Antisymmetric Flows and Edge-connectivity

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Abstract

Let $G = (V, E)$ be a directed graph, let M be an abelian group, and let $f : E \to M$ be a flow. We say that f is antisymmetric if $f(E) \cap -f(E) = \emptyset$. Using a theorem of DeVos, Johnson, and Seymour, we improve upon a result of theirs by showing that every directed graph (without the obvious obstruction) has an antisymmetric flow in the group $\mathbb{Z}_3{}^3 \times \mathbb{Z}_6{}^6$. We also provide some additional theorems proving the existence of an antisymmetric flow in a smaller group, under the added assumption that G has a certain edge-connectivity.

*Supported by Project LN00A056 of the Czech Ministery of Education

1 Introduction

Nowhere-zero flows initiated by Tutte [12] may be viewed as dual to vertex colorings of graphs. Despite of the fact that they are defined by means of an orientation of the graph, the existence of a given M-flow (M a given Abelian group) does not depend on the given orientation, and thus a nowhere-zero flow is a property of the underlying undirected graph.

Motivated by the notion of oriented chromatic number (see [7, 11]) of a graph $G = (V, E)$ we define in [6] the notion of *antisymmetric flow* (or *antiflow* for short) which should be viewed as the dual notion of (strong) oriented coloring. An M-ASF (antisymmetric flow) is any flow $f : E \to M$ (M an abelian group) such that no two arcs $e, e' \in E$ get opposite value $f(e) = -f(e)$.

It follows that any antisymmetric flow is a nowhere-zero flow. However the evidence shows that the antiflow property is substancially harder to handle than nowhere-zero flow. It has been proved in [6] that an antisymmetric flow exits for any oriented graph without an oriented 2-cut and that any orientation of a 3-edge connected planar graph has a ${\mathbb{Z}_6}^5$-ASF (of order 7776). That later property has been proved by the dual connection to (strong) oriented chromatic number.

These results were extended in [1] where it has been proved that any orientation of any 3-edge connected graph has ${\mathbb{Z}_3}^9 \times {\mathbb{Z}_6}^8$-ASF (thus solving the main problem of [6]). The purpose of this paper is to supply some further results which improve bounds given in [1, 6].

The paper is organized as follows: in section 2 we give all definitions needed, in section 3 we give the new bound for 3-edge connected graphs. In section 4 we give some bounds for k-edge connected graphs ($4 \leq k \leq 6$), the section 5 contains concluding remarks.

2 Definitions

Let $G = (V, E)$ be an oriented graph, if $S \subset V$ we denote by $\omega^+(S)$ the set of the edges which begin in S and end in $V \setminus S$. We write $\omega^-(S) = \omega^+(V \setminus S)$. If M is an abelian group (with additive notation), then a M-flow is a mapping ϕ from E to M such that : for all $S \subset V$, $\sum_{e \in \omega^+(S)} \phi(e) - \sum_{e \in \omega^-(S)} \phi(e) = 0$. Now if $B \subset M$, such that $0 \notin B$ and $B = -B$ then a B-nowhere-zero flow (shorter B-NZF) of a graph G is a flow of G where the values are taken in

B , see [2].
As is customary a M-NZF is defined as a B-NZF for $B = M \setminus \{0\}$.
When $M = \mathbb{Z}$ and $B \subseteq [1-k, -1] \cup [1, k-1]$ $(k \geq 2)$, we call a B-NZF a k-NZF. A classical result of Tutte [12] is that for a graph a $\mathbb{Z}_k$-NZF exists if and only if a k-NZF exists.
In [6] we proposed the following definition:
If $B \subset M$ is such that $B \cap -B = \emptyset$, then a flow with values in B will be called an *antisymmetric-flow*. In this case we will say that G has a M-antisymmetric-flow (shortly M-ASF).
If G contains an isthmus then G does not admit a nowhere-zero flow. If G contains an edge-cut consisting of two edges e_1,e_2 between components K_1 and K_2 and if we take an orientation of the graph such that the edges e_1 and e_2 are both oriented from K_1 to K_2 then clearly G with this orientation has no antisymmetric-flow. A 2-edge-cut will be also called a 2-cut. When the arcs of the 2-cut are oriented in the same direction we will call it an oriented 2-cut. The question is now to find the minimum order of a group M such that any oriented graph without isthmus and oriented 2-cut has an M-ASF. We will prove that any graph without isthmus and oriented 2-cut has a $(\mathbb{Z}_3)^3 \times (\mathbb{Z}_6)^6$-ASF.

3 3-edge connected graphs

The purpose of this section is to establish the following theorem:

Theorem 3.1 *Every oriented graph without an isthmus or an oriented 2-cut has a $(\mathbb{Z}_3)^3 \times (\mathbb{Z}_6)^6$-ASF.*

We will use the following result proved in [1]:

Theorem 3.2 (DeVos, Johnson, Seymour) *Let $G = (V, E)$ be a 3-edge connected graph, there exits a partition $\mathcal{A} = \{A_1, A_2, A_3\}$ of E and a partition $\mathcal{A}_i = \{A_i^1, A_i^2, A_i^3\}$ of each A_i for $i \in \{1, 2, 3\}$ such that for any $i \in \{1, 2, 3\}$ the subgraph $G \setminus A_i$ is connected and such that the subgraph $G \setminus A_i^j$ for $i, j \in \{1, 2, 3\}$ is 2-edge connected.*

Proof. By the well-known technique of contracting edges (see [2, 9, 10]), it is easy to see that any minimal counterexample to Theorem 3.1 is a 3-edge connected graph. Thus we may assume that G is 3-edge connected. Let G be

an oriented 3-edge connected graph and let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ be a 9 partition of its edge-set as in Theorem 3.2. For each $i \in \{1,2,3\}$ there exists a $\mathbb{Z}_3$-flow f_i of G such that $f_i(e) = 1$ for $e \in A_i$. Since the graph $G_i^j = (V, E \setminus A_i^j)$ is 2-edge-connected, we may choose a nowhere-zero $\mathbb{Z}_6$- flow g_i^j of G_i^j by Seymour's 6-Flow Theorem [9] for each $i \in \{1,2\}$ and $j \in \{1,2,3\}$. We extend each of these flows to a flow of G by giving the value 0 to every edge in A_i^j.
We define the following ${\mathbb{Z}_3}^3 \times {\mathbb{Z}_6}^6$-NZF of G:

$$\phi = (f_1, g_1^1, g_1^2, g_1^3, f_2, g_2^1, g_2^2, g_2^3, f_3)$$

We summarize the situation by the following table:

	f_1	g_1^1	g_1^2	g_1^3	f_2	g_2^1	g_2^2	g_2^3	f_3
A_1^1	1	0	*	*	?	*	*	*	?
A_1^2	1	*	0	*	?	*	*	*	?
A_1^3	1	*	*	0	?	*	*	*	?
A_2^1	?	*	*	*	1	0	*	*	?
A_2^2	?	*	*	*	1	*	0	*	?
A_2^3	?	*	*	*	1	*	*	0	?
A_3	?	*	*	*	?	*	*	*	1

In this table we put in the first columm the edge sets of the partition and in the first row the components of the flow ϕ. We then put the different possible values of the flow's component on the edges belonging to the corresponding set of the partition. The "*" means that it is a non-zero value of a $\mathbb{Z}_6$-NZF and "?" means that the edge will have a value of a $\mathbb{Z}_3$-flow with possibly 0.
It is now easy to see that the flow ϕ has no opposite values. Indeed if two edges belong to the same A_i^j or to A_3 then the corresponding flow's values will has the value 1 in the same component hence they cannot be opposite, if the two edges belong to the different A_i^j or to A_3 and a A_i^j the corresponding flow's values will has the value 0 in one component of the flow's value of one edge and in the same component the flow's value of the other edge will have a value different from 0. We have then a ${\mathbb{Z}_3}^3 \times {\mathbb{Z}_6}^6$-ASF. This completes the proof. □
By the 4-Color Theorem, every 2-edge connected planar graph has a $\mathbb{Z}_4$-NZF. Using this fact instead of Seymour's theorem in the above proof yields the following result:

Proposition 3.3 *Every oriented planar graph with no isthmus and no oriented 2-cut has a* ${\mathbb{Z}_3}^3 \times {\mathbb{Z}_4}^6$*-ASF.*

We have to notice that this result is not better than the one obtained in [6] by a different method: for any orientation, a 3-edge connected planar graph has a ${\mathbb{Z}_6}^5$-ASF.

4 k-edge-connectivity and antisymmetric flows.

We give now some results linking k-edge-connectivity ($k > 3$) and antisymmetric flows.

Theorem 4.1 *Every oriented 4-edge connected graph has a* $(\mathbb{Z}_3)^2 \times (\mathbb{Z}_6)^2$*-ASF.*

To prove this result we first prove the following result:

Theorem 4.2 *For every 4-edge-connected graph G, there is a partition of $E(G)$ into $\{X_1, X_2, X_3, X_4\}$ so that setting $A_1 = X_1 \cup X_2$ and $A_2 = X_3 \cup X_4$, $G \setminus X_i$ is a 2-edge-connected graph for every i and $G \setminus A_j$ is connected for every j.*

In the proof of Theorem 4.2 we shall make use of the following lemma (see [4] and [8]):

Lemma 4.3 *If $H = (V, E)$ is a hypergraph, and for every subset $\{E_1, E_2, .., E_k\}$ of $E(H)$, the union of these k edges contains at least $k+1$ vertices, then H is 2-colorable.*

Proof of Theorem 4.2.

Let G be a 4-edge-connected graph and let T_1, T_2 be two edge-disjoint spanning trees of G (here we use [5] or [13] or [3]). Now, we construct for $i = 1, 2$ a hypergraph H_i as follows: the vertex set of H_i is $E(G) \setminus E(T_i)$. For every edge e in T_i, if S is the set of edges of $E(G) \setminus E(T_i)$ in the fundamental cocircuit of e, then S is a hyperedge of H_i. We claim that the union of any $k > 0$ edges of H_i contains more than k vertices for $i = 1, 2$. To see this, let $e_1, e_2, ..., e_k$ be edges of T_i, let $S_1, S_2, ..., S_k$ be the corresponding edges of H_i and let R be the union of $S_1, S_2, ..., S_k$. Let G' be the graph obtained from G by contracting the edges $T_i \setminus \{e_1, e_2, ..., e_k\}$ and then removing all of the loop-edges. By construction, $E(G') = R \cup \{e_1, e_2, ..., e_k\}$. Now, $\{e_1, e_2, ..., e_k\}$ is a spanning tree of G', so $|V(G')| = k+1$. Since G' is 4-edge-connected, it has minimum degree at least 4, and by summing degrees we find that $|E(G')| \geq 2k+2$. Therefore, $|R| \geq k+2 > k$ as desired. It follows from the lemma above that H_1 and H_2 are 2-colorable. Let $\{X_i, Y_i\}$ be a 2-coloring of $V(H_i)$. Now, $G \setminus X_i$ and $G \setminus Y_i$ are 2-edge-connected for $i = 1, 2$. Replacing $\{X_1, Y_1, X_2, Y_2\}$ by a partition $\{X_1', Y_1', X_2', Y_2'\}$ so that X_1' is a subset of X_1 etc, gives us the desired partition of $E(G)$. □

Proof of Theorem 4.1: Let G be a 4-edge-connected graph and $\{X_1, X_2, X_3, X_4\}$ a 4-partition of $E(G)$ as in the previous theorem. We define a $(\mathbb{Z}_6)^2 \times (\mathbb{Z}_3)^2$-NZF ϕ as before. We summarize the situation by the following table:

	f_1	g_1^1	g_1^2	f_2
X_1	1	0	*	?
X_2	1	*	0	?
A_2	?	*	*	1

It is easy to check that $\phi = (f_1, f_2, g_1, g_2)$ is a $(\mathbb{Z}_3)^2 \times (\mathbb{Z}_6)^2$-ASF.

For the 5-edge connected graphs we have:

Theorem 4.4 *Every oriented* 5*-edge connected graph has a* ${\mathbb{Z}_3}^5$*-ASF.*

Proof. We replace each edge of the graph $G = (V, E)$ by two parallel edges. Then the new obtained graph G' is 10-edge connected graph, so by Nash-William's Theorem [5] (see also [13] and [3]) it contains 5 edge-disjoint spanning trees. These 5 spanning trees $T_1, T_2, ..., T_5$ are spanning trees of G and one edge of G belongs to at most two different trees. For each T_i we choose a $\mathbb{Z}_3$-flow f_i which is equal to 1 on $E(G) \setminus E(T_i)$. It is easy to see that $\phi = (f_1, f_2, f_3, f_4, f_5)$ is a $(\mathbb{Z}_3)^5$-ASF of G. Indeed, for every pair of edges $e_1, e_2 \in E$, there is a spanning tree T_j with $e_1, e_2 \notin E(T_j)$. Thus $f_j(e_1) = 1 = f_j(e_2)$, and we find that $\phi(e_1) \neq -\phi(e_2)$. It follows from this that ϕ is a $(\mathbb{Z}_3)^5$-ASF as desired □

In the same vein we have:

Theorem 4.5 *Every oriented* 6*-edge connected graph* G *has a* ${\mathbb{Z}_3}^2 \times \mathbb{Z}_2$*-ASF*

Proof. Let $G = (V, E)$ be a 6-edge connected graph. By Nash-William's Theorem [5] (see also [13] and [3]) we know that the graph contains three edge disjoint spanning trees T_1, T_2, T_3. Let $E_1 = E \setminus \{E(T_2) \cup E(T_3)\}$, $E_2 = E(T_2)$, $E_3 = E(T_3)$. Let f_1 be a $\mathbb{Z}_3$-flow such that $f_1(e) = 1$ for every $e \in E_1 \cup E_2$, f_2 be a $\mathbb{Z}_3$-flow such that $f_2(e) = 1$ for every $e \in E_1 \cup E_3$ and g be a $\mathbb{Z}_2$-flow such that $g(e) = 1$ for every $e \in E_2$ and with value 0 on E_3. It is then easy to see that the flow $\phi = (f_1, f_2, g)$ is a ${\mathbb{Z}_3}^2 \times \mathbb{Z}_2$-ASF, we summarize the situation by the following table:

	f_1	f_2	g
E_1	1	1	?
E_2	1	?	1
E_3	?	1	0

5 Concluding remarks

The bound ${\mathbb{Z}_3}^2 \times \mathbb{Z}_2$ for 6-edge connected graphs is already of reasonable size. Currently the best lower bound (obtained via duality and oriented chromatic number) is 16 for 3-edge connected graphs. However this is presently also the best lower bound for antisymmetric flows for planar graphs.
The difficulty of coloring of planar oriented graphs by oriented circulants is perhaps surprising (as illustrated by the gap 7776 versus 80; see [6], [7]).

References

[1] M. DeVos, T. Johnson, P. D. Seymour, *Cut Coloring and Circuit Covering*, manuscript.

[2] F. Jaeger, *Nowhere zero-flow Problems.* Selected topics in Graph Theory 3 Academic Press, London 1988, 71–95.

[3] S. Kundu. *Bounds on the number of disjoint spanning trees.* J. of Combinatorial Theory, Ser. B. **17** (1974) 199–203

[4] L. Lovász, *Combinatorial problems and exercises* Amsterdam - New York - Oxford: North-Holland Publishing Company. 1978

[5] C. St. J. A. Nash-Williams. *Edge-disjoint spanning trees of finite graphs.* J. Lond. Math. Soc. 36, 445-450 (1961).

[6] J. Nešetřil and A. Raspaud, *Antisymmetric Flows and Strong Colourings of Oriented Graphs*, Ann. Inst. Fourier. 49, 3 (1999), 1037-1056.

[7] A. Raspaud and E. Sopena. *Good and semi-strong colorings of oriented planar graphs*. Inf. Processing Letters **51** (1994), 171–174.

[8] P. D. Seymour. *On the two coloring of hypergraphs.* Quart J. Mah. Oxford (3) **25** 1974, 303–12.

[9] P. D. Seymour. *Nowhere-zero 6-flows.* J. Combinatorial Theory (B) **30** 1981, 130-135.

[10] P. D. Seymour. *Nowhere-zero flows*. Handbook of Combinatorics, edited by R. Graham, M. Grötschel and L. Lovász. (1995) 289–299.

[11] E. Sopena. *The chromatic number of oriented graphs*. J. Graph Theory, **25** (1997), 191–205.

[12] W. T. Tutte. *A contribution to the theory of chromatic polynomials.* Canad. J. Math. **6** (1954) 80-91.

[13] W. T. Tutte. *On the problem of decomposing a graph into n connected factors*. J. London Math Soc. **36** (1961), 221-230.