Trying to prove the Kuratowski theorem from below.

Dainis ZEPS * †

ABSTRACT. This note examines possibility to prove the Kuratowski theorem from below, i.e. assuming that Kuratowski-like theorem for free-planar graphs is right. Version of Kuratowski theorem for 3-connected components is proved. Another proof of Kuratowski teorem is added.

Graph is defined as a pair of sets (V, E), where V is the set of vertices and E – the set of edges. For graph G V(G) is its vertex set and $E(G)$ is its edge set. We denote by $G - e$ graph obtained by deleting edge $e \in E(G)$ from G. Similarly, $G - v$ is graph obtained by deleting vertex $v \in V(G)$ from G.

Similarly, $G.e$ is graph obtained by contracting edge $e \in E(G)$ in G. Reverse operation to edge adding and its contraction is the *vertex split* operation $G \odot u$, that is not unique. Thus, if in G by adding and contracting $e \notin E(G)$ appears a new vertex $u \in V(G')$ then there exists such vertex split $G' \odot u$ that we receive back previous graph G.

H is *subgraph* of G (denoting it $H \subset G$) if there is such a graph H' isomorphic to H and $V(H') \subset V(G)$ and $E(H') \subset E(G)$.

H is a *minor* of G (denoting it $H \prec G$) if H can be obtained by edge contractions from some subgraph of G. It is easy to see that if $H \prec G$ then H can be obtained from G by vertex deletions, edge deletions and edge contractions.

A class of graphs A is called *minor closed* if for each graph H belonging to A and arbitrary graph G from $G \prec H$ follows that G is in A.

For a minor closed class A, $F(A)$ is the minimal set of forbidden minors, i.e.

$$F(A) = \lfloor \{G \mid G \notin A\} \rfloor.$$

Here we use a notion $\lfloor B \rfloor$ denoting set which contains only minimal minors of B:

$$\lfloor B \rfloor \triangleq \{G \mid H \in B \wedge H \prec G \Rightarrow H \cong G\}.$$

Analogously, $\lceil B \rceil$ which contains only maximal graphs, i.e. all graphs of B that are minors of $\lceil B \rceil$ is defined as follows:

$$\lceil B \rceil \triangleq \{G \mid H \in B \wedge G \prec H \Rightarrow H \cong G\}.$$

Proposition 1. *For a minor closed class A if G doesn't belong to A there exists such $H \in F(A)$ that $H \prec G$ and conversely.*

Theorem 2. *(Robertson, Seymour): $F(A)$ is finite for any minor closed A.*

Let $\mathcal{G}(n, m)$ denote the set of all graphs with $n, (n > 0)$ vertices and $m, (m \geq 0)$ edges. Let $\mathcal{B}(n, m)$ denote the set of all bigraphs with $n, (n > 0)$ vertices and $m, (m \geq 0)$ edges. Let $\mathcal{C}(n, m)$ be arbitrary subset of $\mathcal{G}(n, m)$ and $\mathcal{D}(n, m)$ be arbitrary subset of $\mathcal{B}(n, m)$. Sometimes we are saying that graph G with n vertices and m edges belongs to a set of graphs (not specifying the set) meaning by this set $\mathcal{G}(n, m)$.

Sets of graphs $\mathcal{C}_1 = \mathcal{C}(n_1, m_1)$ and $\mathcal{C}_2 = \mathcal{C}(n_2, m_2)$ are *non-compatible* if no graph from $\mathcal{C}_1$ is a minor for any graph from $\mathcal{C}_2$ and vice versa.

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†Author's address: Institute of Mathematics and Computer Science, University of Latvia, 29 Rainis blvd., Riga, Latvia. dainize@cclu.lv

Lemma 3. *Sets of graphs $\mathcal{G}_1 = \mathcal{G}(n_1, m_1)$ and $\mathcal{G}_2 = \mathcal{G}(n_2, m_2)$ are non-compatible iff $n_1 > n_2$ and $m_1 < m_2$ or vice versa: $n_1 < n_2$ and $m_1 > m_2$.*

Proof. If there holds $n_1 > n_2$ and $m_1 < m_2$, then it is easy to see that compatibility is impossible.

Let us assume that $\mathcal{G}_1 = \mathcal{G}(n_1, m_1)$ and $\mathcal{G}_2 = \mathcal{G}(n_2, m_2)$ are non-compatible. If graphs with equal number of vertices were allowed then we could take a graph and its subgraph with the same number of vertices and the sets were compatible. If graphs with equal number of edges were allowed then we could take two equal graphs and one replenish with as many as necessary isolated vertices and they were compatible. Thus, the non-compatibility condition must be just as stated by lemma. $\square$

It is easy to see that the statement of the lemma is right also for pairs of sets $\mathcal{B}(n_1, m_1)$ and $\mathcal{B}(n_2, m_2)$ and $\mathcal{B}(n_1, m_1)$ and $\mathcal{G}(n_2, m_2)$ respectively. Evidently, the last case corresponds to Kuratowski graphs, i.e. $\mathcal{B}(6, 9)$ and $\mathcal{G}(5, 10)$ are non-compatible sets of graphs, where $\mathcal{B}(6, 9)$ comprise only $K_{3,3}$ but $\mathcal{G}(5, 10)$ only K_5.

For sets of graphs $\mathcal{G}$ and $\mathcal{S}$ let $\mathcal{G} \mid_{\mathcal{S}}$ denote only these graphs from $\mathcal{G}$ that do not have graphs from $\mathcal{S}$ as minors:

$$\mathcal{G} \mid_{\mathcal{S}} \triangleq \{G \mid G \in \mathcal{G} \wedge \forall H \in \mathcal{S} : H \not\prec G\}.$$

$N_\circ(B)$ denotes the minor closed class with B as its set of forbidden minors, i.e.

$$N_\circ(B) \triangleq \{G \mid \forall H \in B : H \not\prec G\}.$$

In other words, we may say, that $N_\circ(B)$ is a minor closed class generated by its forbidden minors in B. For example, $N_\circ(K_5, K_{3,3})$ is the class of planar graphs, as it is asserted by Kuratowski theorem.

One more denotation for the minor-closure of B:

$$\langle B \rangle \triangleq \{G \mid \exists H \in B : G \prec H\}.$$

If $B = \{G\}$ then we write $\langle G \rangle$ in place of $\langle \{G\} \rangle$.

Let us use some denotations for some small graphs:

Z_k, $k > 0$, is empty graph with k vertices, equal with $Z_k = k \times K_1$.

$X = K_{1,4} + K_1$, i.e. star with four edges and an isolated vertex.

$Y = K_{1,3} + K_{1,1}$, i.e. star with three edges and an isolated edge.

$V = K_3 + 3 \times K_1$, i.e. triangle and three isolated vertices.

U is the cycle of length 5 plus a vertex of degree 2 connected to two non-consecutive vertices of the cycle.

Let us state some simple facts with these and Kuratowski graphs:

Lemma 4. $\lceil N_\circ(\{V, Z_7\}) \rceil = \{K_{3,3}, K_5, K_{2,4}, K_{1,5}, U\}$.

Proof. If graph has 6 vertices it is without triangles, i.e. it is a bigraph. There are four possible maximal such graphs, i.e. $K_{3,3}, K_{2,4}, K_{1,5}$ and U.

If graph has less than 6 vertices, it is arbitrary otherwise, i.e. maximal graph is K_5. $\square$

Lemma 5. $\lceil N_\circ(\{V, X, Y, Z_7\}) \rceil = \{K_{3,3}, K_5\}$.

Proof. X excludes $K_{1,5}$ and Y excludes U and both X and Y exclude $K_{2,4}$. $\square$

If Z_7 is removed from lemmas condition then graphs with components isomorphic to Kuratowski graphs and their subgraphs are allowed.

A planar graph is called free-planar, if after adding an arbitrary edge it remains to be planar. In [4] it is proved, that the class of free-planar graphs is equal to $N_\circ(K_5^-, K_{3,3}^-)$, and its characterization in terms of the permitted 3-connected components is given.

In [2] a generalization of the notion of free-planar graphs is suggested. We denote by $Free(A)$ the class of graphs that consists of all graphs which should belong to A after adding an arbitrary edge to them. It is easy to see, that, if A is minor closed, then $Free(A)$ is minor closed too [2]. Because of this we use to say, that $Free(A)$ is *free-minor-closed-class* for a minor closed class A.

In [2] Kratochvíl proved the theorem:

$$F(Free(A)) = \lfloor F(A)^- \cup F(A)^{\odot} \rfloor,$$

where

$$B^- \triangleq \{G - e \mid G \in B, e \in E(G)\}$$

and

$$B^{\odot} \triangleq \{H \mid H \cong G \odot v, G \in B, v \in V(G)\}.$$

Further, we denote by $Free^k(A)$ repeatedly applied $Free$ k times, i.e.

$Free^0(A) = A$;

$Free^k(A) = Free(Free^{k-1}(A))$.

Let for a minor closed class A $Free^m(A)$ is not consisting of only empty graphs but $Free^{m+1}(A)$ is, then we say that A is *of depth* m.

In the graph G with some vertex v a vertex split $G \odot v$ is called *proper* if both new vertices arising in the result of the split of v are of degree at least two. Otherwise the vertex split is called *non-proper*.

Theorem 6. *Let a class A be of depth m and all graphs of $F(A)$ belong either to mutually non-compatible or coinciding sets of graphs. If there holds*

$$F(Free(A)) = F(A)^-$$

then there holds also

$$F(Free^k(A)) = F(Free^{k-1}(A))^-$$

for $k = 1, ..., m$.

Proof. By induction if m is equal to 1 all is done by theorem's assumption otherwise it suffices to prove that

$$F(Free^{k+1}(A)) = F(Free^k(A))^-$$

for $1 \leq k < m$ assuming that

$$F(Free^k(A)) = F(Free^{k-1}(A))^-$$

is right.

Let us suppose first that $F(Free^k(A))$ consists only from one graph. Then all graphs $F(Free^k(A))^-$ have the same number of edges and they can't be proper minors of each other. Thus, they all are present in $F(Free^{k+1}(A))$.

$F(Free^k(A))^{\odot}$ can not give some contribution to $F(Free^{k+1}(A))$ either. Let us suppose for a moment that it does and some graph $G \in F(Free^k(A))$ is such that G' with some vertex v split ($G' = G \odot v$ giving new vertices v_1 and v_2) were not present in $F(Free^k(A))^-$.

Let us suppose that this vertex split was non-proper. Then a corresponding hanging edge or isolated edge arises, but graph without this edge is already present in $F(Free^k(A))^-$. Thus, non-proper vertex split can not give any new contributor to $F(Free^{k+1}(A))$. Further, let us suppose that this vertex split was proper. Let us find some ancestor H of G in $F(Free^{k-1}(A)$ such that H minus some edge e is equal to G. Let us add $e = (s,t)$ to

G' getting a new graph $H'(= G' + e')$ in this way: if none of ends of e were equal to v then adding $e(= e')$ is possible only in one way; if one of the ends, say s, of e is equal to v then without loss of generality we add a new edge $e' = (v_1, t)$ to one of the new vertices (i.e. v_1) arising in the result of the split of v. Then this graph must be also contributor to $F(Free^k(A))$, because there exists some vertex split $H \odot v$ such that $H \odot v$ is equal to H' but the assumption of the theorem excludes this [...because H' has a proper minor h already present in $F(Free^k(A))^-$. If $e' \in E(h)$ then $h - e'$ should be also a minor of G', thus G' is not a contributor in $F(Free^{k+1}(A))$. If $e' \notin E(h)$ then h minus arbitrary edge is minor of G' and present in $F(Free^k(A))^-$. Thus, in this case too G' is not a contributor in $F(Free^{k+1}(A))$].

Let us suppose that$F(Free^{k-1}(A)$ consists from many graphs. But, because of either non-compatibility or coincidence of the sets to which these graphs belong the same is true for the set $F(Free^k(A))$ too. Truly, non-compatible assumed contributors of $F(Free^k(A))^-$ can not exclude each other. Either can not such that are with equal number of vertices and edges except the cases of isomorphism. Thus distinct descendants of the same level from distinct forbidden graphs can not be proper minors of each other.

The consideration about the contribution of $F(Free^k(A))^{\odot}$ does not change in the general case. □

Theorem 7. *For $A = Planar$ and $0 < k \leq 10$*

$$F(Free^k(Planar)) = \mathcal{B}(6, 9-k) \mid_{\{X,Y\}} \cup \mathcal{G}(5, 10-k).$$

Proof. Graphs X, Y exclude these minors that are present in $K_{2,4}$ but are absent in $K_{3,3}$. Indeed, $\mathcal{B}(6,8) = \{K^-_{3,3}, K_{2,4}\}$ and X, Y as minors are present in $K_{2,4}$ and its descendants and are absent in $K_{3,3}$ and its descendants. □

Corollary 8. *For $A = Planar$ Kratochvíl's theorem has the following appearance*

$$F(Free^k(Planar)) = F(Free^{k-1}(Planar))^-$$

for $k = 1, ..., 10$.

Let for a class A of depth m holds:

$$F(Free^k(A)) = F(Free^{k-1}(A))^-$$

for $k = 1, ..., m$. Then we call this class $\mathcal{M}$-class.

Then we can state:

Corollary 9. *For the case A is M-class of depth m Kratochvíl's theorem has following appearance*

$$F(Free^k(A)) = F(Free^{k-1}(A))^-$$

for $k = 1, ..., m$.

Theorem 10. *Let $B = free(A)$ and B is $\mathcal{M}$-class. A is $\mathcal{M}$-class iff $F(B) = F(A)^-$.*

Proof. If A is $\mathcal{M}$-class then theorem by definiton of $\mathcal{M}$-class $F(B) = F(A)^-$. Conversely, from the definition of the $\mathcal{M}$-class and from facts that $F(B) = F(A)^-$ and B is $\mathcal{M}$-class follows that A is $\mathcal{M}$-class too. □

We would like to apply last theorem in the case when Kuratowski-like theorem for free planar graphs were supposed to be right and it would be necessary to prove Kuratowski theorem itself. But we could do this only in the case we knew that the class *Planar* is $M-$class. Either we could prove Kuratowski theorem from below directly and then conclude that the class *Planar* is $M-$class.

Let us do the latter case.

We would base ourselves on the characteristic of free planar graphs by their 3-connected components [3, 4].

Let us recall some definitions from 3-connectivity and 3-connected components [or, shorter, 3-components] [5, 3]. *Generalized* vertex or *g-vertex* is vertex or two vertices [virtual edge]. *Generalized* edge or *g-edge* is pair generalized vertices. We say that some planar 3-component becomes nonplanar *by closing g-edge* if by merging this 3-component with another proper 3-component [i. e. having K_4 as minor] through this g-edge the resulting graph [or 3-component] is nonplanar. See fig. 2. We say that H is $S-$marked minor of G if $H \prec G$ and S is subset of vertices both of H and G. Let $ord(e)$ be order of g-edge e, i. e. number of vertices involved in the g-edge.

We would hold ourselves to following assumptions. Let G be planar but G plus some edge $e = vw$ becomes nonplanar. Let G be divided into 3-components and P_{vw} be minimal sequence of g-edges $v_1w_1, ..., v_kw_k$, $k > 0$ such that each g-edge belongs to distinct 3-component and $v_1 = v$ and $w_k = w$ and $w_i = v_{i+1}$ for $1 < i < k$. Let 3-component C_i possesses g-edge v_iw_i. We say that this 3-component gives rise to nonplanarity if C_i by closure of g-edge v_iw_i gives graph [or 3-component] that is nonplanar. In general, by fixing such path P of g-edges we would say that each component C_i, $1 < i \leq k$ is marked with the g-edge v_iw_i.

Let us consider possible components of free planar graphs. We would turn our interested to these components with such marked g-edge that by composing the graphs could give rise to nonplanarity. Let us recall from [4] that these components with marked g-edge would be

1) W_3 with marked one rim edge and one spike edge in such a way that they are not touching each other;

2) W_k, $k > 3$, with marked two spike edges not going into common triangle;

3) $\overline{C}_6$ with marked two edges both going into different triangles.

See fig. 1 [cases 1, 2, 3].

Let us suppose the following reduction of the nonplanar graph G such that $G - e$ for some $e \in E(G)$ is planar: let $G - e = G_0, G_1, ...G_i, ...G_m$ for $m \geq 0$ be such sequence of graphs that $G_{i+1} = G_i - e_i$ where not virtual edge $e_i \in E(G_i)$ and e_i belongs to such 3-component of G_i that is not free planar or isomorphic to K_5^- and if it were giving rise to nonplanarity it remained such after elimination of e_i too. Furthermore, m is maximal. Let us call this reduction of the nonplanar graph G with the nonplanar edge e FP-reduction giving in the result graph G_m.

Let us formulate some variation of Kuratowski-like theorem going out from the free planarity:

Theorem 11 (Kuratowski theorem from below)**.** *Let G be planar and $G + e$ nonplanar. Then arbitrary FP-reduction of $G+e$ gives G_m that G_m+e is nonplanar, all 3-components in G_m are free planar graphs or reduced Kuratowski graphs, and at least one 3-component is, according one of four cases of fig. 1, free planar graph with pair of virtual edges giving rise to nonplanarity [cases 1,2,3], or reduced Kuratowski graph with pair of virtual vertices giving rise to nonplanarity [case 4]. In cases 1, 2, 3 $G+e$ should have $K_{3,3}$ as minor and in case 4 $G + e$ should have K_5 as minor. Other cases are excluded.*

Let us first prove a lemma that may be considered as sort of some variation of the Kuratowski theorem applied for 3-components.

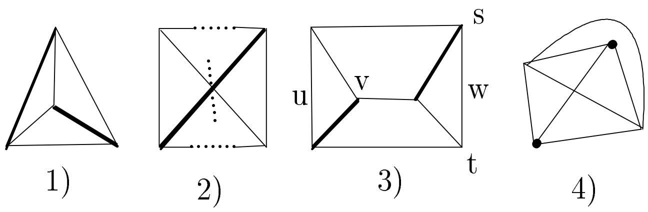

FIGURE 1. Cases of 3-components of G_m in the FP-reduction giving rise to nonplanarity. Marked g-edge vw is drawn bold.

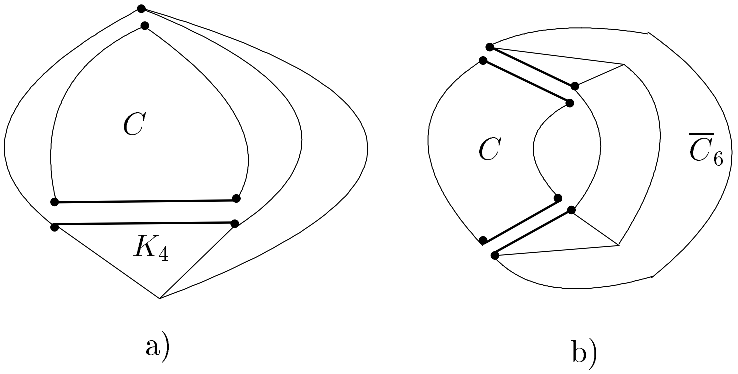

FIGURE 2. Closure of g-edge with 3-component. In case a g-edge of degree three in C is closed with K_4; in case b g-edge of degree four in C is closed with $\overline{C}_6$.

Lemma 12 (Kuratowski theorem for 3-components). *Let C be 3-component with one g-edge $e = vw$ and C be planar but closed by e becomes nonplanar and let C be minimal in the sense that no edges of C are that can be eliminated or contracted that C reduces to some smaller 3-component with the same feature. Then:*
1) if $ord(e) = 4$ in C then C is isomorphic to W_3 with virtual edges non touching each other [case 1 fig. 1];
2) if $ord(e) = 3$ and both g-vertices in e are of degree 2 in C then C is isomorphic to W_4 with virtual edges touching each other but not in the boundary of one face [case 2 fig. 1] ;
3) if $ord(e) = 3$ and one g-vertex in e is of degree 1 in C then such minimal C does not exist ;
4) if $ord(e) = 2$ in C v then C is isomorphic to K_5^- with virtual vertices non seeing each other [case 4 fig. 1]

Proof. Because of 3-connectivity of C there must be in C always $\{v\}-$marked minor K_4 [, say, K_v] and $\{w\}-$marked minor K_4 [, say, K_w] as well as $\{v,w\}-$marked minor K_4 [, say, K_{vw}].

Let us consider four cases:
1) $ord(e) = 4$: K_v may coincide with K_w and K_{vw} thus $W_3 \cong K_{vw}$ otherwise C must have K_4 as $\{v,w\}-$marked minor and at least one edge not belonging to it and C would be removable not violating nonplanarity condition;
2) $ord(e) = 3$ and both g-vertices in e are of degree 2: let common vertex of virtual edges

be s, then $deg(s) >= 4$ because of nonplanarity by closing e: thus C in minimal case is isomorphic to W_4; otherwise C must have W_4 as $e-$marked minor with e as prescribed and at least one edge not belonging to it and C would be reducible not violating nonplanarity condition;
3) $ord(e) = 3$ and one g-vertex in e is of degree 1: then C in minimal case is isomorphic either to W_4 or $\overline{C}_6$. Without loss of generality let us suppose that $deg(v) = 1$ and $deg(w) = 2$. In case 1) let be $w = st$ where, say, $e_1 = vs \in E(C)$; then C is not minimal because e_1 can be eliminated giving case 1. In case 2) edge u can be contracted [see case 3 fig. 1] and thus modified C can be reduced to case 1.
4) $ord(e) = 2$: in K_{vw} edge vw must be chain in C having some middle vertex of degree at least 4 and a chain crossing $v..w$. Then in minimal case C would be isomorphic to K_5^-; otherwise C must have K_5^- as $\{v, w\}-$marked minor plus extra edge [because of three distinct chains from v to w such that v is not seen from w] and C would be reducible not violating nonplanarity condition. □

Proof of the Kuratowski theorem from below. Let e be vw. Firstly, in G_m there exist a chain from v to w, otherwise, if there would be an edge whose elimination would exclude such a chain, then it would comprise a 3-component and its elimination were prohibited.

We must prove that the 3-components of G_m that are giving rise to nonplanarity are just as in fig. 1. Indeed, if we requested free planarity or equality to K_5^- then we must get just these cases that are shown in fig. 1. Requesting additionally minimality [of edges], case 1 can not be reduced and coincides with case 1 of the lemma. Case 2 can be reduced to case 2 of lemma. In case 3 one outer edge can be eliminated giving reduction to case 1. Case 4 coincides with the case 4 of the lemma. FP-reduction was performed in such a way that the edges that were m for nonplanarity were left untouched. Thus both $G + e$ and $G_m + e$ are nonplanar because of these 3-components of G_m that give rise to nonplanarity. Thus $G + e$ should have $K_{3,3}$ as minor in cases 1, 2, 3 and K_5 as minor in case 4. Other cases are excluded and thus Kuratowski theorem is proved. □

After we have done this excursion with 3-components of free planar graphs in order to prove the Kuratowski theorem from below we may conclude what follows.

Theorem 13. *The class Planar is $\mathcal{M}$-class.*

This follows from theorem 10.

What really does mean the fact that class *Planar* is $\mathcal{M}$-class? Let us accomplish following consideration. Let G be planar and $G+e$ – nonplanar and let G be divided into 3-components. Then according previous $G + e$ is nonplanar because of nonplanarity of some 3-components [in case of appropriate closing by g-edge]. Let the number of such 3-components be k, $k \geq 1$. Then eliminating just k edges [i. e. one from each 3-component rising nonplanarity] G becomes free planar and reversely, $G + e + f_i$ is nonplanar, where f_i is such edge from arbitrary 3-component that gives rise to nonplanarity that violates this feature. Thus, only in the case when $k = 1$ we get something resembling directly $\mathcal{M}$-class's main feature that eliminating to edges we cross some free class. But this may occur on arbitrary large graphs and arbitrary large 3-components. Because of this *Planar* is actually $\mathcal{M}$-class.

Further some examples of $\mathcal{M}$-classes. $N_\circ(\{K_{3,3}\})$ is an $\mathcal{M}$-class because proper vertex split operation is not possible. In fig. 3 the forbidden graphs of a non-trivial example of $\mathcal{M}$-class are given. Fig. 4 shows instances of these forbidden graphs after proper vertex split operation is applied to them. Fig. 5 shows the instances of the forbidden graphs of corresponding free class of this $\mathcal{M}$-class where proper vertex split operation is possible.

Fig. 6 shows instances of the graphs of Fig. 5 where proper vertex split operation is applied to them.

One more proof of Kuratowski theorem follows.

Theorem 14 (Kuratowski)**.** *The class Planar is equal to* $N_\circ(\{K_{3,3}, K_5\})$.

Proof. Let us assume that there is some forbidden minor H for class *Planar* distinct from $K_{3,3}$ and K_5 and it is 3-connected and minimal. Then neither $K_{3,3}$ nor K_5 are minors of H.

Let $v \in V(H)$ $deg(v) = k > 3$. Let v be split arbitrary in v_1 and v_2 in such a way that $deg(v_1) = 2$ and $deg(v_2) = k - 2$, thus obtaining graph H_1 from H. Let edges s_1v_1 and t_1v_1 be incident with v_1 in H_1. Let us choose such split of v that H_2 is nonplanar of higher genus than before this split. Such possibility must be always present because H neither has hinges nor is equal to K_5. But $H_1 - v_1t_1$ must be planar because $H - vt_1$ is planar [because of minimality of H]. But one edge $[v_1t_1]$ can not cause higher nonplanarity [than genus one of H_1] from state of planarity [of $H_1 - v_1t_1$]. Contradiction. Thus, if we assume H to have vertices of degree higher than three then we come to contradiction.

Then H must be cubic graph and can not have K_5^- as minor. Then for arbitrary edge e of H, $e = vw$, and arbitrary cycle c in $H - e$ through v and w there must be two chords with respect to c that prevent against embedding c with both chords and e in the plane. They form together $K_{3,3}$. We have come to contradiction that H does not have $K_{3,3}$ as a minor. □

The proof of Kuratowski theorem in [4] is improved in [6].

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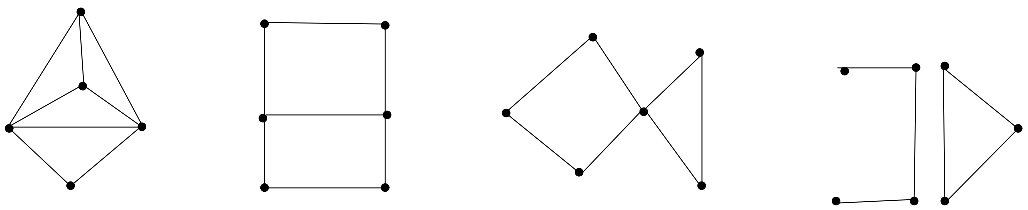

FIGURE 3. Non-trivial example of $\mathcal{M}$-class: Forbidden graphs of an $\mathcal{M}$-class A with one forbidden graph from $\mathcal{G}(5,8)$, two forbidden graphs from $m(6,7)$ and one forbidden graph from $\mathcal{G}(7,6)$.

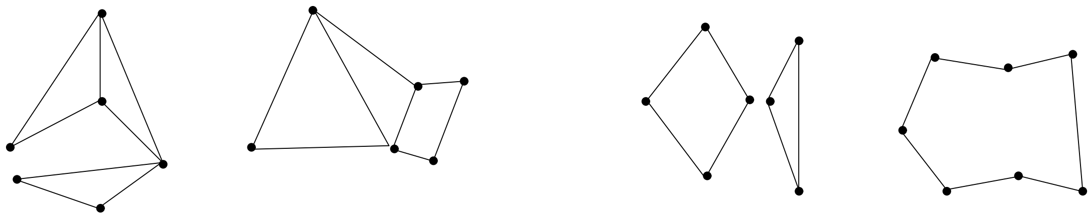

FIGURE 4. Non-trivial example of $\mathcal{M}$-class: Instances of the forbidden graphs of A from fig. 3 after proper vertex split operation is applied to them.

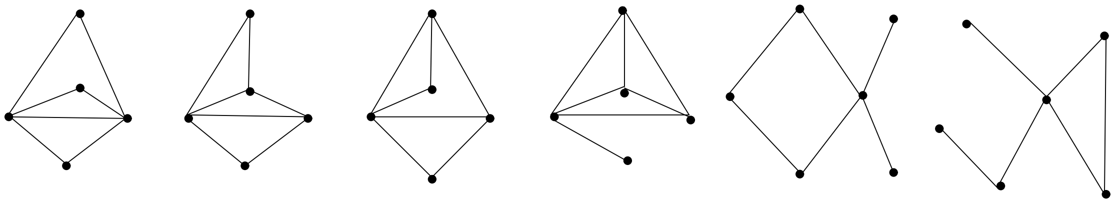

FIGURE 5. Non-trivial example of $\mathcal{M}$-class: Instances of the forbidden graphs of $Free(A)$ where proper vertex split operation is possible.

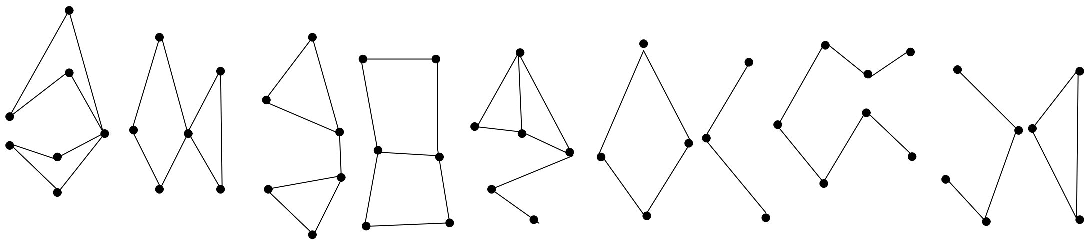

FIGURE 6. Non-trivial example of $\mathcal{M}$-class: Instances of the graphs from fig. 5 with proper vertex split operation applied to them.

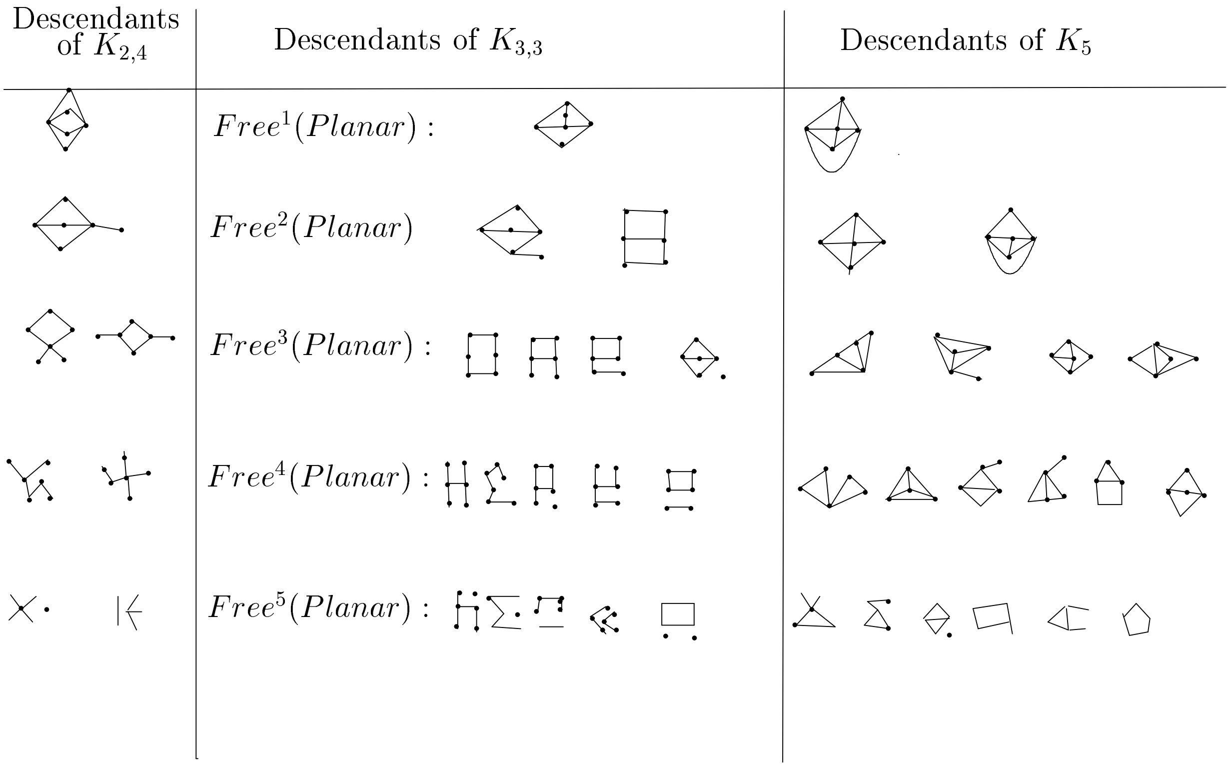

FIGURE 7. In the first column there are elements of $B(6,9)$ that have minors X or Y and do not contribute to $Free^k(Planar)$, $1 \le k \le 5$. In the second and third columns there are elements of $Free^k(Planar)$, $1 \le k \le 5$.