The NP completeness of the edge precoloring extension problem on bipartite graphs

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Abstract

We show that the following problem is NP complete: Let G be a cubic bipartite graph and f be a precoloring of a subset of edges of G using at most three colors. Can f be extended to a proper edge 3-coloring of the entire graph G? This result provides a natural counterpart to classical Holyer's result on edge 3-colorability of cubic graphs and a strengthening of results on precoloring extension of perfect graphs.

1 Introduction

Through the paper $G = (V(G), E(G))$ stands for a finite simple undirected graph, i.e. $V(G)$ is a finite set and $E(G) \subseteq \binom{V(G)}{2}$.

The symbol $deg(u)$ denotes the degree of the vertex u and is equal to the number of edges that contain u. If all vertices of a graph G have the same degree k, we say that the graph is k-regular. The cubic graph is a synonym for a 3-regular graph. The maximum degree of a vertex in the graph G is denoted by $\Delta(G)$.

The chromatic number $\chi(G)$ is the minimum number of distinct colors that are necessary to color the vertex set of G s.t. no edge connects vertices of the same color. A graph G is called bipartite if $\chi(G) \leq 2$.

The chromatic index $\chi'(G)$ is the minimum number s.t. the set $E(G)$ can be colored using $\chi'(G)$ colors and no two edges that share a common vertex get the same color.

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Due to the Vizing theorem, the chromatic index $\chi'(G)$ can be bounded in terms of the maximum degree: $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. The computational complexity of determining the chromatic index was characterized by Holyer who proved that the problem to decide whether $\chi'(G) \leq 3$ is *NP*-complete [4]. The Holyer's construction does not produce bipartite graphs by the because König–Hall marriage theorem implies that the chromatic index of a bipartite graph is equal to its maximum degree and a proper edge coloring can be found in polynomial time by a matching algorithm.

The vertex precoloring extension problem asks whether a graph G which has exposed a family of disjoint independent sets allows a coloring using $\chi(G)$ colors s.t. each given independent set is colored by a unique color.

The question of the existence of a precoloring extension [1] generalizes the classical concept of the graph coloring, hence its computational complexity characterization is interesting (and nontrivial) for graphs whose chromatic number can be computed in a polynomial time.

The complete characterization of the complexity of precoloring extension for the class of perfect graphs was given by Kratochvíl and Sebö [5]. They showed that the precoloring extension is polynomially solvable if at most two colors are used in the precoloring and one of these colors is used on at most one vertex. All other cases are *NP*-complete.

We connect the concepts of edge coloring and precoloring extension to conclude that the precoloring extension problem of perfect graphs persists *NP*-complete even restricted to the class of line graphs of bipartite graphs.

Problem: *Edge precoloring extension*
Input: A graph G, f precoloring of $E' \subseteq E(G)$
Question: Can f be extended to the edge-coloring of G using at most $\chi'(G)$ distinct colors?

One one hand this provides a counterpart to Holyer *NP*-completeness theorem for bipartite graph. On the other hand, since line graphs of bipartite graphs are perfect we present a strengthening of the result of Kratochvíl and Sebö.

2 The main result

Theorem 1 *The edge precoloring extension problem is NP-complete for bipartite graphs of maximum degree three.*

Proof: We show a polynomial reduction from the Not-All-Equal 3-SAT problem [3].

Let Φ be a formula in the normal form and each clause has three (not necessarily distinct) literals. We construct a graph G and define a coloring f on a subset of $E(G)$ s.t. f allows an extension to the entire graph G if and only if Φ is NAE-satisfiable.

We denote the three colors used in the edge-coloring of G by r, g, b and call them red, green and blue.

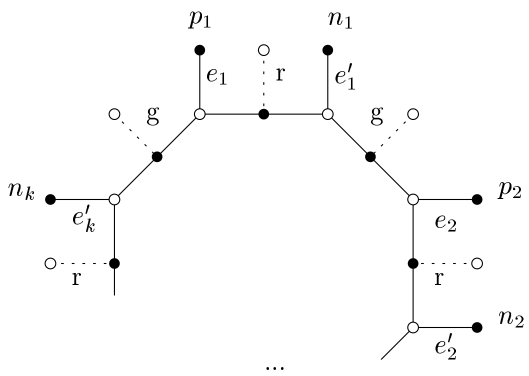

Figure 1: Variable gadget V

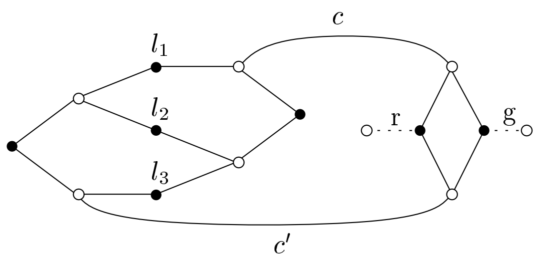

Figure 2: Clause gadget C

Assume that every variable in Φ has at most k positive and at most k negative occurrences. For each variable x put into G an extra copy V^x of the graph depicted in Fig. 1. For each clause $z = (L_1^z \wedge L_2^z \wedge L_3^z)$ of the formula Φ put into G an extra copy of the graph C^z depicted in Fig. 2. We finish the construction of the graph G by a series of unifications:

For every variable x and each literal L_i^z equal to x, unify the corresponding vertex l_i^z with a unique p_j^x not used by other literals. Similarly for each $L_i^z = \neg x$ unify l_i^z with a unique $n_{j'}^x$.

Observe that the graph G is bipartite. The bipartition is indicated by white and black vertex colors.

Define the precoloring f on the dotted edges of G as depicted on Figures 1 and 2.

Consider a proper edge coloring g of the graph G that extends f. On every copy V^x the edges $e_1, .., e_k$ and $e'_1, .., e'_k$ are colored red or green and $g(e_1) = g(e_2) = ... = g(e_k) \neq g(e'_1) = ... = g(e'_k)$.

For each variable x we assign x the true value if $g(e_1^x) = r$ and the false value otherwise. We show that each clause contains both positively and negatively valued literals. For a contradiction assume that a clause z has all three literals

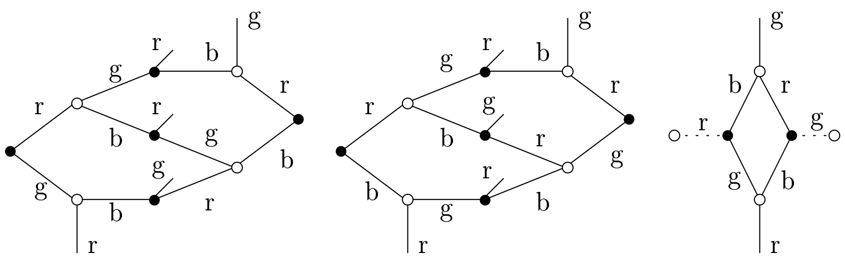

Figure 3: Coloring the clause gadget C

positively valued. In the corresponding graph C^z all three edges connecting vertices l_1^z, l_2^z and l_3^z to the variable gadgets are colored red. The coloring cannot be extended to the entire gadget C^z, since edges c and c' should be colored by the same color but it is impossible due to the right part of the graph. The appearance of three negative valued literals in C^z yields to a contradiction due to the same argument.

In the opposite direction consider a proper assignment of variables of the formula Φ. For each variable x color edge e_1^x red if the variable x has assigned the true value and color it green otherwise. Then each graph V^x has a unique extension of the above coloring. Moreover each clause gadget is connected to the rest of the graph by edges such that at least one edge is red and at least one is green. Fig. 3 shows that the coloring can be extended to the entire edge gadget C^z (the four remaining cases are obtained due to the symmetry of the gadget and by exchange of red and green color). $\square$

Corollary 2 *The edge precoloring extension problem is NP-complete for the class of cubic bipartite graphs.*

Proof: Assume that G does not contain isolated edges. Use two copies of graphs G constructed in the previous proof and merge each pair of the corresponding edges incident with a vertex of degree one into a single edge. In addition join every pair of the corresponding vertices of degree two by an extra new edge. The new graph G' is cubic and the formula Φ has a solution if and only if both copies allow an edge coloring extension. In the opposite direction when a coloring of a single copy of G is given, it can be extended to the entire graph G'. $\square$

3 Conclusion

As far as we know this paper brings the first complexity results on the edge precoloring extension problem. There are several variants of the problem that are still open, e.g. the edge precoloring extension on planar bipartite graphs or perfect graphs.

Our complexity study was motivated by properties of locally injective homomorphism on small graphs (see [2], the graph $P(1,3,5)$).

The same concept also states the problem of determining the computational complexity of the edge-coloring choosability, i.e. the problem asking whether a graph allows a proper edge coloring s.t. every edge is colored by a color belonging to a prescribed set of feasible colors and these sets are specific for each edge.

Acknowledgment

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References

[1] Biro, M., Hujter, M., and Tuza, Z. Precoloring extension. i: Interval graphs. *Discrete Mathematics 100* (1992), 267–279.

[2] Fiala, J., and Kratochvíl, J. Partial covers of graphs. submitted.

[3] Garey, M. R., and Johnson, D. S. *Computers and Intractability*. W. H. Freeman and Co., New York, 1979.

[4] Holyer, I. The np-completeness of edge coloring. *SIAM Journal of Computing 10*, 4 (1981), 718–720.

[5] Kratochvíl, J., and Sebö, A. Coloring precolored perfect graphs. *Journal of Graph Theory 25* (1997), 207–215.