Partial covers of graphs

Jiří Fiala* Jan Kratochvíl†

May 15, 2000

Abstract

Given graphs G and H, a mapping $f : V(G) \to V(H)$ is a *homomorphism* if $f(x)f(y)$ is an edge of H for every edge xy of G. In this paper we initiate the study of computational complexity of *locally injective homomorphisms* called *partial covers* of graphs. We motivate the study of partial covers by showing a correspondence to generalized (2,1)-colorings of graphs, the notion stemming from a highly practical problem of assigning frequencies to transmitters to avoid interference. We compare the problems of deciding existence of partial covers and of full covers (locally bijective homomorphisms) which were previously studied. Then we present first complexity results about partial covers, which include both NP-complete and nontrivial polynomially solvable cases.

1 Introduction

Given graphs G and H, a mapping $f : V(G) \to V(H)$ is a *homomorphism* if $f(x)f(y)$ is an edge of H for every edge xy of G. A homomorphism from G to H is also called an H*-coloring* of G (since homomorphisms to the complete graphs correspond to ordinary colorings). Hell and Nešetřil [13, 14] considered the complexity of deciding if an input graph G allows a homomorphism into a fixed parameter graph H. They proved that this question is polynomially solvable for graphs H which contain a loop or are bipartite, and NP-complete otherwise. Dyer and Greenhill [7] extended the research in this direction and characterized the graphs H for which counting homomorphisms from G into H is $\#P$-complete.

A homomorphism which is a local isomorphism (i.e., for every node x of G, the morphism f maps the neighborhood of x bijectively onto the neighborhood of $f(x)$) is called a *covering projection* (and if such a homomorphism exists, we say that G *covers* H). The motivation for the study of graph covers comes from the topological graph theory [3]. As special cases of covering spaces from algebraic topology [23], graph covers are used in many applications in topological graph theory [11]. Computational applications of graph covers are used by Angluin

*Department of Applied Mathematics, Charles University, Malostranské nám. 25, 118 00 Prague, Czech Republic, e-mail: `fiala@kam.ms.mff.cuni.cz`. Research supported in part by Czech Research grant GAČR 201/99/0242.

†Department of Applied Mathematics, Charles University, Malostranské nám. 25, 118 00 Prague, Czech Republic, e-mail: `honza@kam.ms.mff.cuni.cz`. Research supported in part by Czech Research grants KONTAKT 338/99, KONTAKT 055/97 and GAUK 158/99.

[2] to study "local knowledge" in distributed computing environments, and by Courcelle and Métivier [6] to show that nontrivial minor closed classes of graphs cannot be recognized by local computations. In [1], Abello *et al.* raised the question of computational complexity of H-cover problems, noting that there are both polynomial-time solvable (*easy*) and NP-complete (*difficult*) versions of this problem depending on the parameter graph H. (The H-cover problem asks if a given input graph G covers H, the latter graph is considered a fixed parameter of the problem). The complexity of the H-cover problem was further studied in [17, 19, 18]. Several infinite classes of both polynomial and NP-complete instances were recognized, however, currently no good conjecture about the characterization of graphs H for which the H-cover problem is polynomially solvable is at hand (assuming of course P$\neq$NP).

In this paper we initiate the study of computational complexity of *partial covers*. We say that G is a *partial cover* of H if G is an induced subgraph of a (full) cover of H. In the algebraic setting, a partial covering projection is a *locally injective homomorphism*, i.e., a homomorphisms which maps the neighborhood of every vertex injectively into the neighborhood of the image of this vertex. Partial covers of graphs are used in [17] as a tool in the gadget construction for NP-completeness reduction in the proof of Proposition 3.4.

We believe that the study of the computational complexity of partial covers is worthwhile for two reasons. First, as it is shown in Section 2, partial covers correspond to generalized (2,1)-colorings of graphs, while the notion of a (2,1)-coloring stems from a highly practical problem of assigning frequencies to transmitters to avoid interference. Secondly, as our results show, from the computational complexity point of view partial covers seem to be much more difficult than full covers (already for graphs H containing only two vertices of degree greater than 2 the problem is NP-complete in infinitely many instances). This raises a hope that the number of polynomially solvable instances may be so restricted that it would allow a characterization theorem.

The paper is organized as follows. In Section 2 we reveal the connection to generalized (2,1)-colorings. In Section 3 we compare the problems of deciding existence of full and partial covers. In the last section we present first complexity results about partial covers. We note that already graphs H having only two vertices of degree 2 give rise both to NP-complete instances and to instances solvable in polynomial time by a nontrivial algorithm which may be of independent interest.

2 Generalized distance two colorings of graphs

Roberts proposed the following *distance two constrained labelings of graphs*, a notion stemming from the radio frequency assignment problem (where the task is to assign radio frequencies to transmitters at different locations without interference). Assuming that the distance function of the transmitters can be modelled by a graph distance, it is asked that transmitters that are close to each other receive different channels and transmitters that are *very* close together receive channels that are at least two apart:

A $\lambda_{(2,1)}$-*labeling* of a graph G is an assignment of labels from the set $\{0, \ldots, \lambda\}$ to the vertices of G such that vertices at distance two are assigned different labels and adjacent vertices are assigned labels which differ by at least 2. The minimum value λ for which G admits a $\lambda_{(2,1)}$-labeling is denoted by $\lambda_{(2,1)}(G)$.

An upperbound for $\lambda_{(2,1)}(G)$ in terms of the maximum degree $\Delta(G)$ ($\lambda_{(2,1)}(G) \leq \Delta^2(G) + 2\Delta(G)$) was obtained in [10, 24] and this was improved in [5] to $\lambda_{(2,1)}(G) \leq \Delta^2(G) + \Delta(G)$. The conjecture if $\lambda_{(2,1)}(G) \leq \Delta^2(G)$ is still open, though it was proven true for some special graph classes (chordal graphs, graphs of diameter 2). From the complexity point of view, the problem to decide if a given graph allows a $\lambda_{(2,1)}$-labeling was proven NP-complete in [10, 24] if λ is part of the input, and in [9] for every fixed $\lambda \geq 4$.

A natural generalization of the linear channel assignment problem is considering channel (frequency) spaces with nonlinear metric. This would model the case when e.g., frequencies which are multiples can also interfere. The circular metric in the channel space was actually considered by Leese [20]. Assuming that the distance function in the channel space can be modelled by a graph theoretical distance in a graph whose vertices are the possible channels (frequencies), we arrive to a natural generalization of the concept of $\lambda_{(2,1)}$-labelings:

Definition 2.1 *Let H be a graph. An* $H_{(2,1)}$-labeling *of a graph G is a mapping $f : V(G) \to V(H)$ which satisfies*

1. *$d_H(f(x), f(y)) \geq 2$ for every two adjacent vertices $x, y \in V(G)$;*
2. *$d_H(f(x), f(y)) \geq 1$ for every two vertices $x, y \in V(G)$ such that $d_G(x, y) = 2$.*

Obviously, a graph G allows a $\lambda_{(2,1)}$-labeling if and only if it allows a $(P_\lambda)_{(2,1)}$-labeling (here P_k denotes the path with k edges). Similarly, G allows a circular $\lambda_{(2,1)}$-labeling if and only if it allows a $(C_{\lambda+1})_{(2,1)}$-labeling (C_k denotes the cycle with k edges).

We have shown in [9] that deciding existence of $(P_\lambda)_{(2,1)}$-labeling is NP-complete for every fixed $\lambda \geq 4$. Here we have a similar result for the circular metric (proved in Section 3):

Theorem 2.2 *The $(C_{\lambda+1})_{(2,1)}$-labeling problem is NP-complete for every $\lambda \geq 5$.*

We will now relate the notion of $H_{(2,1)}$-labelings to partial covers of graphs. The following Lemma is straightforward:

Lemma 2.3 *A mapping f from a graph G to a graph H is a partial covering projection if and only if*

1. *$(f(u), f(v)) \in E(H)$ for each edge $(u, v) \in E(G)$;*
2. *$f(v) \neq f(w)$ for any two distinct vertices $u, v \in V(G)$ which have a common neighbor in G.*

In the following observation, $\overline{H}$ denotes the complement of the graph H.

Observation 2.4 *A mapping $f : V(G) \to V(H)$ is an $H_{(2,1)}$-labeling if and only if it is a partial covering projection of G to $\overline{H}$.*

Proof Consider a mapping $f : V(G) \to V(H)$. For any two vertices $u, v \in V(H)$, $d_H(u,v) \geq 1$ if and only if $u \neq v$, and so conditions (2) of Definition 2.1 and Lemma 2.3 are equivalent. Condition (1) of Definition 2.1 states that $xy \in E(G)$ implies $(f(x), f(y)) \notin E(H)$, i.e., $(f(x), f(y)) \in E(\overline{H})$ which is exactly condition (1) of Lemma 2.3 for the complement of H. $\Box$

In view of this observation we now begin the study partial covers of graphs, having in mind the impact of our findings on distance two constrained graph labelings.

3 Partial covers versus full covers

It is well known that the existence of a covering projection between two graphs implies that they have the same degree refinement matrix. The converse is not true, but it is shown in [21] that graphs with the same degree refinement matrix have a common finite cover. In general, partial covers do not preserve the degree refinement matrix. However, if two graphs happen to have the same degree refinement matrix, then any partial covering projection between them necessarily is a full covering (Proposition 3.2). This fact will be used in this subsection to derive first complexity results on partial covers and the generalized channel assignment problem.

The *degree partition* of a graph G is the coarsest partition of the vertex set of G into classes (called blocks) $B_1, B_2, \ldots, B_k$ s.t. for every i, j and any two vertices $u, v \in B_i$, $|N(u) \cap B_j| = |N(v) \cap B_j|$.

The degree partition is unique and can be obtained by a procedure which recursively refines partitions by the numbers of neighbors of vertices in the blocks of the partition. (See the Appendix for the details.)

Having the degree partition, let $r_{ij} = |N(u) \cap B_j|$ for any $u \in B_i$, for $1 \leq i, j \leq k$. The *degree refinement matrix* M_G of the graph G is the k by k square matrix

$$M_G = (r_{ij})_{i,j=1}^{k}.$$

The following theorem is a folklore:

Proposition 3.1 *If a graph G covers a connected graph H then $M_G = M_H$.*

It is obvious that for regular graphs of the same valency, a locally injective homomorphism is a local isomorphism, i.e., every partial covering projection is a full cover. Analogous statement is true for graphs with the same degree refinement matrix:

Proposition 3.2 ([8]) *If connected graphs G and H have the same degree refinement matrix then every partial covering projection of G to H is also a full covering projection.*

The consequence for the complexity of partial covers follows:

Corollary 3.3 *For any connected graph H, the H-cover problem is polynomially reducible to the H-partial cover problem.*

Proof Given a graph G subject to the question if G covers H, determine (in polynomial time) if G has the same degree refinement matrix as H. If not, G cannot cover H. In the affirmative case, G partially covers H is and only if it covers H fully. $\square$

So all NP-completeness results about graph covers carry on to partial covers. We certainly do not wish to restate here all the results of [1, 17, 19, 18, 8], but let us mention just one result which has consequence for the circular channel assignment problem of Leese [20]. Note that in view of Observation 2.4, for any graph H the $H_{(2,1)}$-labeling problem and the $\overline{H}$-partial cover problem are polynomially equivalent.

Proposition 3.4 ([17]) *The H-cover problem is NP-complete when H is k-regular $\lfloor\frac{k+1}{2}\rfloor$-edge connected or k-regular k-edge-colorable, for every $k \geq 3$.*

Since the complement of C_k is $(k-3)$-regular and $(k-3)$-edge-connected, the NP-completeness of $\overline{C_k}$-cover (for $k \geq 6$) follows from Proposition 3.4. By Corollary 3.3, the $\overline{C_k}$-partial cover problem is NP-complete, and Theorem 2.2 follows.

4 Complexity of partial covers of small graphs

In [18], the notion of covers of simple graphs is generalized to covers of colored directed multigraphs, and it is shown that a full characterization of the complexity of the H-cover problem for simple graphs would necessarily provide a full characterization of the complexity of the general covering problem (briefly speaking, vertex colors correspond to trees hanging on vertices, which can be uniquely located, and edge colors and directions then correspond to length and color pattern of paths consisting of vertices of degree 2 only). Thus in the multigraph setting, multigraphs with small numbers of vertices do not necessarily provide easy covering problems. The complexity of H-cover for multigraphs with 2 vertices was fully characterized:

Proposition 4.1 ([18]) *Let H be an undirected multigraph with two vertices L, R and with colored edges. Then the H-cover problem is polynomially solvable if for some color, the numbers of loops of this color at L and R are different, or if for every color, the edges of this color are either loops or edges between L, R. Otherwise the problem is NP-complete.*

In this section we will study the complexity of partial covers for graphs with two vertices of degree higher than 2. For $n \geq 3$ and $1 \leq a_1 \leq a_2 \leq \ldots \leq a_n$, let $P(a_1, a_2, \ldots, a_n)$ be the (multi)graph formed by two vertices x, y connected by n paths of lengths $a_1, a_2, \ldots, a_n$. (We write $P(a^n)$ for $P(a, a, \ldots, a)$, and $P(a^k, b^{n-k})$ for $P(a_1, a_2, \ldots, a_n)$ when $a_1 = \ldots = a_k = a$ and $a_{k+1} = \ldots = a_n = b$.) One may wish to compare the results with Proposition 4.1. Proposition 4.8 shows that the partial cover may be NP-complete already when the target graph has just three paths of different lengths (or even two paths of the same length and one more path of a different length), a case which is trivially polynomial for full covers. On the other hand, Theorem 4.6 provides an example of a nontrivial polynomially solvable instance. Let us remark that the NP-completeness of the $P(1, 2, 3)$-partial cover problem shows that the

2-satisfiability approach for full covers does not extend to partial covers (the degree partition of $P(1,2,3)$ has two blocks of size 2 and one singleton). (It is showed in [19] that the H-cover problem is polynomially solvable if every block of the degree partition of H has at most two vertices.)

Note also that $\overline{P(1,2,3)} = P_5$ the path of length 4, and so the NP-completeness of $4_{(2,1)}$-labeling (proven in [9]) is a special case of Proposition 4.8.

We allow multiple edges (in case $a_1 = a_2 = 1$) and we note that all results on partial and full covers of simple graphs translate to multigraphs as well. One has to be careful with a formal definition of (partial) covers, though. To avoid lengthy description we refer to [18]. E.g., a graph G partially covers $P(1^n)$ if and only if it is bipartite and n-edge colorable. Note that such a graph must have maximum degree $\leq n$, and conversely, every bipartite graph of maximum degree n is n-edge colorable (well known König theorem) and hence partially covers $P(1^n)$. Hence we have observed:

Observation 4.2 *The $P(1^n)$-partial cover is polynomially solvable.*

Next we note a simple but useful observation (proof is in the Appendix):

Observation 4.3 *The problems $P(a_1, a_2, \ldots, a_n)$-partial cover and $P(a_1k, a_2k, \ldots, a_nk)$-partial cover are polynomially equivalent for any positive integers $a_1, a_2, \ldots, a_n$ and any k.*

Corollary 4.4 *The $P(a^n)$-partial cover problem is polynomially solvable for every fixed a.*

We will fully characterize the complexity of the $P(a^k, b^l)$-partial cover problem. The following auxiliary problem may be of independent interest, and it provides a case of a nontrivial polynomial algorithm in our partial cover paradigm.

Lemma 4.5 *Given a graph G and for each vertex u, an interval $I(u) = [a_u, b_u], 0 \leq a_u \leq b_u \leq deg_G(u)$, and for each edge e, a nonempty subset $J_e \subset \{0,1,2\}$, then the question whether there exists a subset of* halfedges *with characteristic function $\chi : \{(u,e), u \in e\} \to \{0,1\}$ s.t.*

$$\forall u \in V(G) : \sum_{e \ni u} \chi(u,e) \in I_u \quad \textit{and} \quad \forall e \in E(G) : \sum_{u \in e} \chi(u,e) \in J_e$$

is solvable in polynomial time.

The intervals $I(u)$ are meant as intervals of integers. Note that if all J_e are equal to $\{0,2\}$ (i.e., none or both halfedges are chosen) and all intervals I_u are equal to $[1,1]$ we have the problem of finding a matching in a graph which is polynomially solvable by the well-known Edmonds's algorithm.

Proof We show a reduction to the Edmonds's algorithm in two steps. The first reduction changes all J_e into sets $\{0,2\}$ and in the second stage we transform all I_u into one-element intervals $[1,1]$.

Subdivide each edge $e = (u, v)$ of G s.t. $J_e \neq \{0, 2\}$ by a new extra vertex $w(e)$ and set $I_{w(e)} = J_e$. (All sets different from $\{0, 2\}$ are intervals.) In the new graph G' ask for a factor (a subset of the edge set) F, s.t. $deg_F(u) \in I_u$ for all vertices of G'. The backward-reduction from the factor to the characteristic function is straightforward.

The second step is due to Tutte [22]. We just recall that each vertex is replaced by a tripartite gadget, where the first part of size $deg_G(u)$ is matched to edges adjacent to u, the second part of size $deg_G(u) - a_u$ is connected by a complete bipartite graph to the first one and the third part of size $b_u - a_u$ is similarly connected by a complete bipartite graph to the second part. Finally form a clique on the vertices from all the third part and, if necessary, enlarge the clique by a new extra vertex to get an even number of vertices in the new graph G''. Any perfect matching of the graph G'' corresponds to a proper factor F we look for (and vice versa). $\square$

Theorem 4.6 *The $P(a^k, b^l)$-partial cover problem is polynomially solvable for every odd a, b.*

Proof The case $a = b$ was already mentioned in Corollary 4.4. We also suppose that a, b are relatively prime due to Observation 4.3.

Let G be an input graph for the $P(a^k, b^l)$-partial cover problem. Replace every maximal path whose inner vertices have degree two by a single edge. In the new graph G' each edge e corresponds to a path of length m_e between two vertices of degree $\neq 2$ in the original graph G.

For any length m we decide whether P_m partially covers $P(a^k, b^l)$ s.t. endpoints of the path are mapped on vertices x and/or y and none, one or both final segments of P_m are mapped into a path of length a in $P(a^k, b^l)$. We denote the set of possible cases by $J(m) \subseteq \{0, 1, 2\}$.

The above computation can be done in a constant time since for $m > (a + b)^2$ all three cases are possible.

Assign $I_u = [\max\{0, deg_{G'}(u) - l\}, \min\{k, deg_{G'}(u)\}]$ to all vertices u of G'. Then set $J_e = \{0, 1, 2\}$ to all edges e of G' that are incident with a vertex of degree one. For any other edge e corresponding to a path of length m set $J_e = J(m)$. Apply Lemma 4.5 and find a feasible set of halfedges S.

Obviously the existence of such a set is a necessary condition for the existence of a partial covering projection from G to $P(a^k, b^l)$. We will show that it is also sufficient.

Now in G we partially cover any path between two vertices of degree at least three into $P(a^k, b^l)$ such that final segments are mapped into a path of length a if and only if they are in S.

We need to define an injective mapping on the neighborhood of any vertex of degree at least three to distinct final segments of the a- and b-paths. For simplicity call the pair of final edges of a path the *ends of a path*.

First consider paths in G such that both ends are mapped into an a-path. We decide in a constant time whether both ends of these paths have to be mapped only into the same a-path (e.g. P_a) or into two different a-paths (e.g. P_{2a}) or if both cases are possible (e.g. P_{2a+b}).

If a path Q has always both ends mapped on the same a-path, then either $Q = P_a$ or $k = 2$ and $Q = P_{(2i+1)a}$, but both cases imply that one endvertex of Q is mapped on x in $P(a^k, b^l)$ and the other one onto y.

On the other hand if a path R has both ends mapped onto different a-paths then either $R = P_{2a}$ or $k = 2$ and $R = P_{(2i)a}$, and both endvertices of R are mapped onto the same vertex of $P(a^k, b^l)$.

We decide the distribution of ends of these paths into $P(a^k, b^l)$ by an edge-coloring argument: Consider a subgraph of G' corresponding to paths of type Q and R, and insert into each edge corresponding to a path of type R an extra new vertex. The new graph is bipartite with maximum degree at most k and hence is k-edge colorable (in a polynomial time). The set of k edge colors corresponds to the set of a-paths in $P(a^k, b^l)$.

We already mapped paths of type Q and R into $P(a^k, b^l)$ such that the mapping is locally injective. Perform the same procedure for b-paths.

Remaining paths between vertices of degree at least three have both ends independent — it means, that when we select a mapping of their ends into $P(a^k, b^l)$ (with respect to a- and b-paths determined by a proper set of halfedges) then we are able to extend the mapping into a partial covering of the entire path.

The partial covering projection can be always extended to paths ending in vertices of degree one. □

Further we list the NP-complete instances. The hardness results are proved by reductions from the following problem: For any fixed positive k, l such that $k + l \geq 3$, it is NP-complete to decide if a given $(k + l)$-regular graph allows a partition of its vertex set into two classes such that every vertex has k neighbors in the same class and l neighbors in the other class. The proofs are gathered in the Appendix.

Theorem 4.7 *The $P(a^k, b^l)$-partial cover problem is NP-complete for every $a \not\equiv b$ mod 2.*

In view of Observation 4.3, Theorems 4.6 and 4.7 provide full characterization of the complexity of the $P(a^k, b^l)$-partial cover problem. The next simplest case is when three different lengths appear, namely $P(a, b, c)$ (where $a < b < c$ is assumed). Here we have only partial results.

Proposition 4.8 *The $P(a, b, c)$-cover is NP-complete problem if c is divisible by $a + b$.*

Proposition 4.9 *The $P(1, 2, c)$-partial cover problem is NP-complete for every c.*

The reductions are described in detail in the Appendix. Here we mention two Lemmas which are the cornerstones of the reductions. Especially Lemma 4.11 is interesting on its own.

Lemma 4.10 *Suppose a, b, c are such that there exists an integer m with the following property: All solutions to the equation*

$$x_1 + x_2 + \ldots + x_k = m \tag{1}$$

such that $x_i \in \{a,b,c\}$ and no two consecutive x_i have the same value, satisfy one of the conditions

1) k is odd and $x_1 = x_k = a$

2) k is even and $x_1 = b$ and $x_k = c$ and vice versa

3) k is even and $x_1 = x_k = b$ or $x_1 = x_k = c$,

4) $x_1 = c$ and there exists no solution different from the case 2) with $x_1 = b$ and vice versa and solutions satisfying 1) and 2) exist. Then $P(a,b,c)$-partial cover is NP-complete. (Note that k in (1) is a variable.)

In the proofs of Propositions 4.8 and 4.9 we present constructions of m's satisfying the conditions of Lemma 4.10. The following lemma justifies the general scheme of the NP-completeness reductions (see the Appendix).

Lemma 4.11 *Let G be a connected (multi)graph and let a set C_u of at least $deg_G(u)$ colors be assigned to every vertex u of G. If*

- *G contains a cycle or*
- *there exists a vertex u s.t. $|C_u| > deg_G(u)$ or*
- *there are two leaves u, u' s.t. $C_u \neq C'_u$*

then there exists (and can be found in polynomial time) a coloring of the half-edges of G which satisfies $c(u,e) \in C_u$ for every vertex u, and which uses different colors for any two half-edges sharing an end-point (a vertex or the virtual middle point of the particular edge).

References

[1] Abello, J., M. R. Fellows and J. C. Stillwell, On the complexity and combinatorics of covering finite complexes. *Australian Journal of Combinatorics* **4**, (1991), pp. 103-112.

[2] D. Angluin, Local and global properties in networks of processors, in *Proceedings of the 12th ACM Symposium on Theory of Computing* (1980), 82-93

[3] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, 1974

[4] N. Biggs, Homological coverings of graphs, *J. London Math. Soc.* (2), 30 (1984), 1-14

[5] Chang, G. J. and D. Kuo, The $L(2,1)$-labeling problem on graphs, *SIAM J. Disc. Math.* **9**, (1996), pp. 309–316.

[6] B. Courcelle and Y. Métivier, Coverings and minors: Applications to local computations in graphs, *European Journal of Combinatorics* 15 (1994), 127-138

[7] M.Dyer and C.Greenhill, The complexity of counting graph homomorphisms, In: Proceedings of the Eleventh annual ACM-SIAM Symposium on Discrete Algorithms SODA'00, San Francisco 2000

[8] Fiala, J., *Graph covers*, Master Thesis, Charles University, 1997, (in czech).

[9] Fiala, J., J. Kratochvíl, and T. Kloks, Fixed-parameter tractability of λ-colorings, In: *Graph-Theoretical Concepts in Computer Science, Proceedings 25th WG '99, Ascona, Lecture Notes in Computer Science 1665*, Springer Verlag, (1999), pp. 350-363.

[10] Griggs, J. R. and R. K. Yeh, Labelling graphs with a condition at distance 2, *SIAM J. Disc. Math.* **5**, (1992), pp. 586–595.

[11] J.L. Gross and T.W. Tucker, *Topological Graph Theory*, J.Wiley and Sons, 1987

[12] J.L. Gross and T.W. Tucker, Generating all graph coverings by permutation voltage assignments, *Discrete Math.* 18 (1977) 273-283

[13] P.Hell and J.Nešetřil, On the complexity of H-coloring, J. Combin. Theory Ser. B 48 (1990) 92-110

[14] P.Hell and J.Nešetřil, The core of a graph, Discrete Math. 109 (1992) 117-126

[15] Heuvel, J. van den, R. A. Leese, and M. A. Shepherd, Graph labeling and radio channel assignment, to appeaer in Journal of Graph Theory.

[16] M Hofmeister, Isomorphisms and automorphisms of graph coverings, *Discrete Math.* 98 (1991) 175-183

[17] Jan Kratochvíl, Andrzej Proskurowski, Jan Arne Telle, Covering regular graphs *Journal of Combinatorial Theory B* **71**, (1997), pp. 1–16.

[18] Jan Kratochvíl, Andrzej Proskurowski, Jan Arne Telle, Covering directed multigraphs I. Colored directed multigraphs, *Graph-Theoretical Concepts in Computer Science, Proceedings 23rd WG '97, Berlin, Lecture Notes in Computer Science 1335*, Springer Verlag, (1997), pp. 242-254.

[19] Kratochvíl, Jan, Andrzej Proskurowski, and Jan Arne Telle, Complexity of graph covering problems *Nordic Journal of Computing* **5** (1998), 173-195

[20] R.A. Leese: A fresh look at channel assignment in uniform networks, EMC97 Symposium, Zurich, pp. 127-130

[21] F.T. Leighton, Finite common coverings of graphs, *Journal of Combinatorial Theory B 33* (1982) 231-238

[22] Lovász, L. and M. D. Plummer, *Matching Theory*, Akadémiai Kiadó, Budapest, 1986.

[23] W. Massey, *Algebraic Topology: An Introduction*, Harcourt, Brace and World, 1967

[24] Yeh, Kwan-Ching, Labeling graphs with a condition at distance two, Ph.D. Thesis, University of South Carolina, 1990.

Appendix

A Proof of Theorem 3.2

Lemma A.1 *Suppose graphs G and H have the same degree refinement matrix and let $B_j(G)$, $B_j(H)$, $j = 1, 2, \ldots, k$ be the degree partitions. Then for every $i \geq 0$ and $j, 1 \leq j \leq k_i$ there exists $X_{ij} \subseteq \{1, 2, \ldots, k\}$ such that $B_{ij}(G) = \bigcup_{h \in X_{ij}} B_j(G)$ and $B_{ij}(H) = \bigcup_{h \in X_{ij}} B_j(H)$. Moreover, $n^i_G(u) = n^i_H(v)$ for every $i \geq 0$ whenever $t_G(u) = t_H(v)$.*

Proof The degree partition of a graph G can be obtained by the following procedure (k_i is the number of classes in the partition after i rounds of refining, $B_{ij}, j = 1, 2, \ldots, k_i$ are the classes of this partition. The *weight* $t(u)$ of a vertex $u \in V(G)$ is the index $t(u) = i$ such that $u \in B_i$.):

1. Define $k_0 = 1$ and $B_{01} = V(G)$.
2. Set $i = 0$
3. Repeat until $k_i = k_{i+1}$:
  3.1. For every vertex $u \in V(G)$, compute the
    neighbor vector $n^i(u) = (n_1, n_2, \ldots, n_{k_i})$ where $n_j = |N(u) \cap B_{ij}|$
  3.2. Define a new partition of $V(G)$ so that u, v belong to the same class
    if and only if $n^i(u) = n^i(v)$
  3.3. Let k_{i+1} be the number of the classes in this partition.
  3.4. Order the classes of the partition lexicographically according to
    the neighbor vectors so that $u \in B_{i+1,j}$ and $v \in B_{i+1,h}$ with $j < h$
    imply $n^i(u) >_{\text{lex}} n^i(v)$.
  3.5. Set $i = i + 1$ and continue Step 3.
4. Set $k = k_i$ and $B_j = B_{ij}$ for $j = 1, 2, \ldots, k$.

Note that the first level partitions the vertex according to the degrees of the vertices, and so $k_1 = 1$ and $B_{11} = V(G)$ for regular graphs, and the degree partition of regular graphs consists of a single block. Note also that this procedure does not only compute the degree partition, but also gives a unique ordering of the blocks of the degree partition (e.g., B_{11} contains the vertices of maximum degree and B_{1,k_1} the vertices of minimum degree). This enables to assign a matrix to the degree partition in a unique way.

Now we prove the statement by induction.

For $i = 0$, $X_{01} = \{1, 2, \ldots, k\}$. If $u \in B_i(G)$ and $v \in B_i(H)$ then $deg_G(u) = \sum_{j=1}^k r_{ij} = deg_H(v)$ and hence $n^0(u) = (deg_G(u)) = (deg_H(v)) = n^0(v)$.

For $i > 0$, the existence of X_{ij} follows from the fact that $n^{i-1}_G(u) = n^{i-1}_H(v)$ whenever $t_G(u) = t_H(v)$. Then $n^i(u) = (\sum_{h \in X_{ij}} r_{th})_{j=1}^{k_i} = n^i(v)$, if $t = t_G(u) = t_H(v)$. □

Theorem: *If connected graphs G and H have the same degree refinement matrix then every partial covering projection of G to H is also a full covering projection.*

Proof Let f be a partial covering projection of G to H. For every vertex $u \in V(G)$, $deg_G(u) \le deg_H(f(u))$ follows from local injectivity of f.

We first prove that $t_G(u) \ge t_H(f(u))$ for every $u \in V(G)$. As in the proof of the preceding theorem, run the procedure DRProc simultaneously on G and H. Let $t^i_G(u)$ denote the weight of vertex u in the partition of G after i-th run of Step 3, and similarly for H. We prove by induction on i that $t^i_G(u) \ge t^i_H(f(u))$ for every $u \in V(G)$.

For $i = 0$, $t^0_G(u) = t^0_H(f(u)) = 1$, since $k_0(G) = k_0(H) = 1$.

Suppose $i > 0$. The weights of u and $f(u)$ depend on their neighbor vectors $n^{i-1}_G(u)$ and $n^{i-1}_H(f(u))$. For every neighbor z of u in G, $f(z)$ is a neighbor of $f(u)$ in H and, by induction hypothesis, $t^{i-1}_G(z) \ge t^{i-1}_H(f(z))$. Hence in the lexicographic ordering $n^{i-1}_G(u) \le_{\text{lex}} n^{i-1}_H(f(u))$. By Step 3.4, this means that $t^i_G(u) \ge t^i_H(f(u))$. Note that here we are implicitly using Lemma A.1, as we are comparing neighbor vectors in G and H.

For $u \in B_1(G)$, $t_G(u) = 1 \ge t_H(f(u))$ and hence $t_H(f(u)) = 1$ as well. Thus vertices from the block $B_1(G)$ are mapped to vertices from $B_1(H)$. For the rest of the proof let v be an arbitrary vertex from $B_1(G)$.

Suppose there exists a vertex $u \in V(G)$ for which $t_G(u) > t_H(f(u))$. Consider a path from v to u in G. This path must contain an edge $(v', u') \in E(G)$ such that $t_G(v') = t_H(f(v'))$ and $t_G(u') > t_H(f(u'))$. Since v' and $f(v')$ have the same number of neighbors of each weight, this implies the existence of a neighbor w' of v' for which $t(w') < t(f(w'))$, a contradiction.

Now for all vertices $u \in V(G)$ we have $t_G(u) = t_H(f(u))$, and in particular $deg_G(u) = deg_H(f(u))$. Hence f is a local epimorphism and thus a full covering projection of G onto H. $\square$

B Proof of Observation 4.3

Observation: *The problems $P(a_1, a_2, \dots, a_n)$-partial cover and $P(a_1k, a_2k, \dots, a_nk)$-partial cover are polynomially equivalent for any positive integers $a_1, a_2, \dots, a_n$ and any k.*

Proof Let G be an input graph for the $P(a_1, a_2, \dots, a_n)$-partial cover problem. Construct a graph G' by replacing every edge of G by a path of length k. We claim that G partially coves $P(a_1, a_2, \dots, a_n)$ if and only if G' partially covers $P(a_1k, a_2k, \dots, a_nk)$. The 'only if' part is obvious, for the 'if' direction note that the vertices of degree at least three in G' must map onto the vertices x, y of $P(a_1k, a_2k, \dots, a_nk)$ in any partial covering projection, and the lengths of the paths between any two such vertices are divisible by k. It follows that the only vertices of degree two which may map onto x or y are at distance divisible by k from the vertices of degree at least three, and thus any partial covering projection from G' to $P(a_1k, a_2k, \dots, a_nk)$ yields a partial covering projection from G to $P(a_1, a_2, \dots, a_n)$.

Conversely, if any two vertices of degree at least three of a graph G' are connected by a path whose length is not divisible by k then G' cannot partially cover $P(a_1k, a_2k, \dots, a_nk)$. Then, construct a graph G by replacing every path connecting two vertices of degree at least three in G' by a path of k-times shorter length. Then G partially covers $P(a_1, a_2, \dots, a_n)$ if and only if G' partially covers $P(a_1k, a_2k, \ldots, a_nk)$ (note that paths consisting of vertices of degree two and ending in leaves are irrelevant for partial covers). $\square$

C Proof of Theorem 4.7

Definition C.1 *A* proper half-edge coloring *of a multigraph is a coloring of edge ends such that every edge has both ends colored by different colors and no two ends incident with a common vertex have the same color.*

More formally a proper halfedge coloring using at most k colors is a function $c : HE \to \{1, ..., k\}$, where $HE = \bigcup_{e=(u,v)\in E(G)}\{(u,e),(v,e)\}$ and the following conditions are satisfied:

- $\forall e \in E(G), e = (u,v) : c(u,e) \neq c(v,e)$
- $\forall e, f \in E(G), e \cap f = u : c(u,e) \neq c(u,f)$

Note that if $e = (u,u)$ is a loop then the element (u,e) appears twice in HE and therefore we regard HE as a multiset.

Lemma C.2 =4.11 *Let G be a connected (multi)graph and let a set C_u of at least $deg_G(u)$ colors be assigned to every vertex u of G. If*

- *G contains a cycle or*
- *there exists a vertex u s.t. $|C_u| > deg_G(u)$ or*
- *there are two leaves u, u' s.t. $C_u \neq C_u'$*

then a proper half-edge coloring satisfying $c(u,e) \in C_u$ always exists and can be found in polynomial time.

Proof W.l.o.g. we assume that G does not contain loops, since having a proper halfedge coloring of G without loops we can color every loop by a pair of remaining colors.

We prove the statement by induction on the number of vertices. If G has one vertex only the situation is trivial. Suppose $V(G) \geq 2$.

Select $u \in V(G)$ so that $G \setminus u$ stays connected. By the induction hypothesis a proper halfedge coloring c of $G \setminus u$ exists. (Each neighbor v of u satisfies $|C_v| > deg_{G\setminus u}(v)$.) Denote $d = deg_G(u)$.

We need to color the halfedges (u,e) and $(v_e,e), e = (u,v_e)$. First color each (v_e,e) by a color from C_{v_e} that is not used around the vertex v_e yet.

For each edge e adjacent to u consider the set of feasible colors of the halfedge (u,e) and ask for a distinct representative (color). By König-Hall's marriage theorem the system does not have a solution if and only if all sets of feasible colors are equal and have $d-1$ colors each. In other words all halfedges (v_e,e) are colored by the same color z.

In this situation we try to recolor halfedges adjacent to a neighbor v_e, so that (v_e,e) gets a color different from z. By the same argument as above this is impossible if and only if the color z is used on all halfedges (w, f) where w is a neighbor of v_e different from u and $f = (v_e, w)$ is an edge of G. Since G is a finite multigraph, this situation cannot repeat for all vertices.

Formally we perform a recoloring by the breadth first search:

- *Init:* Put u into the FIFO queue and mark it as the root of the search tree.
- *Step:* Remove the front vertex v from the queue:
  - *Case 1:* If $deg_G(v) < |C_v|$, then assume the path to the root: $v, v', \cdots, v^k = u$. Recolor the halfedges on the path as follows: Denote $f_i = v_i, v_{i+1}$, and on (v_i, f_i) use arbitrary feasible color different from z, while on (v_{i+1}, f_i) use color z.
  - *Case 2:* If there exists a descendant w of v such that $c(w, (v, w)) \neq z$, then recolor $(v, (v, w))$ by z. Finally recolor the path to the root as in Case 1.
  - *Case 3:* Otherwise consider all descendants of v. None of them is in the queue yet, because at most one halfedge can be colored by z around any vertex. Put all descendants into the queue and continue by the next round of BFS.

The algorithm is finite since it stops when it reaches Case 1 or 2. In Case 3 we process edges that form a tree, which cannot grow into the entire graph G because of the assumptions on the graph G. □

Theorem 4.7 *The $P(a^k, b^l)$-partial cover problem is NP-complete for every $a \not\equiv b \mod 2$.*

Proof We suppose that a and b are relatively prime by Observation 4.3. The proof is independent on the size of a and b therefore we w.l.o.g. suppose that a is odd and b is even.

We show a polynomial reduction from the $BW(k, l)$-coloring problem which is NP-complete whenever $k + l \geq 3$, $k, l > 0$. (See [19] for k even, the case of k odd is similar.) The $BW(k, l)$-coloring problem asks whether the vertices of a $(k + l)$-regular graph can be colored by black and white, such that each vertex has exactly k neighbors of the same color.

Suppose $k \geq 2$, $l \geq 2$. Let G be an input graph for the $BW(k, l)$-coloring problem. Replace each edge of G by a path of length ab, call the new graph G' and suppose it covers $P(a^k, b^l)$. The original vertices of G' have degree $(k + l)$ and hence their mirrors in G' are mapped onto x and y in $P(a^k, b^l)$. Color a vertex of G black if its mirror in G' maps to x and color it white otherwise. We show that this is a solution of the $BW(k, l)$-coloring problem for G.

Consider a partial cover of path P_{ab} into $P(a^k, b^l)$ with both ends mapped onto x, y. It means that ab is a nonnegative linear combination of a and b. The only solutions for the linear combination are $a + a + \cdots + a$ and $b + b + \cdots + b$.

The first solution has an even number of summands and therefore the endvertices of paths covered by this pattern have the same color, while in the second case the sum has an odd number of summands and the corresponding endvertices get different colors.

Any partial covering is locally injective, so among the $(k + l)$ edges adjacent to a vertex u of G' a pattern which starts with a is used on k edges, and a b pattern is used on l edges.

To conclude the polynomial reduction we show that any solution of the $BW(k, l)$-coloring problem for G can be transformed to a partial covering projection $G' \to P(a^k, b^l)$.

Observe that the subgraph of G spanned by edges connecting the set of white vertices with the set of black vertices is bipartite and l-regular. Therefore its edges can be partitioned into l 1-factors. For each 1-factor use in G' a pattern of covering P_{ab} into $P(a^k, b^l)$ that starts and ends with the same b-path and use different b-paths for different 1-factors.

Then consider the graph spanned by edges connecting only black vertices (the subgraph spanned by the white vertices is treated similarly). These edges form a k-regular graph. Assign to every vertex the set $\{1, ..., k\}$ as the set of feasible colors (representing the k distinct a-paths of $P(a^k, b^l)$) and ask for a proper halfedge coloring. By Lemma 4.11 such a coloring exists and as above we assign to each color a unique a-path. Since the number of summands in the corresponding sum is even, for any pair of distinct a-paths there exists a covering pattern of P_{ab} into $P(a^k, b^l)$ such that the end segments of P_{ab} are mapped onto the selected a-paths.

If $l = 1$, then we use a reduction from the $BW(k, 1)$-coloring problem. Replace each edge of graph G by a path of length $m = ab + (a - 1)a$. We consider only solutions of the linear combination $m = ap + bq$ satisfying $q \leq p - 1$ since no two consecutive b-paths appear consecutively in a partial covering of P_m. The only solutions are $m = a + a + \cdots + a$ $(p = a + b - 1, q = 0)$ with an even number of summands and $m = b + a + b + a + \cdots + b$ $(p = a - 1, q = a)$ with an odd number of summands. Then the proof is finished similarly.

For $k = 1$ we reduce the $BW(l, 1)$-coloring problem. As above we replace each edge by P_m where $m = ab + (b - 1)b$. For a solution of the linear combination $m = ap + bq$ we demand $p \leq q - 1$ and the only solutions are $m = b + b + \cdots + b$ and $m = a + b + a + b + \cdots + a$. In contrary to the previous two cases, here b-paths represent edges between vertices with the same color and an a-paths connect vertices of different colors. Finish the proof as above with a and b (k and l) interchanged. □

D Proof of Propositions 4.8 and 4.9:

Lemma D.1 =4.10 *Suppose a, b, c are such that there exists an integer m with the following property: All solutions to the equation*

$$x_1 + x_2 + \ldots + x_k = m(1) \tag{2}$$

such that $x_i \in \{a, b, c\}$ and no two consecutive x_i have the same value, satisfy one of the conditions

1) k is odd and $x_1 = x_k = a$

2) k is even and $x_1 = b$ and $x_k = c$ and vice versa

3) k is even and $x_1 = x_k = b$ or $x_1 = x_k = c$,

4) $x_1 = c$ and there exists no solution different from the case 2) with $x_1 = b$ and vice versa and solutions satisfying 1) and 2) exist. Then $P(a, b, c)$-partial cover is NP-complete. (Note that k in (1) is a variable.)

Proof We show a reduction from the $BW(2, 1)$-coloring problem.

Let G be a cubic graph. Replace every edge of G by a path of length m. We claim that this new graph G' partially covers $P(a, b, c)$ if and only if G allows a $BW(2, 1)$-coloring.

Suppose $f : V(G') \to V(P(a,b,c))$ is a partial covering projection. The vertices of degree 3, i.e., the vertices which belong both to G' and G, must map onto x or y, the only vertices of degree 3 in $P(a,b,c)$. Consider a path of length m which joins two of such vertices. The restriction of f to this path is a partial covering projection to $P(a,b,c)$. Let $v_1, v_2, \ldots, v_{k+1}$ be the vertices on this path (in consecutive order) which also map onto x or y (hence v_1 and v_{k+1} are the endpoints of the path). The segment between v_i and v_{i+1} must fully cover one of the paths of $P(a,b,c)$, and two consecutive segments must be mapped onto different paths. Let x_i be the length of the segment between v_{i-1} and v_i, then the x_i's provide a solution to equation (1). Since in a partial covering projection every vertex u of degree 3 has its neighborhood mapped bijectively onto the neighborhood of $f(u) \in \{x,y\}$, exactly one path leaving u starts with segment $x_1 = a$, exactly one starts with segment $x_1 = b$ and exactly one starts with segment $x_1 = c$. Given the properties 1) and 2) of the solutions of (1), and given the fact that $f(v_{k+1}) = f(v_1)$ if k is even and $f(v_{k+1}) \neq f(v_1)$ if k is odd, we see that exactly two paths leaving u end in a vertex which is mapped onto the same vertex as u, and exactly one path ends in a vertex mapped onto the other one. Thus coloring a vertex u of G black if $f(u) = x$ and white if $f(u) = y$, we get a $BW(2,1)$-coloring.

The solution of type 3) describes that a path with first segment $x_1 = b$ or c leads to a vertex of the same color.

The fourth case is not used in the partial covering schemes, since if the only solution with $x_1 = b$ is of type 2), then at each vertex one path starts with segment $x_1 = c$ and the other end of the path supplies each vertex with segment $x_k = b$.

The converse is now almost trivial, a partial covering projection from G' to $P(a,b,c)$ can be constructed from a $BW(2,1)$-coloring using the assumed existence of solutions of type 1) and 2) of equation (1). The paths which correspond to edges between black and white vertices will be covered according to the pattern given by solution of type 1), the paths which correspond to edges with equally colored endpoints form cycles, and along these cycles a partial covering projection will be constructed by repeating a solution of type 2). □

Proposition 4.8 *The $P(a,b,c)$-partial cover is NP-complete problem if c is divisible by $a+b$.*

Proof Suppose $a+b$ divides c. Then $m = c$ works for Lemma 4.10, the only solutions to equation (1) being

$c = c$ (k odd)

$c = a+b+a+b+\ldots+a+b$,

$c = b+a+b+a+\ldots+b+a$ (k even). □

Proposition 4.9 *The $P(1,2,c)$-partial cover problem is NP-complete for every c.*

Proof Again use Lemma 4.10. First discuss the case c odd. For $c = 3h$, $m = c$ allows the following solutions to equation (1)

$c = c$ (k odd)

$c = 1+2+1+\ldots+2 = 2+1+2+\ldots+1$ (k even).

For $c = 3h+1$, $m = c+1$ allows the following solutions to equation (1)

$c+1 = 2+1+2+\ldots+1+2$ (k odd)

$c+1=1+c=c+1$ (k even).

For $c=3h+2$, $m=c+2$ allows the following solutions to equation (1)

$c+2=1+2+1+\ldots+2+1=1+c+1$ (k odd)

$c+2=2+c=c+2$ (k even).

For even c, $m=c+2$ allows the following solutions to equation (1)

$c+2=1+c+1$ (k odd).

$c+2=2+c=c+2$ (k even).

Note that the following solutions do not affect the application of Lemma 4.10. For $c=3h+2: c+2=1+2+1+\ldots+2+1$ (k odd) and $c=3h+1: c+2=1+2+1+\ldots+2$. $\square$