Cover. While we characterize the later the OPPDC is an interesting open problem.

Key words: graph, cycle double cover, perfect path double cover

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1 Introduction

A CYCLE DOUBLE COVER (CDC) of a graph G is a collection of its cycles such that each edge of G lies in exactly two of the cycles. A well known conjecture of P. D. Seymour asserts that every simple bridgeless graph has a CDC [11].

Several various versions of this conjecture were studied, usually some further conditions on the CDC were added (see e.g. [2], [3], [4],[6], [7], [14]).

Here we concentrate on oriented versions of these problems.

In section 3 we define an Oriented Perfect Path Double Cover (OPPDC) of a simple graph and we try to characterize graphs with an OPPDC. We give two examples of graphs (K_3 and K_5) which fail to have an OPPDC and we show some classes of graphs which admit an OPPDC.

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In section 4 we introduce the notion of Oriented Faithful Cycle Covering of graphs. We prove that a simple graph G admits an oriented faithful cycle cover for every admissible weight vector p if and only if G has no K_4-minor.

2 Notation and Terminology

In this paper we denote by $G = (V, E)$ a finite undirected graph (with no loops or multiple edges) and by $S(G)$ the symmetric orientation of G, that is an oriented graph obtained from G by replacing each edge of G by a pair of oppositely directed arcs (e.g. $V(S(G)) = V(G)$ and $E(S(G)) = \{(u, v) \mid \{u, v\} \in E(G)\}$). A *cycle* (or *even* subgraph) in G is a subset of edges $F \subseteq E$ such that each vertex is incident with an even number of edges. An *oriented cycle* in $S(G)$ is a subset of arcs $F \subseteq E$ such that for each vertex its out-degree equals to its in-degree. A *circuit* (an *oriented circuit*) is a minimal nonempty cycle (oriented cycle). By an *(oriented) path* we always mean a simple (oriented) path.

Any graph obtained from G by successive deletions and contractions of edges is called a *minor* of G (see [13] for details). If H is a cubic graph then H is a minor of G if and only if H is homeomorphic to a subgraph of G.

For a subset U of $V(G)$, the set of edges $\delta(U) = (U, V(G) \setminus U)$ which have exactly one endvertex in U is called an *edge-cut* in G. A *bridge* is an edge-cut of cardinality 1. A graph with no bridges is called *bridgeless.*

3 Oriented Perfect Path Double Cover

An ORIENTED PERFECT PATH DOUBLE COVER (OPPDC) of a graph G is a collection of oriented paths in the symmetric orientation $S(G)$ such that each edge of $S(G)$ lies in exactly one of the paths and for each vertex of G there is a unique path which begins in v (and thus there is also a unique path which ends in v).

The notion of Oriented Perfect Path Double Cover (OPPDC) is a natural strengthening of (unoriented) Perfect Path Double Cover (PPDC) introduced by J. A. Bondy in [4]. It was proved by H. Li [7] that every simple graph admits a PPDC.

Despite of the fact that there is a lot of positive CDC-related results in this area not every graph admits an OPPDC. One can easily show that K_3 has no OPPDC and more tediously that the same is true for K_5 (see e.g. [1] or

[9]). The complete graphs K_3 and K_5 are presently the only known examples of connected graphs which have no OPPDC. An easy construction gives an OPPDC for all K_{2n}. T. Tillson proved in [12] that all K_{2n+1} have an OPPDC for $n \geq 3$. Thus the question for complete graphs is solved.

We believe that the complete graphs K_3 and K_5 are the only exceptions for OPPDC and we conjecture the following:

Conjecture 3.1 *K_3 and K_5 are the only connected graphs which do not have an OPPDC.*

In the remaining part of this section we give some supporting evidence for this conjecture. Some positive results can be found in [9]. For example if we add a new vertex of degree one, two or three to a graph which has an OPPDC then the resulting graph also has an OPPDC. Hence the minimal graph G ($G \neq K_3$, $G \neq K_5$) which has no OPPDC has all degrees at least four.

These results have several interesting consequences (see [9] for proofs):

Proposition 3.2 *If G is a union of two arbitrary trees, $G \neq K_3$ then G has an OPPDC.*

Proposition 3.3 *Every graph $G \neq K_3$ with no adjacent vertices of degree ≥ 3 has an OPPDC.*

Proposition 3.4 *If G is a 2-connected graph with $|E(G)| \leq 2n-1$, $G \neq K_3$, then G has an OPPDC.*

Another construction which preserves the property of having an OPPDC is so-called *arrow construction.*

Definition:
A graph I with two distinguished vertices a, b, $\{a, b\} \notin E(I)$, is called an *indicator*. For a given oriented graph $D = (V, E)$ and an indicator (I, a, b) we define an (undirected) graph $D * (I, a, b) = (W, F)$ as follows:

$$W = (E \times V(I))/\sim$$

where the equivalence $\sim$ is generated by the following pairs:

$$((x, y), a) \sim ((x, y'), a), \quad ((x, y), b) \sim ((x', y), b), \quad ((x, y), b) \sim ((y, z), a).$$

For a pair $(e, x) \in E \times V(I)$ its equivalence class is denoted by $[e, x]$.
We put $\{[e, x], [e', x']\} \in F \iff e = e'$ and $\{x, x'\} \in E(I)$.

This arrow construction is schematically indicated on Fig.1. (One can check that the indicator I on Fig.1 satisfies the assumptions of theorem 3.5.)

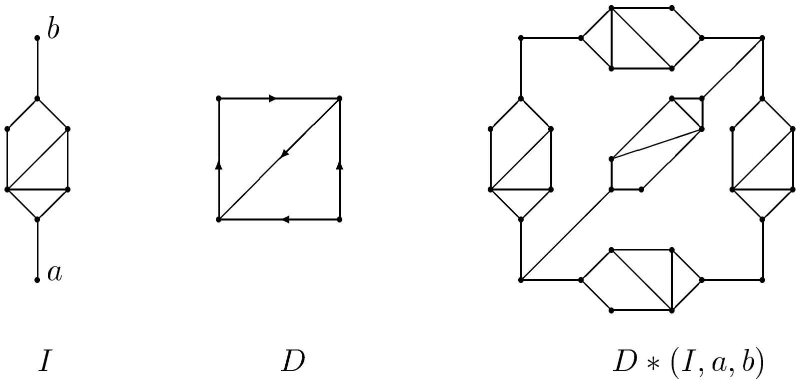

Fig. 1 : Arrow construction

Theorem 3.5 *Suppose an indicator* (I, a, b) *has an OPPDC* Π *containing two paths* $P_1, P_2 \in \Pi$ *such that* P_1 *begins in* a *and ends in* b*, and* P_2 *begins in* b *and ends in* a*. Further suppose* G *has an OPPDC. Then for any orientation* D *of* G *the graph* $D * (I, a, b)$ *has an OPPDC.*

Remark: It is even possible to replace different arcs with different indicators.

Proof (of theorem 3.5): If $\Pi = (P_1, P_2, \ldots, P_t)$ is an OPPDC of I satisfying the assumptions of theorem 3.5 and $\Pi' = (P'_1, P'_2, \ldots, P'_s)$ is an OPPDC of G then symbolically the family of paths $\{[e, P_i] \mid 3 \leq i \leq t,\ e \in E(D)\} \cup \{P'_j * P_{1,2} \mid 1 \leq j \leq s\}$ forms an OPPDC of $D * (I, a, b)$. By $[e, P_i]$ we denote the path $P_i \in \Pi,\ i \geq 3$, in the copy of I which corresponds to $e \in E(D)$ and by $P'_j * P_{1,2}$ we denote the concatenation of the paths P_1 and P_2 along the path P'_j. That is, we replace each arc $(u, v) \in P'_i$ by P_1 or P_2 according to whether $(u, v) \in E(D)$ or $(v, u) \in E(D)$.

This construction has an interesting corollary. By proposition 3.3 there exists a rigid graph I with an OPPDC with the additional properties on P_1 and P_2 as in theorem 3.5 and thus, using standard techniques, graphs with an OPPDC represent all groups and monoids, and in fact there is an embedding of all graphs into graphs with an OPPDC (in the sense of e.g. [10]).

4 Oriented Faithful Cycle Cover

Given a graph $G = (V, E)$ and a nonnegative weight function p on edges of its symmetric orientation $S(G)$ (i.e. $p : E(S(G)) \longrightarrow \mathbf{Z}^+$), we wish to find a collection of cycles in $S(G)$ (of length ≥ 3) such that each arc $e \in E(S(G))$ is contained in exactly $p(e)$ cycles of the collection. Such a collection we call an ORIENTED FAITHFUL CYCLE COVER of (G, p).

For an edge-cut $B = \delta(U) = (U, V \setminus U)$ in the underlying graph G, we denote $\overrightarrow{B_U} = \{(u, v) \mid u \in U,\ v \in V \setminus U\}$ and $\overleftarrow{B_U} = (u, v) \mid u \in V \setminus U,\ v \in U\}$ to distinguish the orientation of arcs in B. Moreover, for $e = (u, v)$ let us denote by $e^- = (v, u)$ the opposite arc to e. We further define $p(B) = \sum_{e \in \overrightarrow{B}} p(e)$.

It's obvious that the weight vector p has to satisfy two necessary conditions:
(i) For each edge-cut $B = (U, V \setminus U)$ in G:

$$p(\overrightarrow{B}) = p(\overleftarrow{B})$$

(ii) For each edge-cut $B = (U, V \setminus U)$ and each arc $e \in \overrightarrow{B}$:

$$p(e) \leq p(\overleftarrow{B} \setminus e^-),$$

$$\text{or equally} \quad p(\overleftarrow{B}) \geq p(e) + p(e^-).$$

A nonnegative integer valued weight vector p is called *admissible* if p satisfies both conditions (i) and (ii).

If equality holds in (ii) then the edge cut B is called a *tight cut* and the arc e is called a *tight cut leader*. It's easy to see that if B is a tight cut with a tight cut leader e then e^- is a tight cut leader as well (for the same tight cut B) and $p(\overrightarrow{B}) = p(\overleftarrow{B}) = p(e) + p(e^-)$.

Our main result is the following theorem.

Theorem 4.1 *G has an oriented weighted cycle cover for each admissible weight vector p if and only if G contains no K_4-minor.*

We were motivated by the analogous problem for undirected weighted graphs stated by P. D. Seymour in [11] who also formulated similar necessary conditions for the weight vector p. B. Alspach, L. Goddyn and C.-Q. Zhang proved in [2] that a graph G has an (undirected) faithful cycle cover for every admissible p if and only if G has no Petersen minor. Note that theorem 4.1 implies that for oriented graphs we have much simpler situation.

Proof (of theorem 4.1):
One implication is easy. It's enough to consider K_4 weighted as on the Fig.2 (arcs of weight 0 are not shown, all shown arcs have weight 1). It's easy to verify that this weighted graph satisfies both conditions (i) and (ii) but has no oriented faithful cycle cover.

Now if G has a K_4-minor then G contains a subdivision of K_4 as a subgraph

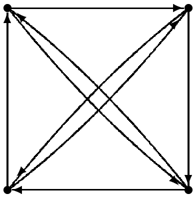

Fig. 2 : K_4 with no OFCC

(since K_4 is cubic). Define p on the arcs of the subdivision of K_4 as on Fig.1 and $p = 0$ on the remaining arcs (out of the subdivision of K_4). We again obtain an admissibly weighted graph G which has no oriented faithful cycle cover.

For the crucial implication we need to show that all K_4-minor-free graphs have an oriented faithful cycle cover for every admissible p.

We may assume that G is 2-connected (otherwise consider blocks of G separately). It is well known (see series-parallel graphs in [11]) that all 2-connected K_4-minor-free graphs can be constructed from C_3 by subdividing edges and by adding a path of length two parallel to an existing edge.

It is clear that C_3 has an oriented faithful cycle cover for every admissible p. The following two statements complete the proof.

Lemma 4.2 *Let G' arise from G by subdividing an edge of G. If G has an oriented faithful cycle cover for every admissible p then G' has an oriented faithful cycle cover for every admissible p as well.*

Proof: Trivial.

Lemma 4.3 *Let G' arise from G by adding a new vertex x and a path uxv parallel to the edge $\{u, v\} \in E(G)$. If G has an oriented faithful cycle cover for every admissible p then G' has an oriented faithful cycle cover for every admissible p as well.*

Proof: Let (G', p) be an admissibly weighted graph. We proceed by induction on $s = p(u, x) + p(x, u) + p(x, v) + p(v, x)$.

If $s = 0$ then p can be viewed as a weight function on $E(S(G))$, the arcs of the original graph, and we are done.

For $s \geq 1$ we consider three cases.

Case 1: If there is an oriented circuit $C = uxv$ or $C = vxu$ such that (G', p') is admissible where $p' = p - 1$ on $E(C)$ and $p' = p$ elsewhere then we are done by induction.

Case 2: If there is no oriented circuit $C = uxv$ or $C = vxu$ (i.e. there is an arc of weight 0 in both of the circuits) then, similarly as for $s = 0$, p can be viewed as a weight function on $E(S(G))$, the arcs of the original graph (except that the edge (u, v) may be subdivided).

Case 3: There is an oriented circuit (without loss of generality $C = uxv$) such that (G', p') is not admissible where again $p' = p - 1$ on $E(C)$ and $p' = p$ elsewhere. It's obvious that only condition (ii) can be violated. This means that two of the arcs (u, x), (x, v) and (v, u) lie in a tight cut $B = (U, V \setminus U)$ in the original weighted graph (G', p). Let b be a tight cut leader of B, that is

$$p(b) = p(\overleftarrow{B} \setminus b^-).$$

$$p(\overrightarrow{B}) = p(b) + p(b^-). \tag{1}$$

We may further asume that $u \in U$ and $v, x \in V \setminus U$, that is $(u, x) \in \overrightarrow{B} \setminus b$ and $(v, u) \in \overleftarrow{B} \setminus b^-$.

We define a new weight vector p' in the following way:
$p'(e) = p(e)$ for all arcs $e \in E(S(G))$, $e \neq (u, v), (u, x), (x, v)$,
$p'(u, v) = p(u, v) + 1$,
$p'(u, x) = p(u, x) - 1$,
$p'(x, v) = p(x, v) - 1$.

If (G', p') is admissible then by induction hypothesis it has an oriented faithful cycle cover which can be modified in an obvious way to an oriented faithful cycle cover of (G', p).

If (G', p') is not admissible then (since again only condition (ii) can be violated) there is a tight cut $D = (U, V \setminus U)$ in the original weighted graph (G', p) with a tight cut leader $d = (u, v)$. Again it holds that

$$p(\overrightarrow{D}) = p(d) + p(d^-). \tag{2}$$

Now we are ready to prove that p was not an admissible weight vector on $E(S(G'))$. Indeed, the edge-cut $B \bigtriangleup D$ does not satisfy the condition (ii), according to (1) and (2):

$$p(\overrightarrow{B \bigtriangleup D}) = p(\overrightarrow{B}) + p(\overrightarrow{D}) - 2p(\overrightarrow{B \cap D}) \leq$$

$$\leq p(b) + p(b^-) + p(d) + p(d^-) - 2\max\{p(d) + p(u, x), p(d^-)\} < p(b) + p(b^-),$$

since $d = (u, v) \in \overrightarrow{B \cap D}$, $(u, x) \in \overrightarrow{B \cap D}$, $e^- = (v, u) \in \overleftarrow{B \cap D}$ and $p(\overrightarrow{B \cap D}) = p(\overleftarrow{B \cap D})$. This completes the proof of theorem 4.1.

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