On Universality of Set Systems

Jaroslav Nešetřil*
Department of Applied Mathematics
and
Institute for Theoretical Computer Science (ITI)
Charles University
Malostranské nám.25, 11800 Praha 1
Czech Republic
Email: nesetril@kam.ms.mff.cuni.cz

Abstract

We introduce here a new structure called *multicuts*. We prove that the class $\mathcal{MC}$ of all multicuts is universal, i.e. that this class represents any partially ordered class. We then show how to represent multicuts by set systems and relations. In the case of infinite relations this is related to variants of the Vopěnka principle and we prove that these variants are always stronger than the new principle $\mathcal{P}(\mathcal{I}^{op})$.

On the other hand the constructions given in this paper present the shortest proof of (proper class) - universality of the partial order of set systems.

1 Introduction and Statement of Results

A *cut* in a partially ordered set $P = (X, \leq)$ is any downward closed subset of X. Cuts (with the inclusion ordering) may be used to represent P as well as to enlarge P into the smallest complete poset. These classical results

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started with Dedekind construction of real numbers and continued with many variants culminating perhaps with the construction of the McNeille completion for arbitrary posets. This is well documented in most textbook on foundations of set theory.

But it is possible to say that these classical ideas are surprisingly pertinent until today (as nicely put by Rota [27]). One witness of this phaenomenon is the number system ("surreal numbers") due to Conway [3] which uses the cuts to generate *on line* the new numbers from the old one.

Our results continue in the same direction, yet our task is more complicated: we want to construct a given (large; even proper class) poset in the situation when the poset is given to us step by step, one vertex at each step, by an adversary (or our enemy).

More precisely, by an *on-line representation* of a poset P we mean that one can construct a representation of P under the circumstances that the elements of P are revealed one by one (without a priori knowledge about the whole poset P).

The on-line representation of can be thought as a game of two players A and B (Bob again and Alice). Bob (the destroyer) selects P and reveals the elements of P one by one to Alice. Whenever an element x of P is revealed, the relation among x and previously revealed elements is also revealed. Alice is required to contstruct a graph (or a structure) G_x. Alice wins if the graphs G_x which she constructed during the game represent poset P. (In most of this paper we want that Alice will win, only in Section 6 we sometimes switch the sides.)

This can be formulated also for partial orders which are large (i.e. which have a proper class of vertices) when such partial orders are indexed by ordinal numbers. However the advantage of this formulation is that On line representation by finite structures is a finite problem (even if the resulting poset is infinite) and On line representation by set structures is a set problem (even if the resulting partail order is a proper class).

From the abstract point of view the on-line representation is nothing else than a proof of representation by induction (or by a transfinite induction). Yet we believe that the concept of on-line representation provides a new aspect to this area. This can be illustrated by the following:

A structure (such as a graph) which represents any countable graph is said to be *(countably) universal* structure. It is well known that universal graphs exists. They are easy to construct by induction by extending a given finite graph with n vertices to an $(n+1)$-vertex graph in all possible ways. This is possible to do as graphs have *(unbounded) extension property.* Similar situation occurs in all examples which we are considering and thus the universal structure always exist. In fact the countable structure with the extension property is uniquelly determined (up to an isomorphism) and this is a part of classical model theory see e.g. [8, 2]. We shall need an extension of these results to structures defined on proper classes of vertices but this is an easy (and folkloristic) variant.

One can sometimes even find explicite examples of universal structures which are easy to describe. For example the graph $\mathcal{R}_\omega$ whose vertices are all finite subsets of positive integers with two sets A, B forming an edge $\{A, B\}$ of $\mathcal{R}$ if and only if $A \in B$. $\mathcal{R}_\omega$ is called *Rado graph*, [25], (or the *Erdös-Rényi Random graph*).

The countable universal oriented graph $\mathcal{O}_\omega$ can be constructed as follows: vertices of $\mathcal{O}_\omega$ are all pairs of finite subsets $A = (A_L, A_R)$ of positive integers with two vertices A, B beeing joined by the arc (A, B) iff either $A \in B_L$ and $A < B$ or $B \in A_R$ and $A > B$ ($<$ is the lexicographic ordering of subsets). There is here more than meet an eye as this particular examples motivates our multicuts.

One can provide further explicite examples of universal graphs and structures. And it is easy to extend these constructions to proper class of vertices.

However, for partially order sets we are facing much more complicated situation and merely the existence of universally partially ordered set and classes was discovered several times [11, 10, 28].

The first concise structure which induces universal partially ordered class has been constructed by Z. Hedrlín [5]. His structure $\mathcal{H}$ is easy to describe:

The vertices of $\mathcal{H}$ are sets $A, B, \ldots$ with partial order defined by $A \leq B$ if there exists a mapping $f : B \to A$ such that $f(x) \subset x$ for every $x \in B$.

Hedrlín proved [5] that $\mathcal{H}$ is universal partially ordered class. This is non-trivial and complicated (and clearly an exercise in "bracketting"). In fact it is not a simple matter to prove it even in the finite case. The representation of a given poset given in [5] cannot be used for the purposes of this paper.

Here we present a new structure which induces universal partially ordered class. It is called *multicuts*, the class of all multicuts will be denoted $\mathcal{MC}$. The structure of multicuts is more special and this will lead us to a more tuned on-line representation of any poset (and of partially ordered class).

The multicuts are described in the section 2. Here we introduce some results:

Theorem 1.1 *The class of all multicuts $\mathcal{MC}$ is universal.*

Explicitely: For every partial ordered class P ordered by $\leq_P$ there exists an injective mapping $\phi : P \to \mathcal{MC}$ such that the following conditions hold (for all $x, y \in P$):

$$x \leq_P y \text{ if and only if } \phi(x) \leq \phi(y)$$

Universality of posets has been studied also in a different context of theory of categories. One of the main goals of this study was to find easy and natural structures which are universal. One of the basic such structures are set systems.

A *set system* (or, shortly, a *system*) S is a pair $(X, \mathcal{E})$ where X is a non-empty set and $\mathcal{E}$ is a subset of the power set 2^X. Motivated by the binary case (i.e. graphs; another name for set systems is *hypergraphs*; as our systems will be mostly infinite we do not use this term in this paper) we call the elements of X *vertices* and elements of the set $\mathcal{E}$ *edges* of S. Although $\mathcal{E}$ may be an empty set we do not allow empty edges.

Set systems will be denoted by $S, S', A, B, \cdots$. For a set system $S = (X, \mathcal{E})$ we also write $X = V(S)$ and $E(S) = \mathcal{E}$.

Given set systems $S = (X, \mathcal{E})$ and $S' = (X', \mathcal{E}')$ a *homomorphism* $f : S \to S'$ is any mapping $f : X \to X'$ which satisfies $\{f(x); x \in M\} \in \mathcal{M}'$ whenever $M \in \mathcal{E}$. Thus homomorphisms are thoese mappings which preserve the edges of set systems. By an abuse of the notation the set $\{f(x); x \in M\}$ will be denoted shortly by $f(M)$.

In this paper we want to study set systems from the point of view of the existence of homomorphism between them. Towards this end we also write $S \leq S'$ if there exists a homomorphism $S \to S'$.

$\leq$ is a quasiorder as clearly we may have set systems which are non isomorphic and which are homomorphic to each other. In this case we say that such systems are *hom-equivalent*.

Consequently, if we consider only mutually non-hom-equivalent set systems then we obtain a partial order. (More precisely, we select from each hom-equivalence class a representative.)

The proper class of all mutually non-homomorphic equivalent set systems will be denoted by $\mathcal{S}$ and thus $\mathcal{S}$ with the relation $\leq$ is a partially ordered class.

In this case of finite set systems $\mathcal{S}$ is a countable partially ordered set.

The class $\mathcal{S}$ has spectacular features (for example it is everywhere dense, see [16] and Section 5) and it is related to some of the key combinatorial notions such as chromatic number, see e.g. [20, 7, 21].

This class has been also studied in the context of category theory in several papers by Hedrlín, Pultr and Trnková and others. The class $\mathcal{S}$ considered with the all homomorphisms betweeen systems is called in [24] category $S(P^+)$ and our partially ordered class is the thinning of this category

One of the main results obtained by Hedrlín and Pultr [24] is the proof of the universality of the category $S(P^+)$ which implies the following:

Theorem 1.2 *$\mathcal{S}$ is universal partially ordered class.*

Explicitely: For every partially ordered class $(\mathcal{X}, \leq_{\mathcal{X}})$ there exists an injective mapping $\phi : \mathcal{X} \to \mathcal{S}$ with the following property:

$$A \leq_{\mathcal{X}} B \text{ if and only if } \phi(A) \leq \phi(B)$$

Let us remark that *no* set-theoretic assumptions (except ZFC) are used in the proof of the above result. This is due to the fact that the set systems have unbounded arities. For first order theories (such as graphs, or relational systems of a given type) the validity of Theorem 1.2 depends on set theoretic assumptions called *Vopěnka Axiom* which can be formulated as follows, see [9, 12]:

$\mathcal{VP}$: Every proper class of graphs contains two homomorphism comparable graphs.

In a combinatorial setting this is a very natural principle. It amounts to say that graphs (and all similar structures) are Class Well Quasi Ordered:

We say a partial order $\leq$ defined on a class $\mathcal{X}$ is *Class Well Quasi Ordered*, shortly ClassWQO, if it does not contain a proper class of incomparable elements and also if it does not contain a proper class descending chain.

Explicitely, there are no elements $x_\alpha, x_\alpha \in \mathcal{X}$, indexed by the class On of all ordinals such that the class $x_\alpha; \alpha \in On$ is ordered in one the following two ways:

i. $x_\alpha \not\leq x_\beta$ whenever α and β are distinct ordinals (i.e. $x_\alpha; \alpha \in On$ form a proper class of incomparable elements);

ii. $x_\alpha \leq x_\beta$ if and only if $\alpha \geq \beta$ (i.e. $x_\alpha; \alpha \in On$ form a proper class of elements with the ordering which is inverse ordering of ordinals).

The Vopěnka Axiom then states that any class of all graphs quasiordered by the existence of a homomorphism is Class Well Quasi Ordered (and indeed that the same is true for any first-order theory). This is a very surprising statement which needs some time to digest (especially when one realizes that we may assume that all the homomorphisms may be assumed injective).

This perhaps should be compared with the notion of *Well Quasi Ordered* set (shortly WQO) which assumes conditions *i.* and *ii.* for sets $x_\alpha \in \mathcal{X}$ indexed by natural numbers. WQO is by now a developed branch which started with [14] and culminated with Robertson - Seymour project solving so called Wagner conjecture (which amount s to say that the class of all finite graphs is WQO with respect to the operation *minor*), see e.g. [26] and with theorems due to Kříž which present the strongest WQO results for infinite graphs [13].

The formal analogy of these two concepts (i.e. of ClassWQO and WQO) does not stop here: For example the condition *ii.* for WQO is satisfied in most "natural" orders and also for ClassWQO in most cases (such as graphs or relational systems) the condition *i.* implies *ii.*

Yet it is perhaps surprising that ClassWQO seem to be a common property of all easily described structures which are bounded in the sense that the relations which are involved in their definition have all arities bounded by a constant (in the case of graphs 2). Note that the class $\mathcal{S}$ does not have bounded arities.

This remarkable Vopěnka's insight was gained during the project ("Prague School") of embedding of categories and universality properties of graphs and other "algebraic" structures in the 60ties, see [24, 5] for the description of this development.

The proof of Theorem 1.2 given in the category theory context is complicated and is spread in several papers (see [6, 5, 4]), see [24] for the complete proof. Even for the finite case this was until recently the only proof available.

We present here an independent proof of Theorem 1.2 which is based on the *extension properties* (of which density is a particular case) of the class S. We believe that this is a more direct and easier proof.

Let us remark that we can use and refine techniques developed in this paper to provide an alternative proof of the universality of the class of all finite graphs. This is going to appear in [19].

The paper is organized as follows:

In Section 2 we introduce Multicuts and the class $\mathcal{MC}$ and prove the universality of $\mathcal{MC}$ (Theorem 1.1).

In Section 3 we relate Universality to the representation of a particular order $\mathcal{I}^{op}$ (of all sets with the reverse inclusion). This will be used in Section 4 to prove Theorem 1.2 for set stystems. In Section 5 we approach the problem from the point of view of extension properties of the class $\mathcal{S}$ and we completely characterize extension properties related to the two ingredients of ClassWQO: density and independence. In the Section 6 we approach the problem from yet another point of view of large cardinals. The problem of embedding of a particular *test* partial order class Y into the class of all graphs $\mathcal{G}$ leads to large cardinal $\kappa(T)$ (which is always non-measurable). We determine the smallest such cardinal by using the techniques developed in Section 3. The Section 7 contains some remarks and open problems.

2 Multicuts

The class of all ordinal numbers will be denoted by On, ordinals will be denoted as customary by $\alpha, \beta, \ldots$. Every ordinal number is a set $\alpha = \{\beta : \beta \in \alpha\}$ which is well ordered by the relation $\in$. The ordering of ordinals will be always denoted by $\in$ to avoid the confusion. The more frequent symbol $\leq$ will be reserved for other orderings (thus we write $3 \in 5$ and $\iota \in \omega$).

Multicuts will be defined as special sets -pointed sets- of ordinals: A set $\{A_\iota; \iota \in I\}$ of non-empty sets of ordinals is said to be *pointed set* if the following holds:

i. Each of the sets A_ι has the maximal element (denoted by $\max A_\iota$);

ii. For distinct sets the maxima are distinct;

iii. If $\max A_\iota \in A_\lambda$ then $A_\lambda \cap \max A_\iota \supseteq A_\iota$.

A *multicut* C is a a set of non-empty pointed sets.

Remark that the empty set is a multicut (however all sets in a multicut are non-empty).

For multicuts $C, C' = \{A_{\iota'}; \iota' \in I'\}$ we write $C \leq C'$ if for every $\iota' \in I'$ there exists $\iota \in I$ such that $A_\iota \subset A_{\iota'}$.

Clearly $\leq$ is a quasiorder. We denote by $\mathcal{MC}$ the class of all multicuts representing each equivalence class of $\leq$ together with the corresponding partial order (denoted again $\leq$).

We shall prove that the partial order $\mathcal{MC}$ represents any partially ordered class (Theorem 1.1). This embedding will also support the name multicut.

Advancing the proof we introduce several definitions and remarks.

Let P be a fixed partially ordered class (by the relation denoted $\leq_P$ defined (without loss of generality) on the class On of all ordinals. For our purposes it is usufull to think that the partial ordered class P is being created consecutively: We are given poset P_1 (induced on the set $1 = \{0\}$, the poset P_2 (induced by P on the set $2 = \{0, 1\}$) and so on by the transfinite induction.

The relation $\in$ gives the order or the *time of creation* of P. The poset P_α induced by P on the set $\{\iota; \iota \in \alpha\}$ is the poset which has been created at time $< \alpha$. This applies also to the class On: we think of $\in$ as the creation time of ordinals.

For an ordinal α we put

$$\alpha_L = \{\beta; \beta \leq_P \alpha \text{ and } \beta \overset{\in}{=} \alpha\}$$

Thus α_L is the set of all predecessors (in P) of the vertex α which were created before α or at the same time as α.

Similarly we put

$$\alpha_R = \{\beta; \beta \geq_P \alpha \text{ and } \beta \overset{\in}{=} \alpha\}$$

Thus α_R is the set of all successors (in P) of the vertex α which were created before α or at the same time as α.

Observe that, for a given partial order P, any set of sets α_L is a pointed set. (However note that for $\alpha \in \alpha'$ we may have $\alpha'_L \cap \alpha \neq \alpha_L$.)

Now we can prove Theorem 1.1:

Proof.

For every α put

$$\phi(\alpha) = \{\beta_L; \beta \in \alpha_R\}$$

(One can check that $\phi(0)$ is formed by the single vertex and single set of size 1 (the loop): $\phi(0) = \{\{0\}\} = \{1\}$.)

Let P_α denotes the restriction of the partial order P to the set α. We shall prove by the transfinite induction that ϕ is an embedding of P_α into $\mathcal{MC}$. Clearly it suffices to consider the non-limit case.

Consider the induction step $\alpha + 1$. The partial order $P_{\alpha+1}$ consists from the partial order P_α and from the vertex α. As ϕ restricted to the set P_α is an embedding it suffices to consider the relations of β and α for $\beta \in \alpha$. In this situation we distinguish four cases:

i. $\beta <_P \alpha$.

We want to prove $\phi(\beta) < \phi(\alpha)$. Thus let $\gamma_L \in \phi(\alpha)$. Then either $\gamma = \alpha$, in which case $\gamma_L = \alpha_L$ and thus $\gamma_L \supseteq \beta_L \in \phi(\beta)$, or $\gamma \in \alpha$, $\gamma \in \alpha_R$. In this later case $\alpha <_P \gamma$ and then also $\beta <_P \gamma, \beta, \gamma \in \alpha$. Applying the induction hypothesis we get $\phi(\beta) < \phi(\gamma)$. As γ_L belongs to both $\phi(\alpha)$ and $\phi(\gamma)$ we get that for γ_L there exists $\delta_L \in \phi(\beta)$ such that $\delta_L \subset \gamma_L$. Thus $\phi(\beta) < \phi(\alpha)$.

ii. $\phi(\beta) < \phi(\alpha)$.

We want to prove $\beta <_P \alpha$. It is $\alpha_L \in \phi(\alpha)$ and thus there exists $\gamma_L \in \phi(\beta)$ such that $\gamma_L \subset \alpha_L$. Thus $\gamma <_P \alpha$ and $\beta \leq_P \gamma, \gamma \in \beta$ (by the definition of $\phi(\beta)$). This then implies $\beta <_P \alpha$.

iii. $\alpha <_P \beta$.

We prove $\phi(\alpha) < \phi(\beta)$. Let $\gamma_L \in \phi(\beta)$ be chosen. Then either $\gamma = \beta$ or $\beta <_P \gamma$ and $\gamma \in \beta$. It follows (by the transitivity of creation $\in$ and $<_P$) that in either case $\gamma \in \alpha$ and $\alpha <_P \gamma$. Thus $\gamma \in \alpha_R$ and $\gamma_L \in \phi(\alpha)$ (as γ_L in time α is the same as γ_L in time β).

iv. $\phi(\alpha) < \phi(\beta)$.

It is $\beta_L \in \phi(\beta)$, thus there exists $\gamma_L \in \phi(\alpha)$ with $\gamma_L \subseteq \beta_L$. This implies $\gamma \leq_P \beta$ and $\alpha \leq_P \gamma$ holds by the construction of $\phi(\alpha)$. Thus $\alpha \leq_P \beta$.

This finishes the proof.

∎

3 Universality of Structures

The proof of the universality of multicuts and the easy structure of multicuts allows us to prove universality of a wide class of structures. In this section we approach this slightly more generally. However this abstract setting will find its (perhaps surprising) justification in Section 5.

Let P be a partially ordered class ordered by the relation $\leq$. By P^{op} we shall denote the partially ordered class with the same vertices ordered by the inverse relation $\leq^{-1}$, i.e. by the relation $\geq$.

Let STR be any class of structures partially ordered by $\leq_{STR}$ with sums.

We say that a structure A is connected if for any two structures B, C holds: $A \leq_{STR} (B + C)$ if and only if either $A \leq_{STR} B$ or $A \leq_{STR} C$.

A mapping $\phi : P \to STR$ is said to be an *embedding* if ϕ is injective and for any two vertices x, y of P holds:

$$x \leq_P y \text{ if and only if } \phi(x) \leq_{STR} \phi(y)$$

The class STR is said to be *universal* if every partially ordered class P is embeddable to STR.

Denote by $\mathcal{I}$ partially ordered class of all subsets of ordinals ordered by the relation inclusion $\subseteq$.

$\mathcal{I}^{op}$ denotes the class of all sets ordered by $\supseteq = \subseteq^{-1}$.

Observe that neither $\mathcal{I}$ nor $\mathcal{I}^{op}$ is an universal class. However the partial order $\mathcal{I}^{op}$ is an important witness for the universality.

Theorem 3.1 *Let STR be any class of structures partially ordered by $\leq_{STR}$. Let STR' denotes the subclass of STR of all connected structures.*

Suppose there is an embedding $\phi : \mathcal{I}^{op} \to STR'$.

Then STR is an universal poset.

Proof. Let $\phi : \mathcal{I}^{op} \to STR'$ be an embedding. Let P be a given partially ordered class (ordered by $\leq_P$) and let $\phi' : P \to \mathcal{MC}$ be an embedding. Define then the mapping $\psi : P \to STR$ as follows.

Given vertex x of P, put explicitely $\phi'(x) = \{A_\iota; \iota \in I\}$. As each of the sets A_ι is a subset of ordinals we can define

$$\psi(x) = \sum_{\iota \in I} \phi(A_\iota)$$

Slightly less formally we could write $\psi = \phi' \circ \phi$. This formula gives us the hint how to deduce that given embeddings ϕ' and ϕ the mapping ψ is an embedding too. We leave it at that. ∎

In the next section we shall provide an example of an embedding ϕ of $\mathcal{I}^{op}$ into the class $\mathcal{S}'$ of all connected set systems.

4 Universality of Set Systems

In this part we show that set systems togother with the existence of a homomorphism represent the partial order $\mathcal{I}^{op}$. First we present the construction.

4.1 Construction

We start again by introducing several auxiliar notions and notations.

Our construction will be simple: the buidlings blocks will be (large) complete graphs with a single edge of higher arity.

The complete graph with the set of vertices α will be denoted K_α. (We may view K_α as the relation $\neq$ on α.)

The *product* of a set $\{G_\iota; \iota \in I\}$ of graphs will be denoted by $\prod_{\iota \in I} G_\iota$: this is the graph with vertices $\prod_{\iota \in I} V(G_\iota)$ where two vertices $(x_\iota; \iota \in I)$ and $(y_\iota; \iota \in I)$ form an edge if and only if x_ι and y_ι for an edge for every $\iota \in I$.

This is a a product (in the sense of category theory) and thus every ι-projection $\pi_\iota : \prod_{\lambda \in I} V(S_\lambda) \to V(S_\iota)$ is a homomorphism $\prod_{\lambda \in I} G_\lambda \longrightarrow G_\iota$.

It follows that the chromatic number (i.e. the minimal number of colors needed to color vertices so that no edge is monochromatic) of $\prod_{\iota \in I} G_\iota$ is at most the minimal chromatic number of one of the factors G_ι. In the case of complete graphs the product $\prod_{\iota\in I} K_\iota$ contains K_λ, $\lambda = \min I$, as a subgraph and thus the chromatic number of $\prod_{\iota\in I} K_\iota$ is equal to λ.

For the product of two graphs G, H we simply write $G \times H$.

Given a graph $G = (V, E)$ we denote by G' the set system (V, E') where $E' = E \cup \{V\}$ (i.e. we added one $|V|$-nary relation containing only one set; this should help to understand not to confuse).

The critical to our proof is the following easy lemma:

Lemma 4.1 Homomorphism Cancelation Lemma

Let $\{G_\iota; \iota \in I$ and G be given graphs. Let for each $\lambda \in I$ the chromatic number of G_λ exceeds the cardinality of the set of all maps $V(\prod_{\iota\in I,\iota<\lambda} G_\iota) \to V(G)$.

Then for each $\lambda \in I$ there exists a homomorphism $(\prod_{\iota\in I,\iota<\lambda} G_\iota) \to V(G)$ if and only there is a homomorphism $(\prod_{\iota\in I} G_\iota) \to G$.

Proof. In the non-trivial direction assume that $f : (\prod_{\iota\in I} G_\iota) \to G$. Think of this homomorphism as $f : B \times C \to A$ where we put $B = \prod_{\iota\in I,\iota<\lambda} G_\iota$ and $C = \prod_{\iota\in I,\iota\geq\lambda} G_\iota$. For every $y \in V(C)$ consider the fibre map $f_y : V(B) \to V(A)$ (defined by $f_y(x) = f(x, y)$). By the cardinality assumption on the chromatic number of C there exists an edge $M \in E(C)$ such that all the mappings $f_y; y \in M$ coincide. Put $f_0 = f_y$ for all $y \in M$. However then f_0 is a homomorphism $B \to A$ (for if $f_0(\{x, x'\}) \notin E(A)$ then for every edge $\{y, y'\} \in E(B)$ we have also $f(\{(x, y), (x', y')\} \notin E(A)$. This is a contradiction).

∎

Define the function $f : On \to Card$ ($Card$ is the class of all cardinals) as follows:

$f(0) = 3$, $f(\alpha) = (\beta)^+$, where β is the cardinality of the set of all mappings $\prod_{\iota\in\alpha} f(\iota) \to \prod_{\iota\in\alpha} f(\iota)$

Given these preparations we can define the mapping $\phi : \mathcal{I}^{op} \longrightarrow \mathcal{S}$ by the following simple formula:

$$\phi(A) = (\prod_{\iota\in A} K_{f(\iota)})'$$

(where A is a set of ordinals).

We prove that the mapping ϕ is an embedding $\mathcal{I}^{op} \longrightarrow \mathcal{S}$.

4.2 Proof of Theorem 1.2

We know that the partial order $\mathcal{MC}$ of all multicuts is universal. As all the sets which occur in these multicuts are pointed (i.e. they have maximal elements with different sets having didstinct maximal elements, see Section 3). Thus to prove the universality of $\mathcal{S}$ it suffices (by the proof of Theorem 3.1) to find an embedding of the subclass of $\mathcal{I}^{op}$ generated by all pointed sets of ordinals.

Clearly each of the systems $\phi(A)$ is connected. Also if $A \supseteq B$ then there is a homomorphism $(\prod_{\iota \in A} K_{f(\iota)}) \longrightarrow (\prod_{\iota \in B} K_{f(\iota)})$ given by the projection. This projection is an onto map which means that it is also a homomorphism $(\prod_{\iota \in A} K_{f(\iota)})' \longrightarrow (\prod_{\iota \in B} K_{f(\iota)})'$.

Conversely, let A, B be pointed sets such that there exists a homomorphism $f : (\prod_{\iota \in A} K_{f(\iota)})' \longrightarrow (\prod_{\iota \in B} K_{f(\iota)})'$.

Put $a = \max A, b = \max B$. Then obviously $a \geq b$ as f is an onto map (this is the only, but crucial step, where unbounded edges are used; see remarks bellow).

If $a = b$ then $A = B$ as A and B are poinetd sets.

Thus assume that $a > b$. Put $I' = \{\iota; \iota \in I, \iota \leq b\}$, $I'' = I - I'$.

We can use the Homomorphism Cancelation Lemma for the following graphs:

$\prod_{\iota \in I'} K_\iota$ and $\prod_{\iota \in I''} K_\iota$ and we obtain that there is a homomorphism

$$\prod_{\iota \in A} K_\iota \longrightarrow \prod_{\iota \in B} K_{f(\iota)}$$

if and only if there exists a homomorphism

$$\prod_{\iota \in I'} K_\iota \longrightarrow \prod_{\iota \in B} K_{f(\iota)}$$

.

It follows that $\max I'$ exists and that $\max I' = b$ (this we get by the cardinality argument as above for $a < b$).

This means that $b \in A$ and thus also $B \subset A$ again by the pointed property of sets A, B.

This finishes the proof of Theorem 1.2.

Particularly we can represent any linearly ordered set or class by set systems. The same is true also for antichains (i.e. independent sets and classes). In these special cases much more is true and this is beeng disscussed in the next section.

5 Set Systems Are Everywhere Dense

Let A, B be systems satisfying $A < B$ (i.e. we are assuming $A \to B \not\to A$). If there is no system C satisfying $A < C < B$ then we say that the pair (A, B) is a *gap*.

We shall prove the following :

Theorem 5.1 *The class $\mathcal{S}$ does not contain any gap.*

Explicitely, given any two systems A, B, $A < B$ there exists a system C such that $A < C < B$.

Theorem 5.1 could be called *Gap Theorem*, and as there are no gaps perhaps a proper name is *Density Theorem*, compare [21].

Proof. Let (A, B) be a pair of systems satisfying $A < B$.

We shall construct the system C satisfying $A < C < B$ in the following form $C = A + (B \times D)$ where $+$ denotes the disjoint union and the *product* $B \times D$ is defined as follows: $V(B \times V(B) \times V(D)$ and

$$E(B \times A) = \{M; \pi_A(M) \in E(B) \text{ and } \pi_B(M) \in E(B)\}$$

Obviously $A \leq C \leq B$ for any choice of D. The strict inequalities will follow after we specify the system D.

Let D be any system of λ-sets (i.e. each edge of D is assumed to have cardinality λ) where λ is bigger than the cardinality of the set $V(B)$ and with chromatic number larger than the cardinality of the set of all functions from the set $V(B)$ to the set $V(A)$. D is easy to find: any uniform system of all subsets of cardinality λ of a sufficiently large set will do.

Note that B contains an edge and thus also C contains an edge. We prove that the set system C has the desired properties.

Claim 1. $B \not\to C$

To see this observe that if M is an edge of B with smallest cardinality then C has no edge of the same or smaller cardinality and thus there is no homomorphism $B \to C$. (This is valid even if B is a loop.)

Claim 2. $C \not\to A$

Assume the contrary: Let $f : C \to A$ be a homomorphism. Thus $f : V(B) \times V(D) \to V(A)$. For every $y \in V(D)$ consider the fibre map $f_y : V(B) \to V(A)$ (defined by $f_y(x) = f(x, y)$). By the cardinality assumption on the chromatic number of D there exists an edge $M \in E(D)$ such that all the mappings $f_y, y \in M$, coincide. Put $f_0 = f_y$ for all $y \in M$. However then f_0 is a homomorphism $B \to A$ (for if $f_0(N) \notin E(A)$ then also $f(N \times M) \notin E(A)$ which is a contradiction).

∎

In fact the class $\mathcal{S}$ does not have any non-trivial cuts. This is formalize by the following:

Corollary 5.1 *Let $\{S_\iota; \iota \in I\}$ and $\{S_\lambda; \lambda \in J\}$ be two sets of set systems. Then the following two statements are equivalent:*

i. There exists a system S such that $S < S_\lambda$ for every $\lambda \in J$ and $S_\iota < S$ for every $\iota \in I$.

ii. For every $\iota \in I, \lambda \in J$ holds $S_\iota < S_\lambda$ and $\prod_{\lambda\in J} S_\lambda \not\to \sum_{\iota\in I} S_\iota$.

Proof. In the non-trivial direction put $A = \sum_{\iota\in I} S_\iota$ and $B = \prod_{\lambda\in J} S_\lambda$ and apply Theorem 5.1. ∎

This gives us the possibility to represent every chain (On^{op} in particular) greedily on line.

However note that the general extension properties of $\mathcal{S}$ are very complicated already in the case of graphs, see [19]. Some particular cases can be solved. Here is another example which is related to the section 6:

A set of systems $\{S_\iota; \iota \in I\}$ is said to be *independent* if $S_\iota \not\leq S_\lambda$ whenever $\iota \neq \lambda \in I$. We are interested in the following particular extension property (compare [16, 19]:

Given an independent set of systems $\{S_\iota; \iota \in I\}$ does there exists a system S such that S together with $\{S_\iota; \iota \in I\}$ is an independent system? For systems this is almost always the case as shown by the following *Independence Theorem*:

Theorem 5.2 *For an independent set system $\{S_\iota; \iota \in I\}$ the following two statements are equivalent:*

i. There exists a system S such that S together with $\{S_\iota; \iota \in I\}$ is an independent system.

ii. The set $\{S_\iota; \iota \in I\}$ does not consists from either the singleton $(\{1\}, \emptyset)$ or from a loop $(\{1\}, \{\{1\}\})$ only.

Proof.

The singleton is the minimal element of $\mathcal{S}$ and the loop is the maximal element of $\mathcal{S}$. Furthemore if an independent set contains singleton (loop, respectively) then it does not contain other systems. Thus i. implies ii.

Now assume ii. and let D (as above) be a system with each of its edges of cardinality greater than any edge of any $S_\iota, \iota \in I$ such that the chromatic number of D exceeds the chromatic number of any $S_\iota, \iota \in I$. Then $S \not\to S_\iota$ by the chromatic number and $S_\iota \not\to S$ by the size of edges in S_ι and in S. ∎

Remark.

This easy situation in Theorem 5.2 with a more complicated situation for finite undirected graphs and with an unsolved problem for relations: There are arbitrary large independent sets of relations which cannot be extended to any larger set. The characterization of such sets is an open problem.

The chains and independent sets will play a crucial role in the next section.

6 Graphs and Large Cardinals

All the above results were related to structures of unbounded arities such as $\mathcal{S}$ and $\mathcal{MC}$. For the first order theories the corresponding questions are interested from model-theoretic and set-theory point of view and they attracted considerable attention in the past. We think that the above results put this are in a yet different perspective.

We shall deal only with graphs, however same results can be obtained for other relational structures, see [12]. Denote by $\mathcal{G}$ the partial order of all graphs with the existence of homomorphism.

Clearly $\mathcal{G}$ is a partial order of $\mathcal{S}$ induced by all graphs.

Let T be a partial order on a given class and consider the following statement - "principle":

$$\mathcal{P}(T) : T \text{ cannot be represented by the partial order } \mathcal{G}$$

Clearly, for $T = (On, =)$ (i.e. for the proper class D of independent elements) the statement $\mathcal{P}(D)$ is just the Vopěnka principle $\mathcal{VP}$ introduced in the Introduction.

Another example is the *Semiweak Vopěnka's Principle* $\mathcal{SWVP}$ which is introduced by Adámek and Rosický in [1]. Semiweak Vopěnka's Principle is equivalent to the principle $\mathcal{P}(On^{op})$.

(The definition given in [1] is the first definition where Vopěnka's Principle is related to a partial order, rather than a category.)

Now for any partial order T (T for *test-partial order*) we can consider the statement $\mathcal{P}(T)$ as a principle of set theory.

This arbitrariness has certain price: many of these principles are simply not true as one can represent the corresponding partial order T by graphs. These include for example the case when T is a partially ordered set (as any small category is representable by graphs via Cayley-type representation). Another such example is On with $\in$ which can be represented , say, by complete graphs.

There are other partially ordered classes T which can be represented by graphs and thus for which one can prove $\neg\mathcal{P}(T)$ (these include e.g. partial order with a set-dimension, or partial orders of bounded height not containing D; some of these cases are non-trivial and will appear elsewhere.)

Clearly if a partial order T contains T' (as induced suborder) then $\mathcal{P}(T)$ implies $\mathcal{P}(T')$.

However the containment is not the only reason for the reducibility. For example one can see easily that $\mathcal{P}(D) = \mathcal{VP}$ implies $\mathcal{P}(On^{op})$.

However the characterization of those partial orders T for which it is relatively consistent to assume that $\mathcal{P}(T)$ holds is an open problem.

This question cannot be formalized in ZFC and we can regard it only as a scheme. Yet it may have an easy solution such as "$\mathcal{P}(T)$ is relatively consistent if and only if T contains either On^{op} or D".

These questions are related to large cardinals. For each partial order T denote by $\kappa(\mathcal{P}(T)) = \kappa(T)$ the smallest cardinal to which the principle $\mathcal{P}(T)$ reflects. (If $\mathcal{P}(T)$ is not true we put $\kappa(T) = \kappa(0 = 1) =$ a symbol larger than any ordinal, see [12],p. 471.)

It is known that all the cardinals $\kappa(T)$ are large:

For any T the cardinal $\kappa(T)$ is greater than any measurable cardinal. This is a result of Kučera and Pultr [15], see e.g. [24, 12].

It is not clear whether every cardinal $\kappa(T)$ is bounded by Vopěnka cardinal $\kappa(\mathcal{VP})$.

However our Theorem 3.1 has a direct (perhaps surprising) bearing here:

Corollary 6.1 *$\kappa(\mathcal{I}^{op})$ is the smallest of all ordinals $\kappa(T)$.*

Proof. It suffices to prove the implication $\neg\mathcal{P}(\mathcal{I}^{op})$ implies $\neg\mathcal{P}(T)$.

To assume $\neg\mathcal{P}(\mathcal{I}^{op})$ amounts to saying that there exists an embedding ϕ of $\mathcal{I}^{op}$ to $\mathcal{G}$.

What we have to prove that this embedding can be modified to an embedding ϕ' which is an embedding of $\mathcal{I}^{op}$ to $\mathcal{G}'$ where $\mathcal{G}'$ is the class of all connected graphs.

However one can find an embedding ϕ' of $\mathcal{G}$ to $\mathcal{G}'$ (see e.g. [24]; such an embedding is easy to find by controlling the cycle structure of graphs; we ommit details here).

The desired embedding is then obtained as the composition of embeddings ϕ' and ϕ. ∎

7 Concluding Remarks and Open Problems

1. In the above proof we used edges of large sizes. That was necessary as for systems with edges of bounded arities there is not an analogous result. We have seen that in fact it is relatively consistent to assume that there is not such a representation.

However for finite sets we know by Theoremone that $\mathcal{MC}_\omega$, i.e. the class of all finite multicuts, is countable universal partial order and that the poset $\mathcal{S}_\omega$ of all finite set systems is universal countable poset.

This follows from our proof.

2. The density theorem 5.1 has direct bearing to problems disscussed in Section 5. Corollary 5.2 may be reformulated by saying that in the partial order $\mathcal{S}$ the only gaps between sets of set-systems are induced by limits (universal constructions) $\sum$ and $\prod$. This implies embedding of On^{op} into $\mathcal{S}$.

Now it is relatively consistent to assume that $\mathcal{G}$ does not contain On^{op} and thus the class of all graphs $\mathcal{G}$ should contain much more gaps. But where are the gaps?

The density theorem *for pairs*, i.e. the analogy of Theorem 5.1 holds for undirected graphs aswell, see [19]. One can also prove that for finite families of graphs there are no other gaps than $\sum$ and $\prod$ (i.e. the analogy of Theorem 5.1). Thus there has to be much more set-gaps for classes of infinite graphs. This in turn indicates that the products of infinite graphs produce these gaps. Their description is an open problem.

3. Is it possible to represent $\mathcal{MC}_\omega$ by finite oriented trees, or even paths?

(This problem is mentioned already in [23].)

4. We can strengthen the above result Theorem **??** to give a short proof of the following of result of Hedrlín and Pultr see [24]:

Theorem 7.1 *Every partially ordered class can be represented by the category of set systems and all their homomorphisms.*

Explicitely: For every partially ordered class $(\mathcal{X}, \leq_{\mathcal{X}})$ *there exists an injective mapping* $\phi : \mathcal{X} \to \mathcal{S}$ *with the following property:*

$A \leq_{\mathcal{X}} B$ *if and only if* $\phi(A) \leq \phi(B)$.

Moreover if $\phi(A) \leq \phi(B)$ *then there is unique homomorphism* $\phi(A) \to \phi(B)$.

This is more technical and it is going to appear elsewhere. There are two essential features to this strengthening: the Homomorphism Cancelation Lemma may be strengthened to the unicity of projections and one every set one can construct a rigid graph, see [29], and [18] for a recent simple proof.

5. The independence problem for graphs depends on set theoretical as-

sumptions (see Section 6). However for finite undirected graphs one has a nice characterization theorem.

For oriented graphs the situation is again complicated both in the finite and infinite case. This follows from [22] and is related to so called *homomorphism dualites*, e.g. [20, 7],[22]

6. For finite relations and undirected graphs one can prove the universality of the countable poset of all finite graphs and existence of homomorphism. This was a motivation of this paper, see [19]. In a sense this is a more complicated situation as already Density Theorem 5.1 for graphs is a more complicated result than for systems. However the short proof of density of undirected graphs ,[16], lies in the heart of the present proof, compare [16, 21].

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