# On the inclusion-exclusion principle.

*Martin Loebl
Department of Applied Mathematics
Charles University
Malostranske n. 25
118 00 Praha 1
CZECH REPUBLIC
loebl@kam.ms.mff.cuni.cz

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**Abstract**

We will be walking for some time where the Inclusion-Exclusion Principle is leading us.

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# 1 Preliminaries.

A graph is a pair $G = (V, E)$ where $V$ is a set of *vertices* and $E$ is a set of unordered pairs of elements of $V$, called *edges*. In this paper we shall consider only graphs with finite number of vertices. If $e = xy$ is an edge then the vertices $x, y$ are called *endvertices* of $e$. The edges will often have *weight*, i.e. we are often given a function $w$ from $E$ to real numbers. If $A \subset E$ then we let $w(A) = \sum w(e); e \in A$.

A graph $G' = (V', E')$ is called a *subgraph* of a graph $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. A *perfect matching* of a graph is a set of disjoint edges, whose union equals the set of the vertices. A subgraph $G' = (V, E')$ is called *even* if each degree of $G'$ is even.

Let $\{v_1, e_1, v_2, e_2, ..., v_i, e_i, v_{i+1}, ..., e_n, v_{n+1}\}$ be a sequence such that each $v_j$ is a vertex of a graph $G$, each $e_j$ is an edge of $G$ and $e_j = v_j v_{j+1}$, and $v_i \neq v_j$ for $i < j$ except if $i = 1$ and $j = n + 1$. If also $v_1 \neq v_{n+1}$ then $P$ is called *a path* of $G$. If $v_1 = v_{n+1}$ then $P$ is called *a cycle* of $G$. In both cases the *length* of $P$ equals $n$. When no confusion arises we shall also denote paths by simply listing their edges, namely $P = (e_1, e_2, \ldots, e_n)$.

A graph $G = (V, E)$ is *connected* if it has a path between any pair of vertices, and it is 2-*connected* if the graph $G_v = (V - \{v\}, \{e \in E; v \notin e\})$ is connected for each vertex $v$ of $G$. Each maximal 2-connected subgraph of $G$ is called a 2-*connected component* of $G$.

An*edge-cut* of a graph $G = (V, E)$ is a partition of its vertices into two disjoint subsets $V_1, V_2 \subset V$, and the implied set of edges between the two parts: $C(V_1, V_2) = \{\{u, v\} \in E : u \in V_1, v \in V_2\}$

The *generating function of edge-cuts* is a polynomial $\mathcal{C}(G, x)$ which equals the sum of $x^{w(C)}$ over all cuts $C$ of $G$.

An *even subgraph* of a graph $G = (V, E)$ is a set of edges $U \subset E$ such that each vertex of $V$ is incident with an even number of edges from $U$.

The *generating function of eulerian subgraphs* is a polynomial $\mathcal{E}(G, x)$ which equals the sum of $x^{w(U)}$ over all eulerian subgraphs $U$ of $G$.

Let $A \Delta B$ denote the symmetric difference of the sets $A$ and $B$ and let $a \stackrel{2}{=} b$ denote $a = b$ modulo 2.

An *orientation* of a graph $G = (V, E)$ is a *digraph* $D = (V, A)$ obtained from $G$ by fixing an orientation of each edge of $G$, i.e., by ordering the elements of each edge of $G$. The elements of $A$ are called *arcs*.

An *embedding* of a graph on a surface is defined in a natural way: the vertices are embedded as points, and each edge is embedded as a continuous non-self-intersecting curve connecting the embeddings of its endvertices. The interiors of the embeddings of the edges are pairwise disjoint and the interiors of the curves embedding edges do not contain points embedding vertices.

A graph is called *planar* if it may be embedded on the plane. A *plane graph* is a planar graph together with its planar embedding. The embedding of a plane graph partitions the plane into connected regions called *faces*. The (unique) unbounded face is called *outer face* and the bounded faces are called *inner faces*.

Let $G$ be a plane graph. A subgraph of $G$ consisting of the vertices and the edges embedded on the boundary of a face will also be called *a face*. If a plane graph is 2-connected then each face is a cycle.

A *matroid* is a pair $(X, M)$ where $X$ is a finite set and $M \subset 2^X$ satisfies some axioms. These notes are concerned with *representable* matroids only. There, set $X$ equals the columns of a matrix $A$ over a field $F$, and $M$ consists of the linearly independent subsets of $X$. We say that such a matroid is represented over $F$ by $A$, and matrix $A$ is its representation over $F$. The matroids represented over $GF[2]$ are called *binary matroids*. If $S \subset X$ then the rank of $S$, denoted by $r(S)$, is defined to be equal to the rank of the set of columns of $A$ corresponding to $S$ over the field $F$.

If $A$ is a matrix representing matroid $M$ then the kernel of $A$ (i.e. $\{x; Ax = 0\}$) is a subspace over $F$ called cyclespace of $M$. The orthogonal complement of the cyclespace is called *cutspace* of $M$. It is generated by the rows of matrix $A$. Note that any subspace may be written as kernel of a matrix. Hence also the cutspace is a kernel of a matrix $D$, and the matroid represented by $D$ is called *dual matroid of* $M$ and denoted by $M^*$.

For binary matroids, the elements of the cyclespace and cutspace are 0,1 vectors and so we may view them as characteristic vectors of subsets of the underlying set $X$. If $G$ is a graph then $calN_G$ will denote the graphic matroid of $G$, i.e. the matroid $(E(G), N)$ where $N$ consists of all acyclic subsets of edges. Note that the graphic matroid is representable over any field $F$:

Let $O_G$ be an oriented incidence matrix of $G$, i.e. an $|V| \times |E|$ matrix obtained from the incidence matrix of $G$ by replacing exactly one '1' of each column by '$-1$'. The columns of $O_G$ represent $\mathcal{N}_G$ over an arbitrary field $F$.

For the graphic matrids, the cycle space consists of the incidence vectors of even subgraphs and the cutspace consists of the incidence vectors of edge-cuts.

# 2 Introduction.

Let us start with the introduction of a paper of Hassler Whitney, which appeared in Annals of Mathematics in August 1932:

"Suppose we have a finite set of objects (for instance books on a table), each of which either has or has not a certain given property A (say of being red). Let n, or $n(1)$, be the total number of objects, $n(A)$ the number with the property $A$, and $n(\bar{A})$ the number without the property $A$. Then obviously $n(\bar{A}) = n - n(A)$. Similarly, if $n(A, B)$ denote the number with both properties A and B, nad $n(\bar{A}, \bar{B})$ the number with neither property, then $n(\bar{A}, \bar{B}) = n - n(A) - n(B) + n(AB)$, which is easily seen to be true.

The extension of these formulas to the general case where any number of properties are considered is quite simple, and is well known to logicians. It should be better known to mathematicians also; we give in this paper several applications which show its usefulness."

Indeed, we all know it, under the name 'inclusion-exclusion principle':

if $A_1, ..., A_n$ are finite sets, and if we let $\cap(A_i; i \in J) = A_J$ then

$$|\cup (A_i; i = 1, ..., n) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{J \in \binom{n}{k}} |A_J|.$$

Whitney mentions in his paper three main applications: on the number of primes smaller than x, on 'problem satnarky' and on the number of ways of coloring a graph.

These notes are concerned with the last application. Let us again follow his article for a while:

"Suppose we have a fixed number z of colors at our disposal. Any way of assigning one of these colors to each vertex of the graph in such a way that any two vertices which are joined by an arc are of different colors, will be called admissible coloring, using z or fewer colors. We wish to find the number $M(z)$ of admissible colorings, using z or fewer colors. ... We shall deduce a formula for $M(z)$ due to Birkhoff.

If there are V vertices in the grapg G, then there are $n = z^V$ possible colorings, formed by giving each vertex in succession any one of z colors. Let R be this set of colorings. Let $A_{ab}$ denote those colorings with the property that a and b are of the same color, etc. Then the set of admissible colorings is

$$M(z) = n - [n(A_{ab}) + n(A_{bd}) + ... + n(A_{cf})]$$
$$+[n(A_{ab}A_{bd}) + ...] - ...$$
$$+(-1)^E n(A_{ab}A_{bd}...A_{cf}).$$

With each property $A_{ab}$ is associated an arc ab of G. In the logical expansion, there is a term corresponding to every possible combination of the properties $A_{pq}$; with this combination we associate the corresponding arcs, forming a subgraph H of G. In particular, the first term corresponds to the subgraph containing no arcs, and the last term corresponds to the whole of G. We let H contain all the vertices of G.

Let us evaluate a typical term $n(A_{ab}A_{ad}...A_{ce})$. This is the number of ways of coloring G in z or fewer colors in such a way that a and b are of the same color, a and d are of the same color, ..., c and e are of the same color. In the corresponding subgraph H, any two vertices that are joined by an arc must be of the same color, and thus all the vertices in a single connected piece in H are of the same color. If there are p connected pieces in H, the value of this term is therefore $z^p$. If there are s arcs in H, the sign of the term is $(-1)^s$. Thus

$$(-1)^s n(A_{ab}A_{bd}...A_{cf}) = (-1)^s z^p.$$

If there are $(p, s)$ (this is Birkhoff's symbol) subgraphs of s arcs in p connected pieces, the corresponding terms contribute to $M(z)$ an amount $(-1)^s(p, s)z^p$. Therefore, summing over all values of p and s, we find the polynomial in z:

$$M(z) = \sum_{p,s} (-1)^s (p, s) z^p."$$

This is our well known chromatic polynomial.

Let $G = (V, E)$ be a connected graph. For $A \subset E$ let $r(A) = |V| - c(A)$, where $c(A)$ denotes the number of connected components (pieces) of $(V, A)$. Then we can write

$$M(z) = z_{c(E)}(-1)^{r(E)} \sum_{A \subset E} (-z)^{r(E)-r(A)}(-1)^{|A|-r(A)}.$$

This lead Whitney to define the two variable polynomial called Whitney rank generating function $R(G, u, v)$ by

$$R(G, u, v) = \sum_{A \subset E} u^{r(E)-r(A)} v^{|A|-r(A)}.$$

Maybe, in the end of the introduction I should apologise to the readers for possible mistakes, misprints and bad formulations. They were partly caused by time pressure. In particular the references are not complete. Time is also the reason I couldnot include an elegant derivation of critical temperature for the square lattice Ising model.

# 3 The Tutte Polynomial.

We start considering the *Tutte polynomial*; it has been defined by Tutte ([22]) and it may be expressed as a minor modification of the Whitney rank generating function ([25]).

$$T(G, x, y) = \sum_{A \subset E} (x - 1)^{r(E)-r(A)}(y - 1)^{|A|-r(A)}.$$

$T(G, x, y)$ is called the *Tutte polynomial* of graph $G$.

Note that for any connected graph G, $T(G, 1, 1)$ counts the number of spanning trees of G: indeed, the only terms that count are those for which $r(A) = r(E) = |A|$. These are exactly the spanning trees of G. Hence, $T(G, 1, 1)$ may be evaluated using a determinant, as you all know from first year Discrete math lecture. How to get the chromatic polynomial is explained above.

More generally, the Tutte polynomial of a matroid is defined as follows.

**Definition 3.1** *Let $M$ be a matroid on set $E$. For $A \subset E$ let $r(A)$ denote the rank of $A$ in $M$. Then let*

$$T(M, x, y) = \sum_{A \subset E} (x - 1)^{r(E)-r(A)}(y - 1)^{|A|-r(A)}.$$

*$T(M, x, y)$ is called the Tutte polynomial of matroid $M$.*

For example if $G$ is a graph and $\mathcal{N}_G$ the graphic matroid of $G$ then $T(G, x, y) = T(\mathcal{N}_G, x, y)$.

If $M$ is a matroid and $M^*$ its dual then $r^*(E) - r^*(A) = |A| - r(A)$ where $r^*$ is the rank function of $M^*$ and we immediately get that $T(M, x, y) = T(M^*, y, x)$. This relation is called **duality of the Tutte polynomial**.

## 3.1 Weight enumerator of a linear code.

Let $V = \mathbf{F}^n$ be a vector space over a field $\mathbf{F}$. Each subspace $C$ of $V$ of dimension $k$ is called a *linear code of length* $n$ *and dimension* $k$. The elements of a linear code are called *codewords*. The *weight* of a codeword is the number of its nonzero entries. The *weight distribution* of $C$ is the sequence $A_0, A_1, \ldots, A_n$ where $A_i$ equals the number of codewords of $C$ of weight $i$, $0 \le i \le n$.

The *dual code* of $C$ is denoted by $C^*$ and consists of all those $n$-tuples $(d_1, \ldots, d_n)$ of $\mathbf{F}^n$ satisfying

$$c_1 d_1 + \ldots + c_n d_n = 0$$

for all codewords $(c_1, \ldots, c_n) \in C$. Hence, $C^*$ is a code of length $n$ and dimension $n - k$.

The *weight enumerator* of $C$ is the polynomial

$$A_C(t) = \sum_{i=0}^{n} A_i t^i.$$

The following theorem was proved by MacWilliams ([16]) and it states a fundamental relation between the weight enumerators of $C$ and of its dual $C^*$.

**Theorem 3.2** *Let $C$ be a linear code of length $n$ and dimension $k$ over $GF[q]$ and $1 + (q-1)t \neq 0$. Then*

$$A_{C^*}(t) = \frac{[1 + (q-1)t]^n}{q^k} A_C\left(\frac{(1-t)}{1 + (q-1)t}\right).$$

If the linear code of length $n$ is given as the row space of a $k \times n$ matrix $A$ over a field $\mathbf{F}$, i.e. $C = \{A^T x; x \in \mathbf{F}^k\}$, then we will denote it as $C(\mathbf{F}, A)$. Moreover, if a matroid $M$ is represented by the columns of $A$, then we let $M = M(\mathbf{F}, C)$ and $C = C(\mathbf{F}, M)$. In this case we have $C^* = \{x \in \mathbf{F}^n; Ax = 0\}$. We have $M(\mathbf{F}, C^*) = M(\mathbf{F}, C)^*$.

The following theorem was proved by Greene ([5]).

**Theorem 3.3** *Let $C$ be a linear code of length $n$ and dimension $k$ over $GF[q]$ and $0 \neq t \neq 1$. Then*

$$A_C(t) = (1-t)^k t^{n-k} T\left(M(GF[q], C), \frac{1 + (q-1)t}{(1-t)}, \frac{1}{t}\right).$$

Note that Theorem 3.2 follows immediately from Theorem 3.3 and the duality of the Tutte polynomial. As an immediate corollary we get

**Corollary 3.4** *Let $M$ be a matroid represented over $GF[q]$. If $(x-1)(y-1) = q$ and $0 \neq y \neq 1$ then*

$$T(M, x, y) = y^n (y-1)^{-k} A_{C(GF[q], M)}(y^{-1}).$$

# 4 The Dualities.

We explained above the duality of the Tutte polynomial, and a seminal theorem of MacWilliams which follows from it.

Consider now the following example. Let $G = (V, E)$ be a graph and let $\mathcal{N}_G$ be the graphic matroid of $G$.

Let $O_G$ be an oriented incidence matrix of $G$, i.e. an $|V| \times |E|$ matrix obtained from the incidence matrix of $G$ by replacing exactly one '1' of each column by '$-1$'. The columns of $O_G$ represent $\mathcal{N}_G$ over an arbitrary field $\mathbf{F}$.

The set of the characteristic vectors of edge-cuts of a graph $G$ (including the empty cut) equals $C(GF[2], \mathcal{N}_G)$ and the set of the characteristic vectors of eulerian subgraphs of $G$ equals $C(GF[2], \mathcal{N}_G)^*$. Using this terminology, the theorem of MacWilliams expresses the generating function of edge-cuts of **any graph G** using the generating function of even subgraphs of **the same graph G**.

This is exciting: **the geometric duality** of embedded graphs which we know about from studying plane graphs works in a different way! Before saying more on this let us give a direct proof to that special case of MacWilliams' theorem : it was in fact proved by van der Waerden in 1941.

The basic idea of van der Waerden was to substitute hyperbolic functions for the exponential terms:

$$x^y = \cosh(x, y) + \sinh(x, y) = \cosh(x, y)(1 + \tanh(x, y))$$

where

$$\cosh(x, y) = \frac{x^y + x^{-y}}{2}, \qquad \sinh(x, y) = \frac{x^y - x^{-y}}{2}, \qquad \tanh(x, y) = \frac{\sinh(x, y)}{\cosh(x, y)}$$

Thus we obtain the generating function in the following form: let $w = (w_{ij}; \{i, j\} \in E(G))$ and $W = \sum_{\{i,j\} \in E(G)} w_{ij}$. Then

$$\mathcal{C}(G, x) = \frac{1}{2} x^{\frac{W}{2}} \sum_{\sigma} \prod_{\{i,j\} \in E} x^{-\frac{1}{2} w_{ij} \sigma_i \sigma_j} =$$

$$= \frac{1}{2} x^{\frac{W}{2}} \sum_{\sigma} \prod_{\{i,j\} \in E} \cosh(x, -\frac{1}{2} w_{ij} \sigma_i \sigma_j)(1 + \tanh(x, -\frac{1}{2} w_{ij} \sigma_i \sigma_j) =$$

$$= \frac{1}{2} x^{\frac{W}{2}} \prod_{\{i,j\} \in E} \cosh(x, -\frac{1}{2} w_{ij}) \sum_{\sigma} \sum_{U \subset E} \prod_{\{i,j\} \in U} \sigma_i \sigma_j \tanh(x, -\frac{1}{2} w_{ij}) =$$

$$= \frac{1}{2} x^{\frac{W}{2}} \prod_{\{i,j\} \in E} \cosh(x, -\frac{1}{2} w_{ij}) \sum_{U \subset E} ( \prod_{\{i,j\} \in U} \tanh(x, -\frac{1}{2} w_{ij}) \sum_{\sigma} \prod_{i \in V} \sigma_i^{d_U(i)})$$

where $d_U(i)$ means the number of edges in $U$ incident with the vertex $i$. If we consider the sum over all assigments $\sigma$ for a given $U \subset E$, we can see that whenever there exists a vertex $i$ incident with an odd number of edges from $U$, the resulting sum will be zero. This is because the terms arising from assigments where $\sigma_i = +1$ will exactly cancel out the corresponding terms with $\sigma_i = -1$. On the other hand, if all vertices have even degrees in $U$, the sign of all contributing terms will be positive. So if $n$ is the total number of vertices:

$$\sum_{\sigma\in\{\pm1\}^n}\prod_{i\in V}\sigma_i^{d_U(i)} = 2^n$$

whenever $U$ is eulerian, and zero otherwise. Finally, we get

$$\mathcal{C}(G,x) = 2^{n-1}x^{\frac{W}{2}}\prod_{\{i,j\}\in E}\cosh(x,-\frac{1}{2}w_{ij})\sum_{U eulerian}\prod_{\{i,j\}\in U}\tanh(x,-\frac{1}{2}w_{ij}) =$$

$$= 2^{n-1}x^{\frac{W}{2}}\prod_{\{i,j\}\in E}\cosh(x,-\frac{1}{2}w_{ij})\mathcal{E}(G',\tanh(x,-\frac{1}{2}w_{ij})).$$

So the generating function of cuts can be expressed as the generating function of eulerian subgraphs of the same graph, with $x^{w_{ij}}$ replaced by $\tanh(x,-\frac{1}{2}w_{ij})$.

If $G$ is a plane graph then let us denote by $G^*$ its geometric dual. In the geometric duality there is a natural bijection between the edge-cuts of $G$ and the even subgraphs of $G^*$. Hence the generating function of even subgraphs of $G^*$ equals the generating function of edge-cuts of $G$. For plane graphs we have two dualities to play with, the duality of the Tutte polynomial and the geometric duality. The game becomes even more interesting when a plane graph is self-dual in the geometric sence. But are there such graphs different from $K_4$? Well, square grids are almost self-dual, and the interplay of the two dualities for square grids will be looked at in the next section.

# 5 The Max Cut Problem And The Ising Problem.

The *MAX-CUT problem* is a combinatorial optimization problem which is easy to define, but surprisingly hard to solve:

*Given a graph, divide its vertices into two parts so that the number of edges between them is as large as possible. More generally, the edges may have arbitrary weights, and we need to maximize the sum of weights over all the edges between the two sets of vertices.*

This combinatorial problem has a history on its own, but what makes it so widely studied is the enormous number of applications it finds in different fields. One of these relevant application comes from the study of the *Ising model*: a theoretical physics model of the nearest-neighbor interactions in a crystal structure.

## 5.1 The Ising model

Let us consider the *Ising model* of a physical system: the vertices of a graph represent particles and the edges describe interactions between pairs of particles. The most common example is a planar square lattice where each particle interacts only with its neighbors. Often, one adds edges connecting the first and last vertex in each row and column, which represent *periodic boundary conditions* in the model. This makes the graph a *toroidal square lattice.*

Now, we assign a factor $J_{ij}$ to each edge $\{i,j\}$; this factor describes the nature of the interaction between particles $i$ and $j$. A physical state of the system is an assignment of $\sigma_i \in \{+1,-1\}$ to each vertex $i$. This describes the two possible spin orientations the particle can take. The *Hamiltonian* (or *energy function*) of the system is then defined as

$$H(\sigma) = -\sum_{\{i,j\}\in E} J_{ij}\sigma_i\sigma_j$$

One of the key questions we may ask about a specific system is:

"What is the lowest possible energy (the *ground state*) of the system?"

Before we seek an answer to this question, we should realize that the physical states (spin assignments) correspond exactly to the cuts of the underlying graph. Let us define:

$$V_1 = \{i \in V; \sigma_i = +1\}$$
$$V_2 = \{i \in V; \sigma_i = -1\}$$

Then this partition of vertices encodes uniquely the assignment of spins to particles. The edges contained in the cut $C(V_1, V_2)$ are those connecting a pair of particles with different spins, and those outside the cut connect pairs with equal spins. This allows us to rewrite the Hamiltonian in the following way:

$$H(\sigma) = \sum_{\{i,j\}\in C} J_{ij} - \sum_{\{i,j\}\in E\setminus C} J_{ij} = 2w(C) - W,$$

where $w(C) = \sum_{\{i,j\}\in C} J_{ij}$ denotes the weight of a cut, and $W = \sum_{\{i,j\}\in E} J_{ij}$ is the sum of all edge weights in the graph.

Clearly, if we find the value of MAX-CUT, we have found the maximum energy of the physical system. Similarly, MIN-CUT (the cut with minimum possible weight) corresponds to the minimum energy of the system. Note that if we allow negative edge weights, we can transform MAX-CUT into MIN-CUT (and vice versa) simply by reversing the value of each edge weight. So the two problems are equivalent (unlike the MIN-CUT problem with positive edge weights arising in the study of network flows, which is significantly easier).

However, the analogy between cuts and physical states goes even deeper than that.

If we assign weight $w_e = J_{ij}$ to each edge $e = \{i, j\}$, the generating function becomes

$$\mathcal{C}(G, x) = \sum_{cutC} x^{w(C)} = \sum_k c_k x^k$$

where $c_k$ denotes the number of cuts of weight $k$. But this is also equal to the number of states with energy $2k - W$. Thus, if we knew the complete generating function, we would also know the distribution of physical states over all possible energy levels. In the language of physics, this information is encapsulated in the *partition function*:

$$Z(\beta) = \sum_\sigma e^{-\beta H(\sigma)}$$

Substituting the equalities above and counting over cuts instead of $\sigma$ assignments yields:

$$Z(\beta) = 2 \sum_{cutC} e^{-\beta(2w(C)-W)} = 2e^{\beta W} \sum_{cutC} e^{-2\beta w(C)} = 2e^{\beta W}\mathcal{C}(G, e^{-2\beta})$$

So we can obtain the value of the partition function simply by substituting $e^{-2\beta}$ into the generating function of cuts. The other way round, multiplying the partition function by $\frac{1}{2}e^{-\beta W}$ and expressing it as a polynomial in $e^{-2\beta}$ yields the generating function of cuts. The difference between the two functions is merely formal; both of them encode the state/energy distribution in a similar way.

Finding the generating function of cuts is generally a harder problem than MAX-CUT, because knowing all the coefficients of the polynomial, we can simply find the highest power $x^{k_{max}}$ with a non-zero coefficient; $k_{max}$ is then the maximum cut value. Similarly, the first non-zero term of the partition function corresponds to the ground state energy, but there is much more information contained in it.

## 5.2 The Ising Problem.

The lattices of interest for physical application are usually the infinite limits of finite lattice graphs. The Ising problem may be formulated as finding properties of the limit when n goes to infinity of $1/nlogZ(\beta)$, where n is the number of vertices of a finite lattice graph. Which properties do we have in mind? For instance in which $\beta$ is the thermodynamic limit not continuous, or its derivative is non-continuous etc. This suggests a critical behaviour. There is a nice illustration for the square lattice, where the interplay of the two dualities is used, but unfortunately I do not have time now to write it down.

## 5.3 How To Solve The Ising Problem.

Well, that is what a lot of people would like to know. We said that a solution is the 'knowledge' of the thermodynamic limit. The most straightforward approach towards this goal is the study of the partition function of finite pieces of the (countably) infinite lattice. Let us describe a discrete approach designed in the beginnig of sixties by Kasteleyn and by Fisher, who reduce the study of the Ising partition function to the study of the generating function of perfect matchings of slightly modified graph. The motivation of Kasteleyn and Fisher and many others in fifties and beginning of sixties was to find other, discrete ways to derive the results of Onsager on the Ising partition function of planar square lattices.

For every finite graph we saw that its Ising partition function is the same thing as the generating function of cuts. This in turn may be obtained from the generating function of the even subgraphs of the same graph using the duality of the Tutte polynomial, as we saw in the theorem of Mac Williams and in the derivation of van der Waerden. Next we show that the generating function of even subgraphs of a graph may be transformed into a generating function of perfect matchings of a slightly modified graph. This may be done in many ways, the transformation given here is based on Fisher's construction described in [2]. It is local in the sense that it only modifies each vertex in a way dependent on its degree. It can also be seen that it preserves the genus of the graph.

A subset $A$ of edges of a graph $G$ is called *perfect matching* if each vertex of $G$ belongs to exactly one element of $A$.

The *generating function of the perfect matchings* of $G$ is the polynomial $\mathcal{P}(G,x)$ which equals the sum of $x(P)$ over all perfect matchings $P$ of $G$.

**Definition 5.1** *Let $G=(V,E)$ be a graph embedded in an orientable surface of genus $g$, and $v \in V$ a vertex. Let $e_1,e_2,...,e_d \in E$ denote the edges incident with $v$, ordered clockwise as they spread out from $v$ in the embedding. Then the even splitting of $v$ is a graph $G'=(V',E')$ where*

- $V' = V \setminus \{v\} \cup \{v_1,...,v_d,v'_1,...,v'_d\}$
- $E' = E \setminus \{e_1,e_2,...,e_d\} \cup \{e'_1,e'_2,...,e'_d\} \cup E^A$
- $E^A = \{\{v_i,v'_i\}; i=1,...,d\} \cup \{\{v_i,v'_{i-1}\}; i=2,...,d\} \cup \{\{v'_i,v'_{i+1}\}; i=1,...,d-1\}$

*The edges $e'_i \in E'$ (*image edges*) are obtained from $e_i \in E$ by replacing the vertex $v$ by $v_i$. The edges $E^A$ will be called* auxiliary.

**Lemma 5.2** *The graph obtained by even splitting can be again embedded in the same surface.*

**Proof.** The transformation replaces a vertex $v \in V$ by a cluster of $2d$ vertices and $3d-2$ edges. The cluster itself is a planar graph which can be embedded in a small neighborhood of the original location of the vertex $v$. The images of the edges incident with $v$ can be embedded in the same way as they were in the original graph. □

**Definition 5.3** *Let $G=(V,E)$ be a graph and $G_s=(V_s,E_s)$ the graph obtained by successive even splitting of all vertices in $V$. If there are weights $w_e$ assigned to edges $e \in E$, we assign the same weights to their images in $E_s$: $w_{e'}=w_e$. The auxiliary edges $f \in E_s$ get assigned $w_f=0$.*

**Theorem 5.4** *If $G$ is a graph of genus $g$, $G_s$ has genus $g$ as well. With the assigment of weights described above, the generating function of perfect matchings of $G_s$ is equal to the generating function of eulerian subgraphs of $G$:*

$$\mathcal{P}(G_s, x) = \mathcal{E}(G, x).$$

**Proof.** From the definition of even splitting and Lemma 5.2, it follows that the resulting graph $G_s$ can be embedded again in the same surface.

If $M$ is a perfect matching in $G_s$, it must cover each of its vertices exactly once. Because the cluster replacing every vertex has an even number of vertices, and any of the auxiliary edges which is in $M$ covers a pair of vertices of the cluster, there remain an even number of vertices to be covered by the image edges incident with the cluster. Therefore, every cluster coincides with an even number of image edges which are in $M$; in other words, these edges form the image of an eulerian subgraph of $G$.

Vice versa, the image of any eulerian subgraph of $G$ can be extended (uniquely) by adding some of the auxiliary edges in $G_s$ to make a perfect matching in $G_s$. Thus, there is a one-to-one correspondence between the perfect matchings of $G_s$ and the eulerian subgraphs of $G$. As all the auxiliary edges have weights equal to 0, the corresponding terms contributing to either of the generating functions are equal. Consequently, the two generating functions are equal. □

Hence it remains to calculate the generating function of perfect matchings, and it is dealt with in the next subsection.

## 5.4 A Theory of Pfaffian Orientations.

The theory of Pfaffian orientations of graphs has been introduced by Kasteleyn [13, 12, 10] in early sixties to solve some enumeration problems arising from statistical physics [9, 19]. He proved fundamental results in the planar case and extended his approach to toroidal grids [10, 12, 13]. His method has been recently extended to general graphs by Galluccio and Loebl, and this is what this subsection will be about.

From now on we introduce slightly more general definition of generating functions. We associate with each edge $e$ of $G$ a variable $x_e$ and we let $x = (x_e : e \in E)$. For each $M \subset E$, let $x(M)$ denote the product of the variables of the edges of $M$.

The *generating function of the perfect matchings* of $G$ is the polynomial $\mathcal{P}(G, x)$ which equals the sum of $x(P)$ over all perfect matchings $P$ of $G$.

The other generating functions we are dealing with are modified accordingly. Do not worry, a confusion may not occur.

Let $M, N$ be two perfect matchings of a graph $G$. Then $M \Delta N$ consists of vertex disjoint cycles of even length. These cycles are called *alternating cycles* of $M$ and $N$.

Let $C$ be a cycle of $G$ of an even length and let $D$ be an orientation of $G$. $C$ is said to be *clockwise even* in $D$ if it has an even number of edges directed in $D$ in agreement with a chosen direction of traversal. Otherwise $C$ is called *clockwise odd.*

**Definition 5.5** *Let $G$ be a graph and let $D$ be an orientation of $G$. Let $M$ be a perfect matching of $G$. For each perfect matching $P$ of $G$ let $sgn(D, M \Delta P) = (-1)^n$ where*

*$n$ is the number of clockwise even alternating cycles of $M$ and $P$, and let $\mathcal{P}(D,M)$ be the sum of $sgn(D, M\Delta P)x(P)$ over all perfect matchings $P$ of $G$.*

**Definition 5.6** *Let $G=(V,E)$ be a graph with $2n$ vertices and $D$ an orientation of $G$. Denote by $A(D)$ the skew-symmetric matrix with the rows and the columns indexed by $V$, where $a_{vw}=x_{vw}$ in case $(v,w)$ is an arc of $D$, $a_{vw}=-x_{vw}$ in case $(w,v)$ is an arc of $D$, and $a_{vw}=0$ otherwise.*

*The Pfaffian of the skew-symmetric matrix $A(D)$ is defined as*

$$Pf(A(D)) = \sum_P s^*(P) a_{i_1 j_1} \cdots a_{i_n j_n}$$

*where $P=\{\{i_1 j_1\},\cdots,\{i_n j_n\}\}$ is a partition of the set $\{1,\ldots,2n\}$ into pairs, $i_k < j_k$ for $k=1,\ldots,n$, and $s^*(P)$ equals the sign of the permutation $i_1 j_1 \ldots i_n j_n$ of $12\ldots(2n)$.*

*Each nonzero term of the expansion of the Pfaffian of $A(D)$ equals $x(P)$ or $-x(P)$ where $P$ is a perfect matching of $G$. If $s(D,P)$ denote the sign of the term $x(P)$, we have that*

$$Pf(A(D)) = \sum_P s(D,P)x(P).$$

The following theorem was proved by Kasteleyn [10].

**Theorem 5.7** *Let $G$ be a graph and $D$ an orientation of $G$. Let $P, M$ be two perfect matchings of $G$. Then*

$$s(D,P) = s(D,M)sgn(D, M\Delta P).$$

*Hence,*

$$Pf(A(D)) = \sum_P s(D,P)x(P) = s(D,M)\sum_P sgn(D,M\Delta P)x(P) = s(D,M)\mathcal{P}(D,M).$$

The relevance of Pfaffians in our context lies in the fact that, despite their similarity with the permanent, the Pfaffian is a determinant-type function. They are polynomial time computable by a variant of Gaussian elimination. In fact, see [1],[13] for a proof of the following classic theorem.

**Theorem 5.8** *Let $G$ be a graph and let $D$ be an orientation of $G$. Then*

$$Pf^2(A(D)) = det(A(D)).$$

In [10] Kasteleyn introduced the following notion:

**Definition 5.9** *A graph $G$ is called Pfaffian if it has a Pfaffian orientation, i.e., an orientation such that all alternating cycles with respect to an arbitrary fixed perfect matching $M$ of $G$ are clockwise odd.*

Hence if a graph $G$ has a Pfaffian orientation $D$ then the signs $s(D,P)$ are equal for all perfect matchings $P$ of $G$ and $\mathcal{P}(G,x)^2 = Pf^2(A(D)) = det(A(D))$.

Kasteleyn [10] observed that the planar graphs have a Pfaffian orientation; more specifically, he proved that

**Theorem 5.10** *Every plane graph has a Pfaffian orientation such that all inner faces are clockwise odd.*

**Proof.** Let $G$ be a plane graph, and let $M$ be its perfect matching. Each alternating cycle of $M$ belongs to a 2-connected component of $G$.

Observe that $G$ has an orientation so that each inner face of each 2-connected component of $G$ is clockwise odd. Each such face 'encircles' no vertex of the corresponding 2-connected component. Let $W$ be a 2-connected component of $G$. Observe that the orientation we constructed has the property that a cycle $C$ of $W$ is clockwise odd if and only if $C$ encircles an even number of vertices of $W$. Let $C$ be an alternating cycle of $M$ and let $W$ be a 2-connected component of $G$ which contains $C$. Then $C$ encircles an even number of vertices of $W$ and hence it is clockwise odd. □

A lot of attention is given to the problem of characterising graphs which admit a Pfaffian orientation. We will speak more about this in the part devoted to permanents, but let us just say here that the problem goes back to Polya and to 1913, and that a characterisation inducing a polynomial-time algorithm for bipartite graphs has been obtained by McCuaig, Robertson, Seymour and Thomas , and for graphs embeddable on arbitrary orientable surface by Galluccio, Loebl.

A general feeling is that there are few graphs which admit a Pfaffian orientation, perhaps they all may be obtained from planar graphs by some local operations.

**Embeddings and Pfaffian orientations**

The *genus* $g$ of a graph $G$ is that of the orientable surface $\mathcal{S} \subset \mathbb{R}^3$ of minimal genus on which $G$ may be embedded. Any orientable surface of genus $g$ has a *polygonal representation* obtained by cutting the $g$ handles of its space model. In what follows we base our working definition of a surface on this concept.

**Definition 5.11** *A surface $S_g$ of genus $g$ consists of a base $B_0$ and $2g$ bridges $B^i_j$, $i = 1, ..., g$ and $j = 1, 2$, where*

*i) $B_0$ is a convex $4g$-gon with vertices $a_1, ..., a_{4g}$ numbered clockwise;*

*ii) $B^i_1$, $i = 1, \dots, g$, is a 4-gon with vertices $x^i_1, x^i_2, x^i_3, x^i_4$ numbered clockwise. It is glued with $B_0$ so that the edge $[x^i_1, x^i_2]$ of $B^i_1$ is identified with the edge $[a_{4(i-1)+1}, a_{4(i-1)+2}]$ of $B_0$ and the edge $[x^i_3, x^i_4]$ of $B^i_1$ is identified with the edge $[a_{4(i-1)+3}, a_{4(i-1)+4}]$ of $B_0$;*

*iii) $B^i_2$, $i = 1, \dots, g$, is a 4-gon with vertices $y^i_1, y^i_2, y^i_3, y^i_4$ numbered clockwise. It is glued with $B_0$ so that the edge $[y^i_1, y^i_2]$ of $B^i_2$ is identified with the edge $[a_{4(i-1)+2}, a_{4(i-1)+3}]$ of $B_0$ and the edge $[y^i_3, y^i_4]$ of $B^i_2$ is identified with the edge $[a_{4(i-1)+4}, a_{4(i-1)+5 (mod 4g)}]$ of $B_0$.*

Observe that in Definition 5.11 we denote by $[a, b]$ edges of polygons and not edges of graphs. The usual representation in the space of an orientable surface $\mathcal{S}$ of genus $g$ may be then obtained from its polygonal representation $S_g$ by the following operation: for each bridge $B$, glue together the two segments which $B$ shares with the boundary of $B_0$, and delete $B$.

**Definition 5.12** *A graph $G$ is called a $g$-graph if it may be embedded on $S_g$ so that all the vertices belong to the base $B_0$, and the embedding of each edge uses at most one bridge. The set of the edges embedded entirely on the base will be denoted by $E_0$ and the set of the edges embedded on the bridge $B^i_j$ will be denoted by $E^i_j$, $i = 1, \dots, g$, $j = 1, 2$. If a $g$-graph $G$ satisfies the following further conditions:*

1. *the outer face of $G_0 = (V, E_0)$ is a cycle, and it is embedded on the boundary of $B_0$,*
2. *if $e \in E^i_1$ then $e$ is embedded entirely on $B^i_1$ and one endvertex of $e$ belongs to $[x^i_1, x^i_2]$ and the other one belongs to $[x^i_3, x^i_4]$. Similarly, if $e \in E^i_2$ then $e$ is embedded entirely on $B^i_2$ and one endvertex of $e$ belongs to $[y^i_1, y^i_2]$ and the other one belongs to $[y^i_3, y^i_4]$.*
3. *each vertex is incident with at most one edge which does not belong to $E_0$,*
4. *$G_0$ has a perfect matching,*

*then we say that $G$ is a proper $g$-graph.*

Given a proper $g$-graph $G$, we denote by $C_0$ the cycle which forms the outer face of $E_0$; then, we fix a perfect matching of $G_0$ and denote it by $M_0$.

**Definition 5.13** *Let $G$ be a proper $g$-graph and let $G^i_j = (V, E_0 \cup E^i_j)$. If we draw $B_0 \cup B^i_j$ on the plane as follows: $B_0$ is unchanged, and the edge $[x^i_1, x^i_4]$ ($[y^i_1, y^i_4]$ respectively) of $B^i_j$ is drawn so that it belongs to the external boundary of $B_0 \cup B^i_j$, we obtain a planar embedding of $G^i_j$. This embedding will be called planar projection of $E^i_j$ outside $B_0$.*

**Definition 5.14** *Let $G = (V, E)$ be a proper $g$-graph. A Pfaffian orientation $D_0$ of $G_0$ such that each inner face of each 2-connected component of $G_0$ is clockwise odd in $D_0$ is called a basic orientation of $G_0$.*

Note that a basic orientation always exists for a planar graph by Theorem 5.10.

**Definition 5.15** *Let $G = (V, E)$ be a proper $g$-graph and $D_0$ a basic orientation of $G_0$. We define the orientation $D^i_j$ of each $G^i_j$ as follows: We consider $G^i_j$ embedded on the plane by the planar projection of $E^i_j$ outside $B_0$ (see Definition 5.13), and complete the basic orientation $D_0$ of $G_0$ to an orientation of $G^i_j$ so that each inner face of each 2-connected component of $G^i_j$ is clockwise odd.*
*The orientation $-D^i_j$ is defined by reversing the orientation $D^i_j$ of $G^i_j$.*

Observe that after fixing a basic orientation $D_0$, the orientation $D_j^i$ is uniquelly determined for each $i, j$.

**Definition 5.16** *Let $G$ be a proper $g$-graph, $g \geq 1$. An orientation $D$ of $G$ which equals the basic orientation $D_0$ on $G_0$ and which equals $D_j^i$ or $-D_j^i$ on $E_j^i$ is called relevant. We define its type $r(D) \in \{+1, -1\}^{2g}$ as follows: For $i = 0, \ldots, g-1$ and $j = 1, 2$, $r(D)_{2i+j}$ equals $+1$ or $-1$ according to the sign of $D_j^{i+1}$ in $D$.*

**Definition 5.17** *Let $G$ be a proper $g$-graph and let $A$ be a subset of its edges. The type of $A$ is a vector $t(A) \in \{0,1\}^{2g}$ defined as follows: For $i = 0, ..., g-1$ and $j = 1, 2$, we let $t(A)_{2i+j}$ equals the number of edges of $A$ which belong to $E_j^{i+1}$, modulo 2.*

*Let $CR(A) \stackrel{2}{=} \sum_{i=0}^{g-1} t(A)_{2i+1} \cdot t(A)_{2i+2}$ denote the number of crossings of the embeddings of the edges of $A$, after making planar projections of $E_j^i$ for all $i, j$.*

*Let $BR(A)$ denote the subset of edges of $A$ which do not belong to $E_0$. For each $e \in BR(A)$, let $d(e) = 2i + j$ if $e \in E_j^{i+1}$.*

We complete the section with a lemma.

**Lemma 5.18** *Let $G$ be a proper $g$-graph. Let $C_1, ..., C_k$ be vertex-disjoint cycles of $G$ and let $\mathcal{C}$ denote their union. Then*

$$CR(\mathcal{C}) \stackrel{2}{=} \sum_{i=1}^{k} CR(C_i).$$

**Proof.** Let us embed the cycles $C_1, ..., C_k$ using the planar projections of $E_j^i$ outside $B_0$ by Definition 5.17. Then $CR(\mathcal{C})$ equals the total number of crossings of $\mathcal{C}$ (modulo 2). Now, each cycle $C_l$, $l = 1, ..., k$ is represented as a closed curve in the plane and each pair of cycles $C_i$ and $C_j$, $i \neq j$, intersects an even number of times. Hence the sum (modulo 2) of the number of crossings between pairs of cycles $C_i$ and $C_j$, $i \neq j$, is 0 and does not affect $CR(\mathcal{C})$. Each of the remaining crossings is a crossing of some $C_l$, $l = 1, ..., k$, with itself and the lemma follows. □

**Perfect matchings**

Through this section, the graph $G$ will be a proper $g$-graph embedded on a fixed surface $S_g$. We also fix a perfect matching $M_0$ of $G_0$.

The aim of this section is to prove that, for any perfect matching $P$, the $sgn(D, M_0 \, \Delta P)$ depends only on the vectors $t(M_0 \Delta P)$ and $r(D)$.

Given an orientation $D$ of $G$ and an even length cycle $C$ of $G$, we denote by $l_D(C)$ the number of arcs of $C$ directed in agreement with any of the two possible ways of traversing $C$, modulo 2. For short, any alternating cycle with respect to $M_0$ will be simply called an *alternating cycle*. In order to prove our statement, we consider first the case that $M_0 \Delta P$ consists of exactly one alternating cycle.

**Theorem 5.19** *Let $G$ be a proper $g$-graph and let $D$ be a relevant orientation of $G$. If $C$ is an alternating cycle of $G$, then*

$$l_D(C) \stackrel{2}{=} |BR(C)| - 1 - CR(C) + \frac{1}{2} \sum_{e \in BR(C)} (r(D)_{d(e)} + 1).$$

**Proof.** We assume without loss of generality that $G = C \cup C_0 \cup M_0$, where $C_0$ is the outer face of $G_0$ and $M_0$ is the fixed perfect matching of $G_0$. Let $D_0$ be the basic orientation of $G_0$.

**Claim 1.** *If $C$ intersects at most one of $E_1^i, E_2^i$, for each $i = 1, ..., g$, then*

$$l_D(C) \stackrel{2}{=} |BR(C)| - 1 + \frac{1}{2} \sum_{e \in BR(C)} (r(D)_{d(e)} + 1).$$

A cycle $C$ satisfying the properties of Claim 1 may be embedded without crossings using the planar projection of each $E_j^i$ outside $B_0$. Hence $l_D(C) = 1$ if and only if $|\{e \in BR(C) : r(D)_{d(e)} = -1\}| \stackrel{2}{=} 0$. **End of Claim 1.**

The proof is by induction on $|BR(C)|$. The case $|BR(C)| = 0$ is proved by Claim 1. By induction we assume that

$$l_W(C') \stackrel{2}{=} |BR(C')| - 1 - CR(C') + \frac{1}{2} \sum_{e \in BR(C')} (r(W)_{d(e)} + 1)$$

for any alternating cycle $C'$ of a proper $g$-graph $H$, with relevant orientation $W$, such that $|BR(C')| < |BR(C)|$.

We distinguish two cases.

**Case 1.** *There exists a bridge $B = B_j^i$ containing more than one edge of $C$.*
Let $e = u_1u_2$ and $f = v_1v_2$ be two edges of $C \cap E_j^i$ which see each other on $B$, i.e., there is no other edge of $C$ drawn between them on $B$. Without loss of generality, let $e$ be nearer to the edge $[a_{2(i-1)+j}, a_{2(i-1)+j+3}]$ of $B = B_j^i$ than $f$ and let $u_1, v_1$ and $u_2, v_2$ belong to the edge $[a_{2(i-1)+j}, a_{2(i-1)+j+1}]$ and $[a_{2(i-1)+j+2}, a_{2(i-1)+j+3}]$, respectively. Since $e, f$ do not belong to $E_0$, they are not edges of $M_0 \subset E_0$.

Let $R_i$ be the subpath of $C_0$ from $u_i$ to $v_i$, $i = 1, 2$, and let $R$ be the cycle of $G$ consisting of $(R_1, f, R_2, e)$. By the choice of $e, f$, the cycle $R$ is the boundary of a face of the planar projection of $G_j^i = (V, E_0 \cup E_j^i)$ outside $B_0$. Observe that $l_W(R) = 1$ for each relevant orientation $W$ of $G$, since $R$ contains two edges embedded outside $B_0$.

Let us introduce a new edge $h$ (not belonging to $G$), between the endvertices of $e, f$ such that one of two cycles $\bar{H}_1, \bar{H}_2$ formed by $h$ and $C$ and containing $h$ is alternating. Without loss of generality, let $h$ have $u_1$ as an endvertex. Hence we have that $h = u_1v_1$ or $h = u_1v_2$.

We may assume without loss of generality that $\bar{H}_2$ is alternating. Hence $\bar{H}_1$ contains both $e, f$. Note that $\bar{H}_1$ consists of an even number of edges. We denote by $h_1, h_2$ the two arcs with the same endvertices as $h$, directed opositely. Let $D' = D \cup \{h_1, h_2\}$. Let $H_i$ be the subdigraph of $D'$ which is the orientation of $\bar{H}_i$ using $h_i$, $i = 1, 2$. Observe that $l_D(C) = l_{D'}(H_1) + l_{D'}(H_2)$.

*Subcase 1.1: $h_1 = u_1v_1$.*

We adjust the boundary of $B_0$ by replacing $\{R_1\}$ with $h_1, h_2$. Observe that $CR(C) \stackrel{2}{=} CR(H_1) + CR(H_2)$: attention should be drawn to the question of how crossings of $C$ with itself are manifested as crossings of $H_1$ or $H_2$, when all $E_j^i$ are projected outside of $B_0$ (see Definition 5.13). If two edges of $C$ cross and they are

not separated in $C$ by the endvertices of $h_1$, then that crossing counts as a crossing with in $H_1$ or $H_2$. We must therefore consider the parity of the number of crossings of $C$ where the crossed edges are separated in $C$ by the endvertices of $h_1$. These crossings are counted as crossings of $H_1$ with $H_2$. If the number of such crossings of $C$ is odd, then there must be an additional crossing of $H_1$ with $H_2$, since the total number of crossings of $H_1$ with $H_2$ must be even. Since $h_1$ and $h_2$ do not cross, this additional crossing must occur at an endvertex of $h_1$. It is easy to see that in the present case there is no such crossing, and so, there are an even number of crossings of $C$ where the crossed edges are separated in $C$ by the ends of $h$. The required congruence therefore follows in this case.
We construct now two digraphs $D_1, D_2$ as follows:

- $D_1$ is obtained from $D-\{e,f\}$ by adding new vertices $u_1', v_1'$ of degree 2, incident with new arcs $e', f', h_1'$. The arcs $e', f', h_1'$ are obtained from $e, f, h_1$ by replacing $u_1$ by $u_1'$ and $v_1$ by $v_1'$. We adjust the boundary of $B_0$ by replacing $\{R_2\}$ with $\{e', f', h_1'\}$. Finally we add $h_1'$ to $M_0$. Let $H_1'$ be the cycle of $D_1$ obtained from $H_1$ by replacing $e, f, h_1$ by $e', f', h_1'$. Then $l_{D_1}(H_1') = l_{D'}(H_1)$ and $CR(H_1') \stackrel{2}{=} CR(H_1)$;

- $D_2$ is obtained from $D-\{e,f\}$ by adding arc $h_2$. We remind that $h_2$ is embedded on the adjusted $B_0$ parallel to $R_1$. Let $H_2' = H_2$. Then $l_{D_2}(H_2') = l_{D'}(H_2)$ and $CR(H_2') \stackrel{2}{=} CR(H_2)$.

We remind that $l_D(R) = 1$. Hence, exactly one of $h_i$ is oriented so that both cycles it makes with $R$ are clockwise odd. Let it be $h_2$. Then $D_2$ is a relevant orientation and $D_1$ becomes relevant after reversing the orientation of $h_1'$: this digraph, obtained from $D_1$ by reversing the orientation of $h_1'$, we denote by $D_1^*$, and its subdigraph corresponding to $H_1'$ we denote by $H_1^*$. Then, $l_{D_1^*}(H_1^*) \stackrel{2}{=} l_{D_1}(H_1') + 1$.

Note that both $D_2$ and $D_1^*$ are relevant orientations of proper $g$-graphs, $H_2'$ is an alternating cycle of $D_2$, $H_1^*$ is an alternating cycle of $D_1^*$ and $CR(H_1^*) < CR(C)$ and $CR(H_2') < CR(C)$. Hence, by the induction assumption, we have that:

$$l_D(C) \stackrel{2}{=} l_{D'}(H_1) + l_{D'}(H_2) \stackrel{2}{=} l_{D_1}(H_1') + l_{D_2}(H_2') \stackrel{2}{=} l_{D_1^*}(H_1^*) + 1 + l_{D_2}(H_2') \stackrel{2}{=}$$

$$|BR(H_1^*)| - 1 - CR(H_1^*) + \frac{1}{2} \sum_{p \in BR(H_1^*)} (r(D_1^*)_{d(p)} + 1) +$$

$$|BR(H_2')| - 1 - CR(H_2') + \frac{1}{2} \sum_{p \in BR(H_2')} (r(D_2)_{d(p)} + 1) + 1.$$

Now, the theorem follows by observing that $|BR(C)| \stackrel{2}{=} |BR(C - \{e,f\})| \stackrel{2}{=} |BR(H_1^*)| + |BR(H_2')| - 2$, $CR(C) \stackrel{2}{=} CR(H_1^*) + CR(H_2')$ and $r(D_1^*)_{d(p)}$, $r(D_2)_{d(p)}$ and $r(D)_{d(p)}$ coincide for any $p \in BR(C) - \{e,f\}$. Hence,

$$l_D(C) \stackrel{2}{=} |BR(C)| - 1 - CR(C) + \frac{1}{2} \sum_{p \in BR(C)} (r(D)_{d(p)} + 1).$$

*(End of Subcase 1.1)*

*Subcase 1.2:* $h_1 = u_1 v_2$.

Let $h_1$ and $h_2$ be embedded on the bridge $B$. Observe that $CR(C) \stackrel{2}{=} CR(H_1) + CR(H_2) + 1$: attention again should be drawn to the question of how crossings of $C$ with itself are manifested as crossings of $H_1$ or $H_2$, when all $E^i_j$ are projected outside of $B_0$ (see Definition 5.13). To see this clearly, we introduce some notation. Let $A$ be a subset of arcs of $H_1$ and $B$ a subset of arcs of $H_2$. We denote by $CR(A \times B)$ the number of crossings between arcs of $A$ and $B$, mod 2. We also denote by $cr(i,j)$ the number of crossings of arcs of $H_i \cap C$ with $h_j$. Hence, we have:

$$CR(H_1) \stackrel{2}{=} CR(H_1 \cap C) + cr(1,1),$$

$$CR(H_2) \stackrel{2}{=} CR(H_2 \cap C) + cr(2,2),$$

$$CR(C) \stackrel{2}{=} CR(H_1 \cap C) + CR(H_2 \cap C) + CR((H_1 \cap C) \times (H_2 \cap C)),$$

$$CR(H_1 \times H_2) \stackrel{2}{=} 0,$$

and

$$\sum_{i,j=1}^{2} cr(i,j) \stackrel{2}{=} 0$$

since each arc which crosses $h_1$ crosses also $h_2$.

Hence it remains to show that

$$CR(H_1 \times H_2) \stackrel{2}{=} CR((H_1 \cap C) \times (H_2 \cap C)) + cr(1,2) + cr(2,1) + 1:$$

this follows since in this case one additional crossing between $H_1$ and $H_2$ must occur at an endvertex of $h$. The required congruence follows.

We construct two digraphs $D_1, D_2$ as follows:

- $D_1$ is obtained from $D - \{e, f\}$ by adding a new arc $h'_1$ between $v_1$ and the endvertex $u_2$ of $e$. If $l_{D'}(fh_1e) = 1$ then we let $h'_1 = (v_1, u_2)$. If $l_{D'}(fh_1e) = 0$ then we let $h'_1 = (u_2, v_1)$.

  We consider $h'_1$ embedded on the bridge $B$. Let $H'_1$ be obtained from $H_1$ by replacing $\{f, h_1, e\}$ by $h'_1$. We have $l_{D'}(H_1) = l_{D_1}(H'_1)$ and $CR(H_1) = CR(H'_1)$.

- $D_2$ is obtained from $D - \{e, f\}$ by adding the arc $h_2$. We consider $h_2$ embedded on the bridge $B$. We let $H_2 = H'_2$. Thus again we have $l_{D'}(H_2) = l_{D_2}(H'_2)$ and $CR(H_2) = CR(H'_2)$.

We remind that $l_D(R) = 1$ and thus exactly one of $h_i$ is oriented so that both cycles it makes with $R$ are clockwise odd. Let it be $h_2$. Let $R_3$ be the subpath of $C_0$ from $v_1$ to $v_2$ such that $(e, R_1, R_3, R_2)$ is a cycle. We have $l_{D_1}(h'_1, R_3, R_2) \stackrel{2}{=} l_{D'}(e, h_1, f, R_3, R_2) \stackrel{2}{=} l_{D'}(f, R_3) + l_{D'}(e, h_1, R_2)$.

We show now that both $D_1$ and $D_2$ are relevant orientations with $r(D_1) = r(D_2) = r(D)$. We only need to show that $h'_1$ and $h_2$ are correctly oriented in $D_1$ and $D_2$. This follows easily for $D_2$, since both cycles $h_2$ makes with $R$ are clockwise odd.

For $D_1$ we distinguish two cases. First, let $r(D)_{2(i-1)+j} = 1$. In this case we have $l_{D'}(f, R_3) = 1$ and $l_{D'}(e, h_2, R_2) = 1$. Hence $l_{D'}(e, h_1, R_2) = 0$. It follows that $l_{D_1}(h'_1, R_3, R_2) = 1$ and $D_1$ is relevant with $r(D_1) = r(D)$. Secondly, let $r(D)_{2(i-1)+j} = -1$. In this case we have $l_{D'}(f, R_3) = 0$ and $l_{D'}(e, h_2, R_2) = 1$. Hence $l_{D'}(e, h_1, R_2) = 0$. It follows that $l_{D_1}(h'_1, R_3, R_2) = 0$ and $D_1$ is relevant with $r(D_1) = r(D)$.

Hence, $D_i$ is a relevant orientation of a proper $g$-graph, $H'_i$ is an alternating cycle of $D_i$ and $|BR(H'_i)| < |BR(C)|$, for $i = 1, 2$, and, by the induction hypothesis, we have that:

$$l_D(C) \stackrel{2}{=} l_{D'}(H_1) + l_{D'}(H_2) \stackrel{2}{=} l_{D_1}(H'_1) + l_{D_2}(H'_2) \stackrel{2}{=}$$

$$|BR(H'_1)| - 1 - CR(H'_1) + \frac{1}{2} \sum_{p \in BR(H'_1)} (r(D_1)_{d(p)} + 1) + \frac{1}{2}(r(D_1)_{d(h_1)} + 1) +$$

$$|BR(H'_2)| - 1 - CR(H'_2) + \frac{1}{2} \sum_{p \in BR(H'_2)} (r(D_2)_{d(p)} + 1) + \frac{1}{2}(r(D_2)_{d(h_2)} + 1).$$

The theorem follows by observing that $|BR(C)| \stackrel{2}{=} |BR(C - \{e, f\})| \stackrel{2}{=} |BR(H'_1)| + |BR(H'_2)| - 2$, $CR(C) + 1 \stackrel{2}{=} CR(H'_1) + CR(H'_2)$ and $r(D_1) = r(D_2) = r(D)$.

*(End of Subcase 1.2)*

**End of Case 1**

**Case 2.** *There exists $i$ such that $C$ contains exactly one edge from both $E_1^i$ and $E_2^i$.* Let $e \in E_1^i$ and $f \in E_2^i$ and let $C_1$ and $C_2$ be two paths such that $C = (C_1, e, C_2, f)$. The endvertices of $e, f$ belong to $C_0$. Let us assume that along the boundary of $B_0$ from $a_{4(i-1)+1}$ to $a_{4i+1}$, the endvertices of $e, f$ appear in the order $v_1, u_1, v_2, u_2$ where $e = u_1u_2$ and $f = v_1v_2$.

Let $R_1, R_2$ be the two disjoint subpaths of the segment of $C_0$ between $a_{4(i-1)+1}$ and $a_{4i+1}$, which cover the endvertices of $e, f$. Note that $R_1, R_2$ contain no other vertex of $G$ incident with an edge out of $E_0$, by the choice of $i$. Let $R$ denote the cycle $(R_1, e, R_2, f)$ and let $R_3$ denote the segment of $C_0$ between $u_1$ and $v_2$.

Let us introduce a new edge $h$ (not belonging to $G$), between endvertices of $e, f$ such that one of two cycles $\bar{I}_1, \bar{I}_2$ formed by $h$ and $C$ and containing $h$ is alternating.

Without loss of generality let $h$ have $u_1$ as an endvertex. Hence we have that $h = u_1v_1$ or $h = u_1v_2$. We may also assume without loss of generality that $\bar{I}_2$ is alternating. Hence $\bar{I}_1$ contains both $e, f$. Note that $\bar{I}_1$ consists of an even number of edges.

We denote by $h_1, h_2$ the two arcs with the same endvertices as $h$, directed opositely. Let $D' = D \cup \{h_1, h_2\}$. Let $I_i$ be the subdigraph of $D'$ which is the orientation of $\bar{I}_i$ using $h_i$, $i = 1, 2$. Observe that $l_D(C) = l_{D'}(I_1) + l_{D'}(I_2)$.

Again we distinguish two subcases.

*Subcase 2.1: $h_1 = u_1v_1$.*

In this case $h$ forms a cycle with $R_1$.

As in Subcase 1.1, we extend $B_0$ along $R_1$ and consider $h_1, h_2$ as embedded on the extended $B_0$.

Observe that $CR(C) \stackrel{2}{=} CR(I_1) + CR(I_2)$: attention should be drawn to the question of how crossings of $C$ with itself are manifested as crossings of $I_1$ or $I_2$, when all $E^i_j$ are projected outside of $B_0$ (see Definition 5.13). In this case, the arguments are identical to those used in the proof of Subcase 1.1, and we omit them.

We construct two digraphs $D_1, D_2$ as follows:

- $D_1$ is obtained from $D-\{e,f\}$ by adding new vertices $u'_1, v'_1$ of degree 2, incident with new arcs $e', f', h'_1$. The arcs $e'f', h'_1$ are obtained from $e, f, h_1$ by replacing $u_1$ by $u'_1$ and $v_1$ by $v'_1$. We extend $B_0$ along $R_2$ and we embed the path $(e', f', h'_1)$ on the extended $B_0$. Finally we add $h'_1$ to $M_0$. Let $I'_1$ be the cycle of $D_1$ obtained from $I_1$ by replacing $e, f, g_1$ by $e', f', g'_1$. We have $l_{D'}(I_1) = l_{D_1}(I'_1)$ and $CR(I_1) - 1 \stackrel{2}{=} CR(I'_1)$.

- $D_2$ is obtained from $D-\{e,f\}$ by adding the arc $h_2$. We consider $h_2$ embedded on the extended $B_0$ along $R_1$. We let $I'_2 = I_2$. Hence, $l_{D'}(I_2) = l_{D_2}(I'_2)$ and $CR(I_2) \stackrel{2}{=} CR(I'_2)$.

Hence, for $i = 1, 2$, $D_i$ is an orientation of a proper $g$-graph and $I'_i$ is an alternating cycle of $D_i$. Moreover $|BR(I'_i)| < |BR(C)|$.

Let us assume without loss of generality that $h_2$ is directed so that the cycle $l_{D'}(R_1, h_2) = 1$. Hence $D_2$ is a relevant orientation with $r(D_2) = r(D)$.

We show now that $D_1$ is a relevant orientation with $r(D) = r(D_1)$ if and only if $r(D)_{d(e)} = r(D)_{d(f)}$. We first prove that if $r(D)_{d(e)} = r(D)_{d(f)} = 1$ then $D_1$ is a relevant orientation.

In this case it suffices to show that $l_{D_1}(R_2, f', h'_1, e') \stackrel{2}{=} 1$. We have $l_{D_1}(h'_1, f', R_3) \stackrel{2}{=} l_{D'}(h_1, f, R_3) \stackrel{2}{=} l_{D'}(h_2, f, R_3) + 1 \stackrel{2}{=} l_{D_2}(h_2, f, R_3) + 1 \stackrel{2}{=} 0$, since $r(D_2)_{d(f)} = r(D)d(f) = 1$, and thus, $l_{D_2}(h_2, f, R_3) = 1$.

Moreover $l_{D_1}(R_2, R_3, e') = l_{D'}(R_2, R_3, e) = 1$, since $r(D)_{d(e)} = 1$ and $D$ is a relevant orientation. Replacing $f'h'_1$ for $R_3$ gives what we claimed.

Similarly, we can prove that if $r(D)_{d(e)} = r(D)_{d(f)} = -1$ then again $l_{D_1}(R_2, f', h'_1, e') = 1$, and so, $D_1$ is a relevant orientation.

On the other hand, if $r(D)_{d(e)} \neq r(D)_{d(f)}$ then $D_1$ is obtained from a relevant orientation by reversing one arc, and so, it is not relevant.

Summarizing, if $r(D)_{d(e)} = r(D)_{d(f)}$ then $D_1$ is a relevant orientation with $r(D) = r(D_1)$, and if $r(D)_{d(e)} \neq r(D)_{d(f)}$ then $D_1$ becomes relevant after reversing the orientation of $h'_1$: this digraph, obtained from $D_1$ by reversing the orientation of $h'_1$, we denote by $D^*_1$, and its subdigraph corresponding to $H'_1$ we denote by $H^*_1$. Then $l_{D^*_1}(I'_1) \stackrel{2}{=} l_{D_1}(I'_1) + 1$.

Using the induction assumption of 5.19 for $D^*, I^*_1$, $D_1, I'_1$ and $D_2, I'_2$ we get:

$$l_D(C) \stackrel{2}{=} l_{D'}(I_1) + l_{D'}(I_2) \stackrel{2}{=} l_{D_1}(I'_1) + l_{D_2}(I'_2) \stackrel{2}{=}$$

$$|BR(C)| - 4 - CR(C) + 1 + \frac{1}{2} \sum_{p \in BR(C)-\{e,f\}} (r(D)_{d(p)} + 1) + \frac{1}{2}(r(D)_{d(e)} - 1 + r(D)_{d(f)} - 1) \stackrel{2}{=}$$

$$|BR(C)| - 1 - CR(C) + \frac{1}{2}\sum_{p\in BR(C)} (r(D)_{d(p)} + 1).$$

*(End of Subcase 2.1)*

*Subcase 2.2:* $h = u_1v_2$.

In this case $h$ forms a cycle with $R_3$. We extend $B_0$ along $R_3$ and consider $h_1, h_2$ embedded on the extended $B_0$.

Observe that $CR(C) \stackrel{2}{=} CR(I_1)+CR(I_2)$:attention should be drawn to the question of how crossings of $C$ with itself are manifested as crossings of $I_1$ or $I_2$, when all $E^i_j$ are projected outside of $B_0$ (see Definition 5.13). In this case, the arguments are identical to those used in the proof of Subcase 1.1 and Subcase 2.1, and we omit them.

We construct two digraphs $D_1, D_2$ as follows:

- $D_1$ is obtained from $D-\{e,f\}$ by adding new vertices $u'_1, v'_2$ of degree 2, incident with new arcs $e', f', h'_1$. The arcs $e', f', h'_1$ are obtained from $e, f, h_1$ by replacing $u_1$ by $u'_1$ and $v_2$ by $v'_2$. We extend $B_0$ along $R_1R_3R_2$ and we embed $e', f', h'_1$ on the extended $B_0$. Finally we add $h'_1$ to $M_0$. Let $I'_1$ be the cycle of $D_1$ obtained from $I_1$ by replacing $e, f, h_1$ by $e', f', h'_1$. We have that $l_{D'}(I_1) \stackrel{2}{=} l_{D_1}(I'_1)$ and $CR(I'_1) \stackrel{2}{=} CR(I_1) - 1$.
- $D_2$ is obtained from $D - \{e, f, \}$ by adding arc $h_2$. We again extend $B_0$ along $R_3$ and consider $h_2$ embedded on the extended $B_0$. We let $I'_2 = I_2$. Hence, $l_{D'}(I_2) \stackrel{2}{=} l_{D_2}(I'_2)$ and $CR(I'_2) \stackrel{2}{=} CR(I_2)$.

Hence for $i = 1, 2$, $D_i$ is an orientation of a proper $g$-graph and $I'_i$ is an alternating cycle of $D_i$. Moreover $|BR(I'_i)| < |BR(C)|$.

Let us assume without loss of generality that $h_2$ is directed so that $l(R_3, h_2) = 1$. Hence $D_2$ is a relevant orientation with $r(D_2) = r(D)$.

As in Subcase 2.1, we shall show that $D_1$ is a relevant orientation if and only if $r(D)_{d(e)} = r(D)_{d(f)}$: It again suffices to consider the case that $r(D)_{d(e)} = r(D)_{d(f)} = 1$. In this case it suffices to show that $l_{D_1}(R_2, R_3, R_1, f', h'_1, e') = 1$. In fact, we have $l_{D_2}(R_1, f, h_2) = 1$ since $r(D_{d(f)}) = 1$ and $D_2$ is a relevant orientation. Hence $l_{D_1}(R_1, f', h'_1) = 0$. Moreover $l_{D_1}(R_2, R_3, e') \stackrel{2}{=} l_D(R_2, R_3, e) = 1$ since $r(D)_{d(e)} = 1$. Hence $l_{D_1}(R_2, R_3, R_1, f', h'_1, e') = 1$.

Summarizing, if $r(D)_{d(e)} = r(D)_{d(f)}$ then $D_1$ is a relevant orientation with $r(D) = r(D_1)$, and if $r(D)_{d(e)} \neq r(D)_{d(f)}$ then $D_1$ becomes relevant after reversing the orientation of $h'_1$.

The proof then proceeds analogously as in Subcase 2.1. *(End of Subcase 2.2)*

**End of Case 2**

It is not difficult to see that the two cases complete the proof. □

Next we show that a statement analougous to that of Theorem 5.19 holds for the set of the alternating cycles of $M_0 \Delta P$ as well.

**Theorem 5.20** *Let $G$ be a proper $g$-graph and let $D$ be a relevant orientation of $G$. Let $P$ be a perfect matching of $G$. Then*

$$sgn(D, M_0 \Delta P) = (-1)^q,$$

*where*

$$q \stackrel{2}{=} |BR(M_0\Delta P)| - CR(M_0\Delta P) + \frac{1}{2} \sum_{e \in BR(M_0\Delta P)} (r(D)_{d(e)} + 1).$$

**Proof.** Let $C_1, ..., C_k$ be the alternating cycles of $M_0\Delta P$. We have that $sgn(D, M_0\Delta P) = (-1)^q$, where $q \stackrel{2}{=} l(C_1) + ... + l(C_k) - k$.

Using Theorem 5.19 for $C_1, ..., C_k$, it remains to show that:

$$CR(M_0\Delta P) \stackrel{2}{=} \sum_{j=1}^{k} CR(C_j),$$

but this holds by Lemma 5.18 and the theorem follows. □

**Corollary 5.21** *Let $G$ be a proper 1-graph and $D$ a relevant orientation of $G$. Let $\mathcal{C}$ be a set of disjoint alternating cycles of $M_0$. Then:*

1. *If $r(D) = (1, 1)$ then $sgn(D, \mathcal{C}) = 1$ if and only if $t(\mathcal{C}) \in \{(0, 0), (0, 1), (1, 0)\}$.*
2. *If $r(D) = (1, -1)$ then $sgn(D, \mathcal{C}) = 1$ if and only if $t(\mathcal{C}) \in \{(0, 0), (1, 1), (1, 0)\}$.*
3. *If $r(D) = (-1, 1)$ then $sgn(D, \mathcal{C}) = 1$ if and only if $t(\mathcal{C}) \in \{(0, 0), (0, 1), (1, 1)\}$.*
4. *If $r(D) = (-1, -1)$ then $sgn(D, \mathcal{C}) = 1$ if and only if $t(\mathcal{C}) = (0, 0)$.*

**Definition 5.22** *Let $G$ be a proper $g$-graph and $D$ a relevant orientation of $G$. Let $r(D) = (r_1, ..., r_{2g})$. We let $c(r(D))$ equal to the product of $c_i$, $i = 0, ..., g-1$, where $c_i = c(r_{2i+1}, r_{2i+2})$ and $c(1, 1) = c(1, -1) = c(-1, 1) = 1/2$ and $c(-1, -1) = -1/2$.*

Observe that $c(r(D)) = (-1)^n 2^{-g}$, where $n = |\{i; r_{2i+1} = r_{2i+2} = -1\}|$.

**Corollary 5.23** *Let $G$ be a proper 1-graph. Let $D_1, D_2, D_3, D_4$ be the relevant orientations of $G$. Then*

$$\mathcal{P}(G, x) = \sum_{i=1}^{4} c(r(D_i))\mathcal{P}(D_i, M_0).$$

A result analogous to Corollary 5.23 holds for all proper $g$-graphs, $g > 1$. In order to deduce it we start with another corollary of Theorem 5.20.

**Corollary 5.24** *Let $G$ be a proper $g$-graph and $D$ a relevant orientation of $G$. Let $P$ be a perfect matching of $G$. Then $sgn(D, M_0 \Delta P)$ is a function of $r(D)$ and $t(M_0 \Delta P)$ only. Let us denote this function by $\sigma(r(D), t(M_0 \Delta P))$.*

**Lemma 5.25** *Let $r = (r_1, \ldots, r_{2g})$ and $t = (t_1, \ldots, t_{2g})$ be $2g$-dimensional vectors. Let $r(j) = (r_{2j+1}, r_{2j+2})$ and $t(j) = (t_{2j+1}, t_{2j+2})$, $j = 0, \ldots, g-1$. Then*

$$\sigma(r, t) = \prod_{j=0}^{g-1} \sigma(r(j), t(j)).$$

**Proof.** By Corollary 5.24, we have that $sgn(D, \mathcal{C}) = sgn(D', \mathcal{C}')$ if and only if $r(D) = r(D')$ and $t(\mathcal{C}) = t(\mathcal{C}')$. This implies that we can restrict ourselves to consider the following case: $G = C_0 \cup M_0 \cup \mathcal{C}$ is a proper $g$-graph, $D$ is a relevant orientation of $G$ such that $r(D) = r$ and $\mathcal{C}$ consists of a set of vertex-disjoint cycles $C_1, \ldots, C_k$ satisfying the following properties:

1. each $C_i$ is alternating with respect to the perfect matching $M_0$,
2. for each $i, j$ $|E_j^i| \leq 1$,
3. for each $i$ there is at most one $j$ such that $|C_j \cap (E_1^i \cup E_2^i)| \geq 1$,
4. for each $C_j$ there is exactly one $i$ such that $C_j$ intersects $E_1^i \cup E_2^i$,
5. $t(\mathcal{C}) = t$.

Hence,

$$\sigma(r, t) = sgn(D, \mathcal{C}) = \prod_{i=1}^{k} sgn(D, C_i) = \prod_{i=1}^{k} sgn(D_i, C_i)$$

where $D_i$ is the restriction of $D$ to $C_0 \cup C_i$. Observe that, by Corollary 5.21, $\sigma(z_1, z_2) = 1$ if $z_2 = (0, 0)$. Hence, using Corollary 5.24, we have that $\prod_{i=1}^{k} sgn(D_i, C_i) = \prod_{j=0}^{g-1} \sigma(r^j, t^j)$ as claimed.

□

**Theorem 5.26** *Let $G$ be a proper $g$-graph. Then*

$$\mathcal{P}(G, x) = \mathcal{L}_g(G, x) = \sum_{i=1}^{4^g} c(r(D_i)) \mathcal{P}(D_i, M_0)$$

*where $D_i$, $i = 1, \ldots, 4^g$, are the relevant orientations of $G$.*

**Proof.** Let $P$ be a perfect matching of $G$. In each term $\mathcal{P}(D_i, M_0)$, the coefficient of $x(P)$ is $sgn(D_i, M_0 \Delta P)$. By Corollary 5.24, $sgn(D_i, M_0 \Delta P) = \sigma(r(D_i), t(M_0 \Delta P))$.

Let

$$\mathcal{K}_g(t(M_0 \Delta P)) = \sum_{i=1}^{4^g} c(r(D_i)) \sigma(r(D_i), t(M_0 \Delta P))$$

denote the coefficient of $x(P)$ in $\mathcal{L}_g(G, x)$.

To prove the theorem it suffices to prove the following claim:

**Claim.** $\mathcal{K}_g(t(M_0\Delta P)) = 1$ *for each* $t(M_0\Delta P)$.

The proof of the claim is by induction on $g$. The basis of the induction when $g = 1$ is proved in Corollary 5.23.

To prove the inductive step we introduce the following notation: if $z$ is a $2g$-dimensional vector then we let $z = (z(0), \ldots, z(g-1))$ where $z(i) = (z_{2i+1}, z_{2i+2})$.

We call two relevant orientations $D$ and $D'$ of $G$ *equivalent* if $(r(D)(1), \ldots, r(D)(g-1)) = (r(D')(1), \ldots, r(D')(g-1))$. Clearly, the equivalence classes consist of 4 elements; let $\mathcal{R}_1, \ldots, \mathcal{R}_{4^{g-1}}$ be the equivalence classes of the relevant orientations of $G$ and let $\mathcal{R}_j = \{D_1^j, D_2^j, D_3^j, D_4^j\}$, $j = 1, \ldots, 4^{g-1}$.

Finally let $r(D_i^j)(k) = r_i^j(k)$, $k = 0, \ldots, g-1$ and let $t = t(M_0\Delta P)$. We have that

$$\mathcal{K}_g(t) = \sum_{j=1}^{4^{g-1}} \sum_{i=1}^{4} c(r(D_i^j))\sigma(r(D_i^j), t).$$

Now, by Lemma 5.25, this equals

$$\sum_{j=1}^{4^{g-1}} \sum_{i=1}^{4} c(r_i^j(0))c(r_i^j(1), \ldots, r_i^j(g-1)) \prod_{k=0}^{g-1} \sigma(r_i^j(k), t(k)).$$

By the definition of the equivalence classes, $r_1^j(k) = r_2^j(k) = r_3^j(k) = r_4^j(k)$ for $k \geq 1$ and $j = 1, \ldots, 4^{g-1}$. Hence, we let $r_i^j(k) = r^j(k)$ and write the above summation as:

$$\sum_{j=1}^{4^{g-1}} c(r^j(1), \ldots, r^j(g-1)) \prod_{k=1}^{g-1} \sigma(r^j(k), t(k)) \sum_{i=1}^{4} c(r_i^j(0))\sigma(r_i^j(0), t(0))$$

.

The internal sum equals to 1 for each $j = 1, \ldots, 4^{g-1}$ by the basis step of the induction, and hence, using Lemma 5.25 in the external sum, we can write the above summation as

$$\sum_{j=1}^{4^{g-1}} c(r^j(1), \ldots, r^j(g-1))\sigma((r^j(1), \ldots, r^j(g-1)), (t(1), \ldots, t(g-1))) =$$

$$\mathcal{K}_{g-1}(t(1), \ldots, t(g-1)) = 1,$$

by the inductive hypothesis for $g-1$. **End of Claim.**

□

As a consequence of Theorem 5.7 and Theorem 5.26, we have:

**Corollary 5.27** *Let $G$ be a proper $g$-graph. Then $s(D_i, M_0) = s(D_j, M_0)$ for each $i, j \in \{1, \ldots, 4^g\}$ and*

$$\mathcal{P}(G, x) = \mathcal{L}_g(G, x) = s(D_1, M_0) \sum_{i=1}^{4^g} c(r(D_i))Pf(A(D_i))$$

*where $D_i$, $i = 1, \ldots, 4^g$, are the relevant orientations of $G$.*

**Theorem 5.28** *Let $G$ be a graph embeddable on an orientable surface of genus $g$. Then $\mathcal{P}(G,x)$ may be expressed as a linear combination of $4^g$ Pfaffians of matrices $A(D)$, where each $D$ is an orientation of $G$.*

**Proof.** As observed in the previous section, any orientable surface $\mathcal{S}$ of genus $g$ may be obtained from its polygonal representation $S_g$ as follows: for each bridge $B$, glue together the two segments in which $B$ intersects the boundary of $B_0$, and delete $B$.

If a graph $G$ is embedded on an orientable surface $\mathcal{S}$ of genus $g$, then without loss of generality no vertex belongs to the boundary of $B_0$. In this way we get an embedding of $G$ on $S_g$ such that all vertices of $G$ belong to $B_0$ but the embeddings of some edges may use several bridges.

We construct a graph $G'$ by replacing each edge $e = uv$ which uses $k$ bridges, $k \geq 1$, by a path $P_e = (u, e_1, v_1, \ldots, v_{2k}, e_{2k+1}, v)$. The new vertices $v_1, ..., v_{2k}$ are embedded on the embedding of $e$ so that each new edge uses at most one bridge. Moreover, we let $x'_{e_1} = x_e$ and $x'_{e_i} = 1$ for each $i > 1$. We do a similar construction when $G_0$ has no perfect matching. In fact, take any perfect matching $M$ of $G$ and replace any edge $e = uv \in M$ embedded on a bridge by a path $u, e_1, y, e_2, z, e_3, v$ and let $x'_{e_1} = x_e$ and $x'_{e_2} = x'_{e_3} = 1$. Then leave the only edge $e_2$ to be embedded on the bridge $B$.

Finally, we add edges so that the outer face of the planar part is a cycle and we let $x'_e = 0$ for each such edge $e$.

It is easy to see that $G'$ is a proper $g$-graph and that $\mathcal{P}(G', x') = \mathcal{P}(G, x)$.

Now, by Theorem 5.26, $\mathcal{P}(G', x')$ may be written as a linear combination of $4^g$ terms $Pf(A(D'))$, where each $D'$ is a relevant orientation of $G'$.

It remains to show that for each relevant orientation $D'$ of $G'$ there is an orientation $D$ of $G$ such that $Pf(A(D')) = Pf(A(D))$ or $Pf(A(D')) = -Pf(A(D))$.

We construct $D$ from $D'$ in two steps:

1. delete the edges $e$ of $G' - G$ with $x'_e = 0$,

2. for each edge $e$ of $G$ which was changed into a path $P_e$ of odd length in the construction of $G'$, orient $e$ in the direction in which an odd number of edges of $P_e$ is directed in $D'$: this is uniquelly determined since $P_e$ has an odd length.

If $P$ is a perfect matching of $G$ then there is a unique perfect matching $P'$ of $G'$ such that $x(P) = x'(P')$.

Observe that $sgn(D, P\Delta Q) = sgn(D', P'\Delta Q')$ for each pair of perfect matchings $P, Q$ of $G$. The claim now follows from Theorem 5.7.

This finishes the proof of the theorem. □

Hence, here is a method to solve the Ising problem for finite graphs: translate the problem first to the problem of enumeration of the perfect matchings of a slightly modified graph, and then draw that graph on an orientable surface and use the theorem of Galluccio and Loebl. Now, how realistic is this approach? Given the complex nature of the procedure, we doesnot feel obliged to argue against (it is easy, isn't it?). So let us argue for it:

1. This approach lead to a very successful implementation (by Jan Vondrak) of Ising partition function calculation for toroidal graphs. In fact, it even gives the best performance for detecting maximum cuts for these graphs, when the weights are from $\{1, -1\}$. Just imagine it: for toroidal square grids, the best known computer program to detect max cut finds the whole density as well, i.e. the number of max cuts, the number of cuts which are second largest, etc.

2. There is a good chance that it will be applicable in knot theory, where some animals live on surfaces of bounded genus.

3. The lattices important to physics typically do not have bounded genus. Look for instance at the cubic lattice: its genus is proportional to its number of edges. However still we can apply the method to the cubic lattice, in principle. I include my derivation in the subsection below. I really do not know whether the expression it gives will be of any use: but, it is sufficiently crazy and beautiful and anyway, there doesnot seem to be any method with a slightest success towards solving the 3d Ising problem, so perhaps 'less straightforward' methods may be discussed.

## 5.5 3-dimensional dimer problem.

We present a new expression for the dimer problem, i.e. for the enumeration of dimer arrangements (perfect matchings) of the 3-dimensional cubic lattice. As a consequence this provides a new expression also for the 3-dimensional Ising problem.

The *close-packed dimer problem* of statistical mechanics can be stated as follws. One considers a set of sites and a set of bonds connecting certain pairs of sites. Each bond $b$ can absorb a 'dimer' (which represents a diatomic molecule) with corresponding energy $E_b$. It is required that each site is occupied exactly once by one of the atoms of a dimer. A state $s$ is an arrangement of dimers which meets this requirement, and its energy $E(s)$ is $\sum E_b$ where the sum is taken over all bonds $b$ which absorb a dimer. Then the partition function may be viewed simply as a density function of energy levels.

The dimer problem was first considered by Roberts ([20]) in 1935, and by Fowler and Rushbrook ([3]). The dimer problem for 2-dimensional lattices appears in calculations of the thermodynamic properties of a system of diatomic molecules-dimers. It was solved by Kasteleyn ([11]) and by Temperley, Fisher ([8]). The same problem for 3-dimensional lattices remains an important open problem of statistical physics (see [14] for references). Many fundamental observations about the dimer and monomer-dimer problem in general graphs are given by Heilmann, Lieb ([6, 7]).

We may reformulate the problem in graph theoretical terms as follows. A graph is a pair $G = (V, E)$ where $V$ is a set of *vertices* and $E$ is a set of unordered pairs of elements of $V$, called *edges*. A graph with some regularity properties is often called a *lattice*. We associate with each edge $e$ of $G$ a variable $x_e$ and we let $x = (x_e : e \in E)$. For each $M \subset E$, let $x(M)$ denote the product of the variables of the edges of $M$.

A subset of edges $P \subset E$ is called *perfect matching* or *dimer arrangement* if each vertex belongs to exactly one element of $P$. The dimer problem may be formulated as the problem to find polynomial $\mathcal{P}(G,x)$ which equals the sum of $x(P)$ over all perfect matchings $P$ of $G$.

Let $m$ be odd positive integer and $k$ even positive integer. The cubic lattice $Q_{m,m,k}$ is the following graph:

$Q_{mmk}$ has vertices $V_{xyz}(Q_{mmk})$, $x,y = 1,...,m$, $z = 1,...,k$, and the following edges:

1. The edges $v_{xyz}(Q_{mmk}) = \{V_{xyz}(Q_{mmk}), V_{xy(z+1)}(Q_{mmk})\}$, $z = 1,...,k-1$, called *vertical*,
2. The edges $w_{xyz}(Q_{mmk}) = \{V_{xyz}(Q_{mmk}), V_{x(y+1)z}(Q_{mmk})\}$, $y = 1,...,m-1$, called *width*,
3. The edges $h_{xyz}(Q_{mmk}) = \{V_{xyz}(Q_{mmk}), V_{(x+1)yz}(Q_{mmk})\}$, $x = 1,...m-1$, called *horisontal.*

Let us denote the ordered set $(V_{xy1}(Q_{mmk}),...,V_{xyk}(Q_{mmk}))$ by $V_{xy}(Q_{mmk})$. $V_{xy}(Q_{mmk})$ will also stand for the vertical path of $Q_{mmk}$ from $V_{xy1}(Q_{mmk})$ to $V_{xyk}(Q_{mmk})$. Sometimes we let $V_{xy}(Q_{mmk}) = V_{xy}$ if no confusion may arise and we denote by $\bar{V}_{xy}$ the oppositely ordered set $(V_{xyk}(Q_{mmk}),...,V_{xy1}(Q_{mmk}))$

$Q_{mmk}$ is a bipartite graph, which means that its vertices may be partitioned into two sets $Z_1, Z_2$ such that if $e$ is an edge of $Q_{mmk}$ then $|e \cap Z_1| = |e \cap Z_2| = 1$. Moreover, we have also that $|Z_1| = |Z_2| = mmk/2$. Let $\mathcal{Z}$ be square $(Z_1 \times Z_2)$ matrix defined by $\mathcal{Z}_{ij} = x_{ij}$ if $ij$ is an edge of $Q_{mmk}$ and $\mathcal{Z}_{ij} = 0$ otherwise.

We will consider matrix $\mathcal{Z}$ with its rows and columns ordered in aggreement with the order $(V_{11}(Q_{mmk})V_{12}(Q_{mmk})....V_{1m}(Q_{mmk})V_{21}(Q_{mmk}).....V_{mm}(Q_{mmk}))$ and we will assume that $V_{111}(Q_{mmk}) \in Z_1$.

Note that $\mathcal{P}(Q_{mmk},x)$ equals the permanent of $\mathcal{Z}$.

A seminal observation of Heilmann and Lieb [6, 7] asserts that $\mathcal{P}(Q_{mmk},x)$ equals the average of $(det(Z))^2$ over ALL signings $Z$ of $\mathcal{Z}$, where a *signing* of a matrix is obtained by multiplying some of its entries by $-1$.A proof of this observation included below is taken from the monograph [15]. We say that signing $Z$ of $\mathcal{Z}$ corresponds to orientation $D$ of $Q_{mmk}$ if $Z_{ij} = x_{ij}$ if $(ji) \in E(D)$, $Z_{ij} = -x_{ij}$ if $(ij) \in E(D)$, and $Z_{ij} = 0$ otherwise.

**Theorem 5.29** *$\mathcal{P}(Q_{mmk},x)$ equals the average of $(det(Z))^2$ over ALL signings $Z$ of $\mathcal{Z}$.*

**Proof.** If $D$ is an orientation of $Q_{mmk}$ then let $A(D)$ denote the skew-symmetric adjacency matrix of $D$, i.e. matrix consisting of 4 blocks where both blocks on the main diagonal are 0-matrices and the remaining two blocks equal $Z$ and $-Z$, where $Z$ is the signing of $\mathcal{Z}$ corresponding to $D$. Clearly $det(A(D)) = (det(Z))^2$, hence we need to show that $\mathcal{P}(Q_{mmk},x)$ equals the expectation of $det(A(D))$ over all orientations $D$ of $Q_{mmk}$. For the expectation we have

$$E(det(A(D))) = \sum sgn(\pi)E(a_{1\pi(1)}...a_{n\pi(n)})$$

where $n = mmk$ and $A(D) = (a_{ij})$. If $\pi$ is a permutation having a fix point or such that $i$ and $\pi(i)$ are non-adjacent for some $i \leq n$ then the term corresponding to $\pi$ equals 0. If there is $i$ such that $\pi(\pi(i)) \neq i$ then the random variable $a_{i\pi(i)}$ occurs in the term corresponding to $\pi$ but the random variable $a_{\pi(i)i}$ doesnot. Hence $E(a_{1\pi(1)}...a_{n\pi(n)}) = E(a_{i\pi(i)})E(a_{1\pi(1)}...a_{(i-1)\pi(i-1)}a_{(i+1)\pi(i+1)}...a_{n\pi(n)}) = 0$. So we are left with the terms corresponding to those permutations which have no fix point, for which $i$ and $\pi(i)$ are adjacent and $(\pi)^2$ is the identity. Such permutations uniquelly correspond to perfect matchings of $Q_{mmk}$ and the signs turn out correct. $\square$

Here we show that $\mathcal{P}(Q_{mmk}, x)$ may be computed from the average of determinants of CERTAIN signings of $\mathcal{Z}$.

A difference of our expression with the result of Heilmann and Lieb is that we replace the average of a multiquadratic function by the average of a multilinear function, with a restricted range though. This may help to solve the dimer problem and the Ising problem in three dimensions.

The proof of our result is involved: we embed the three-dimensional cubic lattice to a 2-dimensional orientable surface, use a Theorem of Galluccio and Loebl ([4]) to express $\mathcal{P}(Q_{mmk}, x)$ as a linear combination of Pfaffians of matrices associated with orientations of $Q_{mmk}$ and finally characterize the coefficients of this linear combination.

**Statement of the main result.**

If $D = (V, A)$ is an orientation of $Q_{mmk}$ and $e \in A$ then we say that $e$ is oriented according to the *natural ordering* if its orientation is in aggreement with the ordering $(V_{11}(Q_{mmk})V_{12}(Q_{mmk})....V_{1m}(Q_{mmk})V_{21}(Q_{mmk}).....V_{mm}(Q_{mmk}))$ of the vertices of $Q_{mmk}$.

An orientation $D$ is called *stable* if the edges of each $V_{xy}(Q_{mmk})$ are oriented in $D$ in aggreement with the natural ordering.

Let $D$ be a stable orientation. We define orientation $\bar{D}$ as follows:

1. For each $x, y, z$ such that $y < m$ odd do the following: let $n(xyz)$ be the number of arcs $w_{xab}$, $a \leq y$ odd and $b = z$ or $b = z+1$, oriented in $D$ against the natural ordering. If $n(xyz)$ odd then we orient $w_{xyz}$ in $\bar{D}$ against the natural ordering, else according to the natural ordering.

2. For each $x, y, z$ such that $x < m$ odd do the following: let $n(xyz)$ be the number of arcs $h_{abc}$ oriented in $D$ against the natural ordering. Here $(abc)$ are the triples of indices satisfying $a \leq x$ odd and $(b, c) = (y, z)$ or $(b, c) = (y', z')$ where $(y', z')$ is such that vertex $V_{x,y',z'}$ is immediate successor of vertex $V_{x,y,z}$ in the following ordering of vertices of $Q_{mmk}$: $(V_{11}, \bar{V}_{12}, V_{13}, ...V_{1m}, \bar{V}_{2m}, V_{2(m-1)}, ..., \bar{V}_{21}, V_{31}, ..., V_{mm})$.

   If $n(xyz)$ odd then we orient $h_{xyz}$ in $\bar{D}$ against the natural ordering, else according to the natural ordering.

3. All the remaining arcs orient in $\bar{D}$ in the same way as in $D$.

Let $D$ be a stable orientation. Then we let

$$sgn(D)) = (-1)^{h+\sum_{x=1}^{m} w(x)},$$

where $w(x) = |\{(yz); y$ even and both $w_{xyz}, w_{x,(y-1),z}$ are oriented against the natural ordering in $\bar{D}\}|$; $h = |\{(xyz); x$ even and both $w_{xyz}, w_{(x-1),y,z}$ are oriented against the natural ordering in $\bar{D}\}|$.

Let $D$ be an orientation of $Q_{mmk}$. Let us associate a signing $Z(D)$ of $\mathcal{Z}$ with it such that $Z(D)_{ij} = x_{ij}$ if $(ji) \in E(D)$, $Z(D)_{ij} = -x_{ij}$ if $(ij) \in E(D)$, and $Z(D)_{ij} = 0$ otherwise.

Now we state the main results of the paper. Let $\alpha = \prod_{e \in M} x_e$ where $M$ is unique perfect matching of the collection of paths $V_{11}(Q_{mmk}) \cup ... \cup V_{mm}(Q_{mmk})$.

**Theorem 5.30** *$\mathcal{P}(Q_{mmk}, x) = -2^{C_r}\alpha + \beta(2^{C_r} + 1)$, where $\beta$ equals the average of $det(Z(D))$, $D$ stable orientation of $Q_{mmk}$ with $sgn(D) = +1$, and $C_r = km(m-1)$.*

**Some basic notions and facts.**

**Definition 5.31** *Let $G$ be a graph and let $G'$ be an even subdivision of $G$. Let $D'$ be an orientation of $G'$. An orientation $D$ of $G$ induced by $D$ is constructed as follows: for each edge $e$ of $G$ which was changed into a path $P_e$ in the construction of $G'$, orient $e$ in the direction in which an odd number of edges of $P_e$ is directed in $D'$: this is uniquelly determined since $P_e$ has an odd length.*

Let $G$ be a graph and let $x$ be a vector of variables associated with the edges of $G$. Let $G'$ be an even subdivision of $G$ and let $x'$ be a vector of variables associated with the edges of $G'$ induced by $x$. If $P$ is a perfect matching of $G$ then observe there is a unique perfect matching $P'$ of $G'$ such that $x(P) = x'(P')$.

Observe that $sgn(D, P\Delta Q) = sgn(D', P'\Delta Q')$ for each pair of perfect matchings $P, Q$ of $G$. Hence the following theorem follows from Theorem 5.7.

**Theorem 5.32** *Let $G$ be a graph and let $x$ be a vector of variables associated with the edges of $G$. Let $G'$ be an even subdivision of $G$ and let $x'$ be a vector of variables associated with the edges of $G'$ induced by $x$. Let $D'$ be an orientation of $G'$ and let $D$ be the orientation of $G$ induced by $D'$. Let $M$ be an arbitrary perfect matching of $G$. Then $s(D, M)s(D', M')Pf(A(D')) = Pf(A(D))$.*

We remind that the following theorem is proved in Galluccio, Loebl [4].

**Theorem 5.33** *Let $G$ be a $g$-graph with a perfect matching $M_0 \subset E_0$. If we order the vertices of $G$ so that $s(D_0, M_0)$ is positive then*

$$\mathcal{P}(G, x) = \sum_{i=1}^{4^g} c(r(D_i))Pf(A(D_i))$$

*where $D_i$, $i = 1, \ldots, 4^g$, are the relevant orientations of $G$.*

We need a generalisation of the notion of a $g$-graph.

**Definition 5.34** *Any graph $G$ obtained by the following construction will be called generalised g-graph.*

1. *Let $g = g_1 + ... + g_n$ be a partition of $g$ into positive integers.*

2. *Let $S_{g_i}$ be a surface of genus $g_i$, $i = 1, ..., n$. Let us denote the basis and the bridges of $S_{g_i}$ by $B_0^i$ and $B_{j,k}^i$, $i = 1, ..., n$, $j = 1, ..., g_i$ and $k = 1, 2$.*

3. *For $i = 1, ..., n$ let $H_i$ be a $g_i$-graph with the property that the subgraph of $H_i$ embedded on $B_0^i$ is a cycle, embedded on the boundary of $B_0^i$. Let us denote it by $C^i$.*

4. *Let $G_0$ be a 2-connected plane graph and let $F_1, ..., F_n$ be a subset of faces of $G_0$. Let $K^i$ be the cycle boundering $F_i$, $i = 1, ..., n$. Let each $K^i$ be isomorphic to $C^i$.*

5. *Then $G$ is obtained by glueing the $H_i$'s into $G_0$ so that each $K^i$ is identified with $C_i$.*

For each generalised g-graph $G$ we can define $4^g$ relevant orientations $D_1, ..., D_{4^g}$ with respect to a fixed basic orientation of $G_0$, and coefficients $c(r(D_i))$, $i = 1, ..., n$ in the same way as for a g-graph. The following theorem can be proved in the same way as Theorem 5.26.

**Theorem 5.35** *Let $G$ be a generalised g-graph with a perfect matching $M_0$ of $G_0$. Let $D_0$ be a basic orientation of $G_0$. If we order the vertices of $G$ so that $s(D_0, M_0)$ is positive then*

$$\mathcal{P}(G, x) = \sum_{i=1}^{4^g} c(r(D_i)) Pf(A(D_i))$$

*where $D_i$, $i = 1, \ldots, 4^g$, are the relevant orientations of $G$.*

**Cubic lattices as generalised g-graphs.**

In this section we will describe how to draw 3−dimensional cubic lattices as generalised g-graphs.

**The Cubic lattice.**

Let $m, n$ be odd positive integers such that $k = (n-1)/2$ is even. The cubic lattice $Q = Q_{m,m,n}$ is the following graph: It has vertices $V_{xyz}(Q) = V_{xyz}$, $x, y = 1, ..., m$, $z = 1, ..., n$, and the following edges:

1. The edges $\{V_{xyz}, V_{xy(z+1)}\} = v_{xyz}(Q) = v_{xyz}$, $z = 1, ..., n-1$, called *vertical*,

2. The edges $\{V_{xyz}, V_{x(y+1)z}\} = w_{xyz}(Q) = w_{xyz}$, $y = 1, ..., m-1$, called *width*,

3. The edges $\{V_{xyz}, V_{(x+1)yz}\} = h_{xyz}(Q) = h_{xyz}$, $x = 1, ...m-1$, called *horisontal*.

Let us denote vertical subpath $(V_{xy1}, ..., V_{xyn})$ by $V_{xy}(Q) = V_{xy}$ and let $\bar{V}_{xy}$ denote $V_{xy}$ traversed in the oposite direction.

Let $H_x(Q) = H_x = \{h_{xyz}; z = 1, ..., n, y = 1, ..., m\}$ and $W_{xy}(Q) = W_{xy} = \{w_{xyz}; z = 1, ...n\}$.

**Next we describe a drawing of $Q$ on the plane.**

First draw the paths $V_{xy}$ along a cycle in the following order:

$$V_{11}, \bar{V}_{12}, V_{13}, ...V_{1m}, \bar{V}_{2m}, V_{2(m-1)}, ..., \bar{V}_{21}, V_{31}, ..., V_{mm}.$$

Next, draw the horisontal edges inside this cycle, and the width edges outside of this cycle as depicted in Fig. 1 below.

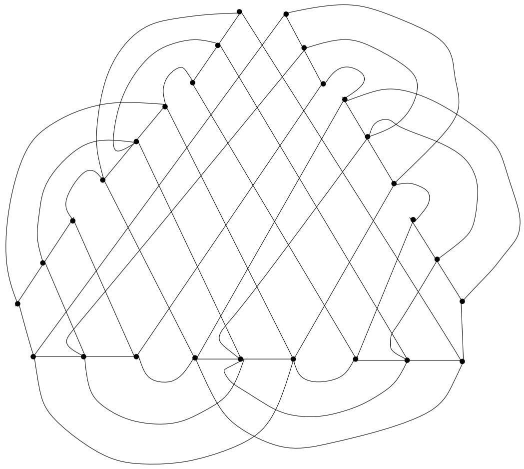

Figure 1

For each $x = 1, ..., m-1$ the curves representing the edges of $H_x$ are pairwise disjoint and for $x = 2, ..., m-2$ the curves representing the edges of $H_x$ intersect the curves representing the edges of $H_{x-1}$ and $H_{x+1}$. We keep the following rule: the interiors of the curves representing $h_{xyz}$ and $h_{(x+1)yz}$ intersect if and only if $z$ is even.

For each $x = 1, ..., m$ and $y = 1, ..., (m-1)$ the curves representing the edges of $W_{xy}$ are pairwise disjoint and for $y = 2, ..., m-2$ the curves representing the edges of $W_{xy}$ intersect the curves representing the edges of $W_{x(y-1)}$ and $W_{x(y+1)}$. We again keep the rule that the interiors of the curves representing $w_{xyz}$ and $w_{x(y+1)z}$ intersect if and only if $z$ is even. The curve representing an edge $e$ will be denoted by $C(e)$.

**Now we modify $Q$ into a generalised g-graph $Q'$.**

First we describe the modification for $W_x$, $x = 1, ..., m$. The modification is illustrated in Fig. 2 where the construction is illustrated on edges among $V_{x(y-1)}, V_{xy}$ and $V_{x(y+1)}$ for $y < m-1$ even.

For each $x = 1, ..., m$ perform the following construction:

1. For each $y$ even let $Aux_1 = \{w_{xyz}; z \text{ odd }\}$. For each edge $e$ of $Aux_1$ introduce a new vertex to each intersection of $C(e)$ with the curves representing the edges of $W_{x(y-1)} \cup W$, where $W = W_{x(y+1)}$ in case $y < m-1$ and $W = \emptyset$ otherwise.

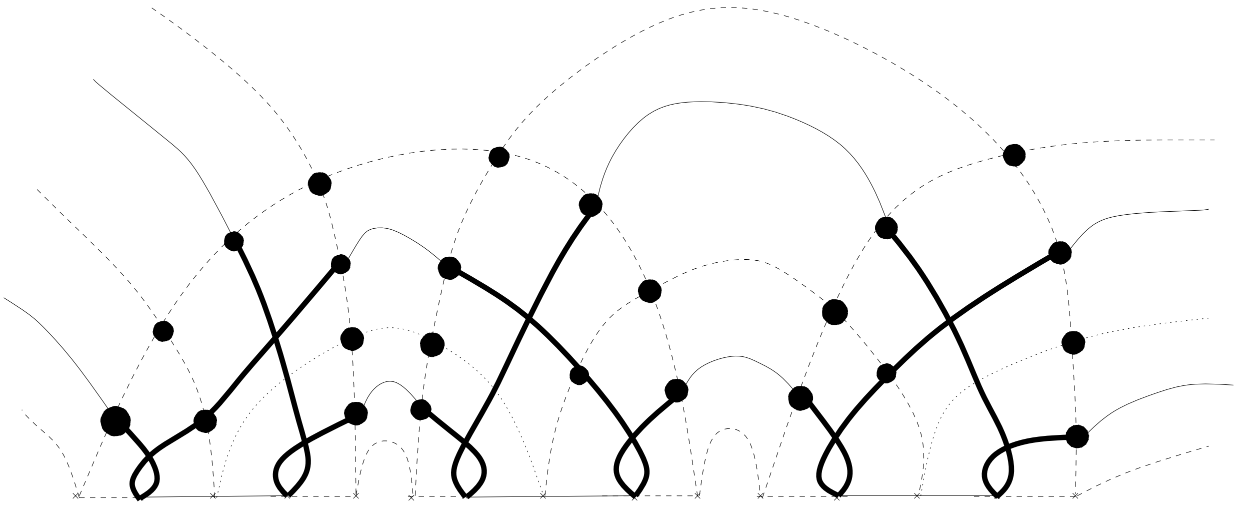

Figure 2

By this operation, each $e \in Aux_1$ is replaced by a path. Call each edge of this path *auxiliary.*

2. For each $y$ even let $Aux_2 = \{w_{x(y-1)1}, w_{x(y-1)n}\} \cup A$, where $A = \{w_{x,(y+1)1}, w_{x(y+1)n}\}$ in case $y < m-1$ and $A = \emptyset$ otherwise. For each edge $e$ of $Aux_2$ introduce a new vertex to each intersection of $C(e)$ with the curves representing the edges of $W_{xy}$. Hence each $e \in Aux_2$ is replaced by a path. Call each edge of this path *auxiliary.*

   For each $y$ even the edges $v_{xy1}, v_{xy(n-1)}$ and also $v_{x(y+1)1}, v_{x(y+1)(n-1)}$ will also be called *auxiliary.*

   In Figure 2, the auxiliary edges are represented by dashed lines.

3. We introduce a new variable $a$ and let $x_e = a$ for each auxiliary edge $e$.

4. The edges $w_{xyz}$, $y$ even and $z$ even will be called *relevant* for $Q$. If $y < m-1$ then the relevant edges are subdivided by two vertices (added in 2.) into three edges of $Q'$. The middle one will be called *special* and the other two *long.*

   If $y = m-1$ then the relevant edge $w_{xyz}$ is subdivided by one vertex into two edges of $Q'$. The one incident to $V_{xm}$ will be called *special* and the other one *long.*

   If $e$ is a relevant edge of $Q$, then we choose a corresponding long edge $f$ and we let $x_e = x_f$. We let the variable of the special edge and of the remaining long edge be equal to 1.

5. The edges of $W_{x(y-1)} \cup W$ also got subdivided by new vertices introduced in step 2 and step 3.

6. We delete all edges of the paths obtained from $w_{x(y-1)z}$ and $w_{x(y+1)z}$, $1 < z < n$ odd.

In Figure 2, the deleted edges are represented by dotted lines.

7. Each edge $w \in \{w_{x(y-1)z}, w_{x(y+1)z}; z \text{ even }\}$, is subdivided by new vertices introduced in 2. into a path. We let the variables assigned to the edges of the path equal to 1 except of one initial edge whose variable is let equal $x_w$. The edge $e$ of this path such that the interior of $C(e)$ doesnot intersect interior of any curve representing a long edge will also be *special*.

8. All vertical edges which are not auxiliary (see 2.) will be called *special*.

   In Figure 2, the special edges are represented by normal lines.

This finishes the construction for the width edges. In Figure 2, the edges which are neither auxiliary nor special nor deleted are represented by fat lines.

Now we perform an analogous construction with the horisontal edges of $Q$.

1. For each $x$ even let $Aux_3 = \{h_{xyz}; z \text{ odd }\}$. For each edge $e$ of $Aux_3$ introduce a new vertex to each intersection of $C(e)$ with the curves representing the edges of $H_{x-1} \cup K$, where $K = H_{x+1}$ in case $x < m-1$ and $K = \emptyset$ otherwise. By this operation, each $e \in Aux_3$ is replaced by a path. Call each edge of this path *auxiliary*.

2. For each $x$ even let $Aux_4 = \{h_{(x-1)11}, h_{(x-1)nn}\} \cup B$, where $B = \{h_{(x+1)11}, h_{(x+1)nn}\}$ in case $x < m-1$ and $B = \emptyset$ otherwise. For each edge $e$ of $Aux_4$ introduce a new vertex to each intersection of $C(e)$ with the curves representing the edges of $H_x$. Hence each $e \in Aux_4$ is replaced by a path. Call each edge of this path *auxiliary*.

3. We let $x_e = a$ for each auxiliary edge $e$.

4. The edges $h_{xyz}$, $x$ even and $z$ even will be called *relevant* for $Q$. If $x < m-1$ then the relevant edges are subdivided by two vertices (added in 2.) into three edges of $Q'$. The middle one will be called *special* and the other two *long*.

   If $x = m-1$ then the relevant edge $h_{xyz}$ is subdivided by one vertex into two edges of $Q'$. The one incident to $V_m$ . will be called *special* and the other one *long*.

   If $e$ is a relevant edge of $Q$, then we choose a corresponding long edge $f$ and we let $x_e = x_f$. We let the variable of the special edge and of the remaining long edge equal 1.

5. The edges of $H_{x-1} \cup K$ also got subdivided by new vertices introduced in step 2 and step 3.

6. We delete all edges of the paths obtained from $h_{(x-1)yz}$ and $h_{(x+1)yz}$, $1 < z < n$ odd.

7. Each edge $h \in \{h_{(x-1)yz}, h_{(x+1)yz};\ z \text{ even }\}$, is subdivided by new vertices introduced in 2. into a path. We let the variables assigned to the edges of the path equal to 1 except of one initial edge whose variable is let equal $x_h$. Each edge $e$ of this path such that the interior of $C(e)$ doesnot intersect interior of any curve representing a long edge will also be *special*.

Finally, let $Aux$ denote the set of all auxiliary edges. Then $Q' - Aux$ is a subdivision of $Q_{mmk}$. We conclude the construction of $Q'$ by subdividing some special edges so that the graph $\mathcal{Q} = Q' - Aux$ is an even subdivision of $Q_{mmk}$.

This finishes the construction of $Q'$.

**Some properties of $Q'$.**

1. Each edge $e$ of $Q'$ such that $C(e)$ doesnot intersect any curve representing other edge in its interior is auxiliary or special. Let us denote the plane subgraph of $Q'$ formed by the auxiliary and special edges by $Q^p$.

2. Any other edge of $Q'$ is drawn on a face of $Q^p$. Moreover, the edges drawn on a face of $Q^p$ may be drawn onto a pair of bridges above this face, where one of the bridges contains one long edge, and the other bridge contains the remaining edges. Hence, we may view $Q'$ as a generalised g-graph with the planar part equaled to $Q^p$.

3 The special edges form an acyclic subgraph of $Q^p$ (see Fig. 2). Hence any orientation of the special edges may be extended into a basic orientation of $Q^p$. We will choose basic orientation $D^p$ of $Q^p$ with the following properties:

   1. $D^p$ on special edges is in aggreement with the ordering $(V_{11}V_{12}....V_{1m}V_{21}.....V_{mm})$,
   2. $Q^p$ has a perfect matching $M$ whose sign in $Pf(A(D^p))$ is positive,
   3. The orientation of edges on a bridge has positive sign if and only if it is in aggreement with the ordering $(V_{11}V_{12}....V_{1m}V_{21}.....V_{mm})$.

4. We constructed $Q'$ so that $\mathcal{Q} = Q' - Aux$ is an even subdivision of $Q_{mmk}$. If $x$ is a vector of variables associated with $Q_{mmk}$ then let $x'$ be the vector of variables associated with $\mathcal{Q}$ and induced by $x$ and let $x''$ be a vector of variables associated with $Q'$ which equals $x'$ on the edges of $\mathcal{Q}$ and $x''_e = a$ for each special edge of $Q'$. If we let $a = 0$ then $\mathcal{P}(Q', x'') = \mathcal{P}(Q_{mmk}, x)$.

We have described how to view $Q_{mmk}$, $m$ odd and $k$ even, as a generalised g-graph $Q'$. Now we can use Theorem 5.35 for $Q'$ to compute $\mathcal{P}(Q_{mmk}, x)$.

**The relevant orientations of $Q'$.**

Each relevant edge of $Q$ corresponds to unique edge of $Q_{mmk}$; this unique edge will also be called *relevant* in $Q_{mmk}$. Hence the relevant edges of $Q_{mmk}$ are: $w_{xyz}(Q_{mmk})$, $x = 1, ..., m$, $y$ even and and $h_{xyz}(Q_{mmk})$, $x$ even. Hence there are $1/2km(m-1)$ relevant width edges and $1/2(m-1)mk$ relevant horisontal edges in $Q_{mmk}$.

We let $\mathcal{R}$ be the set of relevant edges of $Q_{mmk}$ and $C_r = |\mathcal{R}| = km(m-1)$ denote the number of relevant edges of $Q_{mmk}$.

The set $\mathcal{S}$ of the edges of $V_{ij}(Q_{mmk})$, $i, j = 1, ...m$, corresponds to a subset of special edges of $Q'$. The orientation $D^p$ induces orientation $\mathcal{S}^d$ of $\mathcal{S}$ which is in aggreement with the ordering $(V_{11}(Q_{mmk})V_{12}(Q_{mmk})....V_{1m}(Q_{mmk})V_{21}(Q_{mmk}).....V_{mm}(Q_{mmk}))$.

Each relevant orientation $D'$ of $Q'$ is determined by the fixed basic orientation $D^p$ of $Q^p$, and by a pair of signs for each pair of bridges. Each pair of bridges is associated with a long edge of $Q'$. Hence these signs may be given by specifying $(d^1_{D'}(e), d^2_{D'}(e)) \in \{+-\}^2$, for each long edge $e$, where $d^1_{D'}(e)$ denotes the sign of the bridge containing $e$, and $d^2_{D'}(e)$ denotes the sign of the other bridge.

The long edges of $Q'$ are associated with relevant edges of $Q$, and hence also with relevant edges of $Q_{mmk}$.

The relevant edges $w_{x(m-1)z}(Q_{mmk})$ and $h_{(m-1)yz}(Q_{mmk})$ are associated with only one long edge of $Q'$. If $e$ is such relevant edge of $Q_{mmk}$, we will call $e$ *border edge* and we denote by $e_1$ the corresponding long edge. We let $d_{D'}(e) = (d^1_{D'}(e_1), d^2_{D'}(e_1), +, +)$.

Let $C_b = 2mk$ denote the number of border edges.

Each relevant non-border edge $e$ of $Q_{mmk}$ has two long edges $e_1, e_2$ associated with it. We let $d_{D'}(e) = (d^1_{D'}(e_1), d^2_{D'}(e_1), d^1_{D'}(e_2), d^2_{D'}(e_2)))$.

A *relevant vector* is any element $r$ of $[\{+,-\}^4]^{\mathcal{R}}$ such that $r(e)_3 = r(e)_4 = 1$ for each relevant border edge $e$ of $Q_{mmk}$. Hence there are $4^{2C_r - C_b}$ relevant vectors.

There is a bijection between relevant orientations of $Q'$ and relevant vectors. If $s$ is a relevant vector, then let $D'(s)$ denote the corresponding relevant orientation of $Q'$ and let $sgn(s)$ equals 1 or $-1$ according to the sign of $c(r(D'(s)))$. The sign $sgn(s)$ of a relevant vector $s$ may be calculated according to Theorem 5.35 as follows: $sgn(s) = (-1)^{|\{(e,i); i=0,1, s(e)_{2i+1} = s(e)_{2i+2} = -1\}|}$.

If $D'(s)$ is a relevant orientation of $Q'$ then let $D(s)$ denote the orientation of $Q_{mmk}$ induced by $D'(s)$ (see 5.31). An orientation of $Q_{mmk}$ will be called *relevant* if it equals $D(s)$ for some relevant vector $s$.

Hence using Theorem 5.35 and Theorem 5.32 we have the following.

**Theorem 5.36**

$$\mathcal{P}(Q_{mmk}) = 2^{-2C_r + C_b} \sum sgn(s) Pf(A(D(s)))$$

*where the sum is over all relevant vectors.*

Note that possibly $D(r) = D(r')$ for relevant vectors $r \neq r'$. Next we clarify this.

**Definition 5.37** *We define an equivalence $*$ on the relevant vectors as follows. $r * s$ if the following holds: there is exactly one relevant non-border edge $e$ such that $r(e) \neq s(e)$ and $r(f) = s(f)$ for each $f \neq e$. Moreover, $r(e)_1 \neq s(e)_1$, $r(e)_3 \neq s(e)_3$, $r(e)_2 = s(e)_2 \neq r(e)_4 = s(e)_4$.*

**Proposition 5.38** *If $r * s$ then $D(r) = D(s)$ and $sgn(r) \neq sgn(s)$.*

**Proof.** If $r * s$ then $D(r) = D(s)$ by the definition of '*'. Moreover $sgn(r) \neq sgn(s)$ since $r(e)_2 = s(e)_2 \neq r(e)_4 = s(e)_4$ where $e$ is the only relevant edge for which $r(e) \neq s(e)$. □

**Definition 5.39** *A relevant vector is called stable if it forms a one-element class w.r.t. equivalence $*$.*

**Corollary 5.40**

$$\mathcal{P}(Q_{mmk}, x) = 2^{-2C_r+C_b} \sum sgn(r)Pf(A(D(r)))$$

*where the sum is over all stable vectors $r$.*

**Definition 5.41** *If $r, s$ are stable vectors we write $r**s$ if $D(r) = D(s)$.*

**Proposition 5.42** *1. Each equivalence class of '$**$' has $2^{C_r-C_b}$ elements.*

*2. If $r**s$ then $sgn(r) = sgn(s)$.*

**Proof.** Let $r$ be a stable vector. Then $D(r)$ determines uniquelly $r(e)_2$ and $r(e)_4$ for each relevant edge $e$ and also $r(f)_1$ and $r(f)_3$ for each relevant border edge $f$. Hence $D(r)$ determines uniquelly $r(f)$ for each relevant border edge $f$. Moreover $D(r)$ determines uniquelly the product $r(e)_1 \times r(e)_3$ for each relevant non-border edge $e$. Since there are $C_r - C_b$ relevant non-border edges, each equivalence class of '$**$' has $2^{C_r-C_b}$ elements.

Let $r, s$ be stable and let $r**s$. Then $r(e)_2 = s(e)_2 = s(e)_4 = r(e)_4$ for each relevant non-border edge $e$ and $r(e)_2 = s(e)_2$ and $s(e)_1 = r(e)_1$ for each relevant border edge. This implies that $sgn(r) = sgn(s)$. □

**Proposition 5.43** *If $D$ is an orientation of $Q_{mmk}$ that extends $\mathcal{S}^d$ then there is uniquelly determined class $C$ of equivalence $**$ such that $D = D(r)$ for each $r \in C$.*

Hence, given an orientation $D$ of $Q_{mmk}$ that extends $\mathcal{S}^d$, let us call it *stable orientation* and let us define its sign $sgn(D)$ to be equaled to $sgn(r)$ for any stable vector $r$ such that $D = D(r)$. This is a correct definition by Proposition 5.42.

Now we can formulate the main result of the paper.

**Theorem 5.44**

$$\mathcal{P}(Q_{mmk}, x) = 2^{-2C_r+C_b+C_r-C_b} \sum sgn(D)Pf(A(D))$$

*over all stable orientations $D$.*

We continue by characterising $sgn(D)$.

As we noticed before, the sign $sgn(r)$ of a relevant vector $r$ may be calculated according to Theorem 5.35 as follows: $sgn(r) = (-1)^{|\{(e,i);i=0,1,r(e)_{2i+1}=r(e)_{2i+2}=-1\}|}$. If $r$ is a stable vector then $r(e)_2 = r(e)_4$ for each relevant edge $e$ and we get the following observation.

**Proposition 5.45** *Let $r$ be a stable vector. Then $sgn(r) = (-1)^{|\{e;r(e)_1\times r(e)_3=-1,r(e)_2=-1\}|}$.*

**Definition 5.46** *Let $D$ be a stable orientation. We define orientation $\bar{D}$ as follows:*

1. *For each $x, y, z$ such that $y < m$ odd do the following: let $n(xyz)$ be the number of arcs $w_{xab}$, $a \leq y$ odd and $b = z$ or $b = z + 1$, oriented in $D$ against the natural ordering. If $n(xyz)$ odd then we orient $w_{xyz}$ in $\bar{D}$ against the natural ordering, else according to the natural ordering.*

2. *For each $x, y, z$ such that $x < m$ odd do the following: let $n(xyz)$ be the number of arcs $h_{abc}$ oriented in $D$ against the natural ordering. Here $(abc)$ are the triples of indices satisfying $a \leq x$ odd and $(b, c) = (y, z)$ or $(b, c) = (y', z')$ where $(y', z')$ is such that vertex $V_{x,y',z'}$ is immediate successor of vertex $V_{x,y,z}$ in the following ordering of vertices of $Q_{mmk}$: $(V_{11}, \bar{V}_{12}, V_{13}, ...V_{1m}, \bar{V}_{2m}, V_{2(m-1)}, ..., \bar{V}_{21}, V_{31}, ..., V_{mm})$.*

   *If $n(xyz)$ odd then we orient $h_{xyz}$ in $\bar{D}$ against the natural ordering, else according to the natural ordering.*

3. *All the remaining arcs orient in $\bar{D}$ in the same way as in $D$.*

Note that relevant edges are oriented in the same way in both $D$ and $\bar{D}$.

**Proposition 5.47** *Let $D$ be a stable orientation and let $r$ be a stable vector such that $D = D(r)$. Let $e$ be a relevant edge of $Q_{mmk}$. Then*

1. *$r(e)_1 \times r(e)_3 = -1$ if and only if $e$ is oriented in $D$ (and hence also in $\bar{D}$ against the natural ordering.*

2. *If $e = w_{xyz}$, $y$ even then $r(e)_2 = -1$ if and only if $w_{x(y-1)z}$ is oriented in $\bar{D}$ against the natural ordering. If $e = h_{xyz}$, $x$ even then $r(e)_2 = -1$ if and only if $h_{(x-1)yz}$ is oriented in $\bar{D}$ against the natural ordering.*

**Proof.** $r(e)_1 \times r(e)_3 = -1$ if and only if exactly one long edge of $Q'$ corresponding to $e$ is oriented in $D'$ (we remind that $D$ is induced by orientation $D'$ of $Q'$) against the natural ordering. Since $D$ is induced by $D'$, this happens if and only if $e$ is oriented in both $D$ and $\bar{D}$ against the natural ordering.

It remains to prove 2. We will show the case $e = w_{xyz}$ since the other case is completely analogous. If $f$ is an edge of $Q_{mmk}$ then we let $f(D) = 1$ if $f$ is oriented in $D$ according to the natural ordering, and we let $f(D) = -1$ otherwise. We proceed by induction on $(y, k - z)$.

Firstly assume $y = 2$ and $z = k$. In this simplest case $r(e)_2 = w_{x1k}(D)$ (see Fig. 2). Moreover the orientation of $w_{x1k}$ is the same in both $D$ and $\bar{D}$.

Secondly let $y = 2$ and $z < k$. Then $w_{x1z}(D) = w_{x1(z+1)}(D) \times r(e)_2$ (see Fig. 2). It follows from Definition 5.46 that $r(e)_2 = -1$ if and only if $w_{x1z}$ is oriented in $\bar{D}$ against the natural ordering.

Thirdly, let $y = 4$ and $z = k$. Then $w_{x3k}(D) = w_{x1k}(D) \times r(e)_2$ (see Fig. 2). It follows from Definition 5.46 that $r(e)_2 = -1$ if and only if $w_{x3k}$ is oriented in $\bar{D}$ against the natural ordering.

Forthly, let $y = 4$ and $z < k$. Then $w_{x3z}(D) = w_{x3(z+1)}(D) \times r(e)_2 \times r(w_{x2z})_2 = w_{x3(z+1)}(D) \times r(e)_2 \times w_{x1z}(D) \times w_{x1(z+1)}(D)$ (see Fig. 2). It follows from Definition 5.46 that $r(e)_2 = -1$ if and only if $w_{x3z}$ is oriented in $\bar{D}$ against the natural ordering.

In general if $e = w_{xyz}$, $y$ even and $z = k$ then $w_{x(y-1)k}(D) = r(w_{x(y-2)k})_2 \times r(e)_2 = r(e)_2 \times \prod(w_{xy'k}(D); y' < y-1 \text{ odd })$. It follows from Definition 5.46 that $r(e)_2 = -1$ if and only if $w_{x(y-1)k}$ is oriented in $\bar{D}$ against the natural ordering.

Finally if $e = w_{xyz}$, $y$ even and $z < k$ then $w_{x(y-1)z}(D) = w_{x(y-1)(z+1)}(D) \times r(e)_2 \times r(w_{x(y-2)z})_2 = w_{x(y-1)(z+1)}(D) \times r(e)_2 \times \prod(w_{xy'z}(D); y' < y-1 \text{ odd }) \times \prod(w_{xy'(z+1)}(D); y' < y-1 \text{ odd })$. It follows from Definition 5.46 that $r(e)_2 = -1$ if and only if $w_{x3z}$ is oriented in $\bar{D}$ against the natural ordering. □

This proposition has the following corollary.

**Corollary 5.48** *Let $D$ be a stable orientation. Then $sgn(D)) = (-1)^{h+\sum_{x=1}^{m} w(x)}$, where $w(x) = |\{(yz); y$ even and both $w_{xyz}, w_{x,(y-1),z}$ are oriented against the natural ordering in $\bar{D}\}|$; $h = |\{(xyz); x$ even and both $w_{xyz}, w_{(x-1),y,z}$ are oriented against the natural ordering in $\bar{D}\}|$.*

**Proposition 5.49** *There are $2^{2C_r}$ stable orientations. There are $2^{C_r-1}(2^{C_r}+1)$ stable orientations with positive sign.*

**Proof.** The first statement follows directly from the definition of a stable orientation. For $Q_{132}$ there are 16 stable orientations, from which 10 have positive sign. For $Q_{152}$ there are $4^4$ stable orientations from which $10 \times 10 + 6 \times 6$ have positive sign and $2(6 \times 10) = 120$ have negative sign. For $Q_{1(2a+1)2}$ there are $4^{2a}$ stable orientations, and the difference between the number of stable orientations of positive sign and stable orientations of negative sign is $2^{2a}$. For $Q_{mmk}$ the difference between the number of stable orientations of positive sign and those of negative sign equals $2^{(m-1)km} = 2^{C_r}$. From this Proposition follows. □

**From Pfaffians to determinants.**

In the introduction we let $\mathcal{Z}$ be square $(Z_1 \times Z_2)$ matrix defined by $\mathcal{Z}_{ij} = x_{ij}$ if $ij$ is an edge of $Q_{mmk}$ and $\mathcal{Z}_{ij} = 0$ otherwise.

Let $D$ be an orientation of $Q_{mmk}$. In the introduction we associate a signing $Z(D)$ of $\mathcal{Z}$ with it such that $Z(D)_{ij} = x_{ij}$ if $(ji) \in E(D)$, $Z(D)_{ij} = -x_{ij}$ if $(ij) \in E(D)$, and $Z(D)_{ij} = 0$ otherwise.

Note that $Pf(A(D)) = det(Z(D))$. Hence we can reformulate Theorem 5.44:

**Corollary 5.50** $\mathcal{P}(Q_{mmk}, x) = 2^{-C_r} \sum sgn(D)det(Z(D))$ *where the sum is over all stable orientations $D$ of $Q_{mmk}$.*

Let $\alpha = \prod_{e \in M} x_e$ where $M$ is unique perfect matching of the collection of paths $V_{11}(Q_{mmk}) \cup ... \cup V_{mm}(Q_{mmk})$.

**Proposition 5.51** *The average of $det(Z(D))$, $D$ stable, equals $\alpha$.*

Hence finally we can formulate Theorem 5.44 as follows.

**Theorem 5.52** *$\mathcal{P}(Q_{mmk}, x) = -2^{C_r}\alpha + \beta(2^{C_r}+1)$, where $\beta$ equals the average of $det(Z(D))$, $D$ stable orientation with $sgn(D) = +1$, and $C_r = km(m-1)$.*

**Proof.** By Corollary 5.50 and Proposition 5.52 we have that
$\mathcal{P}(Q_{mmk}, x) = 2^{-C_r}[-2^{2C_r}\alpha + 2\beta(2^{C_r-1}(2^{C_r}+1)] = -2^{-C_r+2C_r}\alpha + \beta(2^{-C_r+1+C_r-1})(2^{C_r}+1).$
□

Well, this may be a good end of the section on how to solve the Ising problem. In the next short section let us turn our attention to calculation of permanents.

# 6 On permanents.

If $A = (a_{ij})$ is a square $(n \times n)$ matrix then *permanent* of $A$ is defined by

$$per(A) = \sum_{\pi} \prod_{i=1}^{n} a_{i\pi(i)}$$

where $\pi$ ranges over the permutations of $1, ..., n$. In other words, permanent is defined in a 'similar' way as determinant, one only 'forgets' the signs of the terms. The signs make the two notions completely different from the computational point of view (and most of other views as well).

There is an efficient algorithm to compute the determinant, while Valiant proved that the problem of computing the permanent of a $(0,1)$-matrix is $\#P$-complete [24].

In 1913, Pólya [18] suggested computing the permanent of a matrix $A$ by changing the signs of some entries of $A$ so that the determinant of the resulting matrix equals the permanent of $A$. A matrix obtained from $A$ by changing the signs of some entries will be called *signing of $A$*. Let us call a matrix $A$ *convertible* if such a change is possible.

Szegö [21] pointed out in the same year that not all matrices are convertible.

The computational problem of recognising the convertible matrices had been studied extensively and it was proved recently to admit a polynomial algorithm in a seminal work by McCuaig, Robertson, Seymour and Thomas [17]. The problem of recognizing convertible matrices is known to be equivalent to many computational problems. Let us mention just one of them here, the one which gave it a name. It is the *Even Cycle Problem*: given a directed graph, decide whether it contains a directed cycle of even length.

Let $A$ be a square matrix. Denote by $G(A)$ the bipartite graph whose two bipartition classes are indexed by the rows and the columns of $A$, and for each edge $ij$, $a_{ij} = x_{ij}$. Then $per(A) = \mathcal{P}(G(A), x)$.

Let $A$ be a matrix. A seminal observation of Heilmann and Lieb [6, 7], theorem5.29 asserts that $per(A) = \mathcal{P}(G(A), x)$ equals the average of $(det(Z))^2$ over ALL signings $Z$ of A.

It was proved by Galluccio and Loebl [4] that the method of Pólya may be completed as follows:

**Theorem 6.1** *Let $A$ be a square matrix. Then $per(A)$ may be expressed as a linear combination of terms of the form $det(A^i)$, $i = 1, ..., 4^g$, where each $A^i$ is a signing of $A$ and $g$ is the genus of $G(A)$.*

This theorem immediately follows from the considerations of the previous section.

Again, let us note that a difference of this expression with the result of Heilmann and Lieb is that the average of a multiquadratic function is replaced by a linear combination of multilinear functions.

# 7 Back to binary codes.

In this last section let us return back to the linear codes.

**Definition 7.1** *Let $k \geq 1$ and let $G = (V, E)$ be a graph. A pair $(\{V_1, ..., V_k\}, E')$ is called $k$-cut if $\{V_1, ..., V_k\}$ is a partition of $V$ into $k$ non-empty disjoint subsets and $E'$ is the set of all edges with the endvertices in different parts $V_i$, $i = 1, ..., k$.*

**Definition 7.2** *The generating function of the $k$-cuts is the polynomial $\mathcal{C}_k(G, x)$ which equals the sum of $x(C)$ over all $k$-cuts $(\{V_1, ..., V_k\}, C)$ of $G$.*

First let us extend the considerations of van der Waerden from the section of dualities:

We use the following notation:

$$sinh(z, x) = \frac{z^x - z^{-x}}{2}, \quad cosh(z, x) = \frac{z^x + z^{-x}}{2}, \quad th(z, x) = \frac{sinh(z, x)}{cosh(z, x)}.$$

Note that $sinh(x) = sinh(e, x)$ and $cosh(x) = cosh(e, x)$.

Given a graph $G = (V, E)$, we denote by $\sigma \in \{1, \ldots, k\}^V$ a $|V|$-dimensional vector whose components $\sigma_i$, $i = 1, \ldots, |V|$, take values in the set $\{1, \ldots, k\}$. Clearly, any such vector identifies a partition of $V$ into $i \leq k$ disjoint sets and, consequently, an $i$-cut of $G$.

Let us denote by $\delta$ the vector indexed by the edges of $G$ whose component $\delta_{ij} = \delta(\sigma_i \sigma_j)$, $ij \in E$, equals 1 if $\sigma_i = \sigma_j$ and $-1$ otherwise.

Moreover, for any $A \subset E$ we let

$$U_k((V, A)) = \sum_{\sigma \in \{1, \ldots, k\}^V} \prod_{ij \in A} \delta(\sigma_i \sigma_j).$$

**Theorem 7.3** *Let $G = (V, E)$ be a graph, $z$ a variable and $k > 1$. Then*

$$z^{\sum_{f \in E} x_f}[k + \sum_{i=2}^{k} i! \binom{k}{i} \mathcal{C}_i(G, (z^{-2x_f} : f \in E))] =$$

$$(\prod_{f \in E} cosh(z, x_f)) \sum_{A \subseteq E} U_k((V, A)) \prod_{f \in A} th(z, x_f).$$

**Proof.** Using the identity

$$z^{x\delta(\sigma_i\sigma_j)} = cosh(z,x) + \delta(\sigma_i\sigma_j)sinh(z,x)$$

the result follows after some algebraic manipulations. In fact,

$$z^{\sum_{f\in E} x_f}[k + \sum_{i=2}^{k} i!\binom{k}{i}\mathcal{C}_i(G,(z^{-2x_f} : f \in E))] = \sum_{\sigma\in\{1,\ldots,k\}^V} (\prod_{ij\in E} z^{\delta(\sigma_i\sigma_j)x_{ij}}) =$$

$$\sum_{\sigma\in\{1,\ldots,k\}^V} (\prod_{ij\in E} (cosh(z,x_{ij}) + \delta(\sigma_i\sigma_j)sinh(z,x_{ij}))) =$$

$$(\prod_{f\in E} cosh(z,x_f)) \sum_{\sigma\in\{1,\ldots,k\}^V} (\prod_{ij\in E} (1 + \delta(\sigma_i\sigma_j)th(z,x_{ij}))) =$$

$$(\prod_{f\in E} cosh(z,x_f)) \sum_{\sigma\in\{1,\ldots,k\}^V} \sum_{A\subseteq E} (\prod_{ij\in A} \delta(\sigma_i\sigma_j)th(z,x_{ij})) =$$

$$(\prod_{f\in E} cosh(z,x_f)) \sum_{A\subseteq E} U_k((V,A)) \prod_{f\in A} th(z,x_f)$$

where $\prod_{ij\in A}\delta(\sigma_i\sigma_j)th(z,x_{ij}) = 1$ if $A = \emptyset$. Hence, the theorem follows. □

Specializing the above result to the case of edge-cuts, i.e. $k = 2$, we obtain the following result:

**Theorem 7.4** *Let $G$ be a graph, $z$ a variable and let $\mathcal{C}'_2(G,x) = \mathcal{C}_2(G,x) + 1$. Then*

$$2z^{\sum_{f\in E} x_f}\mathcal{C}'_2(G,(z^{-2x_f} : f \in E)) = (\prod_{f\in E} cosh(z,x_f))2^{|V|}\mathcal{E}(G,(th(z,x_f) : f \in E)).$$

**Proof.** We have, from Theorem 7.3, that

$$2z^{\sum_{f\in E} x_f}\mathcal{C}'_2(G,(z^{-2x_f} : f \in E)) = (\prod_{f\in E} cosh(z,x_f)) \sum_{A\subseteq E} U_2((V,A)) \prod_{f\in A} th(z,x_f).$$

Now observe that if $A \subset E$ is a cycle and $\sigma \in \{1,2\}^V$ arbitrary then $\prod_{ij\in A}\delta(\sigma_i\sigma_j) = 1$. Hence $U_2((V,A)) = 2^{|V|}$ when $(V,A)$ is an eulerian subgraph. Moreover, if $(V,A)$ is not an eulerian subgraph, then observe that $U_2((V,A)) = 0$.

□

**Theorem 7.5** *Let $G = (V,E)$ be a connected graph. Then*

$$A_{C(GF[q],\mathcal{N}_G)}(t) = 1 + \sum_{i=2}^{q}(q-1)\ldots(q-i+1)\mathcal{C}_i(G,(t,\ldots,t)).$$

**Proof.** Let $C = C(GF[q], \mathcal{N}_G)$. We have $C = \{O_G^T x; x \in GF[q]^V\}$ and $A_C(t)$ is the weight enumerator of $C$. Let us define an equivalence on $GF[q]^V$ by $x \equiv y$ if $O_G^T x = O_G^T y$. Observe that each equivalence class consists of $q$ elements since $O_G^T x = O_G^T y$ if and only if $x - y$ is a constant vector, i.e. $(x-y)_i = (x-y)_j$ for each $i, j \in \{1, ..., |V|\}$. Let $C^+$ be the system (in contrast with *a set*, some elements of a system may appear several times) defined by $C^+ = (O_G^T x; x \in GF[q]^V)$. Let $A_{C^+}(t) = \sum_{i=0}^{|E|} A_i^+ t^i$, where $A_i^+$ equals the number of vectors of $C^+$ with $i$ non-zero components. Since each equivalence class of $\equiv$ consists of $q$ elements we have $A_{C^+}(t) = qA_C(t)$.

If $x = (x_1, ..., x_{|V|}) \in GF[q]^V$ then let $l_x$ be the number of different values of $x_i$, $i = 1, ..., |V|$ and let $V_1^x, ..., V_{l_x}^x$ be the partition of $V$ into $l_x$ non-empty classes such that the components of $x$ are equal in each class.

Let $cut(x)$ be the subset of $E$ formed by those edges having endvertices in different sets $V_i^x$, $i = 1, ..., l_x$ and let $Cut(x) = (\{V_1, ..., V_{l_x}\}, cut(x))$.

Each $Cut(x)$ is an $l_x$-cut of $G$ and the weight of the codeword $O_G^T x$ equals $|cut(x)|$. Let $C^{++}$ be the system defined by $C^{++} = (cut(x); x \in GF[q]^V)$. Let $A_{C^{++}}(t) = \sum_{W \in C^{++}} t^{|W|}$. We have $A_{C^+}(t) = A_{C^{++}}(t)$.

For $i = 1, ..., q$ let $X_i = \{x \in GF[q]^V; l_x = i\}$. Define an equivalence on $X_i$ by $x \equiv^* y$ if $Cut(x) = Cut(y)$. Observe that each equivalence class of $\equiv^*$ consists of $q(q-1)...(q-i+1)$ elements. Hence

$$qA_C(t) = A_{C^+}(t) = A_{C^{++}}(t) = q + \sum_{i=2}^{q} q(q-1)...(q-i+1)\mathcal{C}_i(G, (t, ..., t)).$$

This proves the Theorem.

□

By Corollary 3.4 and Theorem 7.5 we have

**Corollary 7.6** *Let $G = (V, E)$ be a connected graph and let $\mathcal{N}_G$ be the graphic matroid of $G$. If $(x-1)(y-1) = 2$ and $0 \neq y \neq 1$ then*

$$T(G, x, y) = T(\mathcal{N}_G, x, y) = y^{|E|}(y-1)^{1-|V|}[1 + \mathcal{C}_2(G, (y^{-1}, ..., y^{-1})].$$

It follows that the Tutte polynomial of a graph of genus $g$ may be expressed along the hyperbola $(x-1)(y-1) = 2$ as a linear combination of $4^g$ Pfaffians, and hence it may be determined efficiently for the graphs embeddable on an arbitrary fixed orientable surface.

It is natural to ask whether there is an analogy of this statement for binary matroids.

## 7.1 Flow polynomial of graphs.

The dual notion to coloring is nowhere-zero flow. There are very nice facts known about it, and interestingly this notion seem to lead naturally to applications in knot theory and in statistical physics. This will not be touched here.

We have $C(GF[q], \mathcal{N}_G)^* = \{z \in GF[q]^E; O_G z = 0\}$. The elements of $C(GF[q], \mathcal{N}_G)^*$ are *flows* on $G$ with values in $GF[q]$. An element of $C(GF[q], \mathcal{N}_G)^*$ is called a *nowhere-zero flow* if its weight equals $|E|$. Let $F'(G,q)$ be the subset of $C(GF[q], \mathcal{N}_G)^*$ consisting of nowhere-zero flows. $F(G,q) = |F'(G,q)|$ is called the *flow polynomial* of $G$.

Theorems 3.2, 7.5 express a duality between flows and cuts of a graph. It is a duality of the Tutte polynomial.

In fact, there was a paper partly on this subject for Jarni skola last year: J. Nesetril, A. Raspaud, Duality, Nowhere-Zero Flows, and Cycle Covers.

The following theorem was proved by Tutte ([23]).

**Theorem 7.7** *Let $G = (V, E)$ be a graph and let $q$ be a power of a prime. Then*

$$F(G,q) = (-1)^{|V|-|E|-c(G)} T(G, 0, 1-q).$$

We give an interesting expression for $F(G,q)$ which is new as far as we know.

**Theorem 7.8** *Let $G = (V, E)$ be a graph. Then*

$$F(G,q) = q^{-|V|} 2^{-|E|} \sum_{A \subset E} U_q((V,A)) q^{|A|} (q-2)^{|E|-|A|}.$$

**Proof.** Let $C = C(GF[q], \mathcal{N}_G)$ and let $D = C(GF[q], \mathcal{N}_G)^*$. By Theorem 3.2 we have

$$A_D(t) = q^{1-|V|}[1+(q-1)t]^{|E|} A_C((1-t)(1+(q-1)t)^{-1}).$$

From Theorem 7.5 and Theorem 7.3 we get for $z > 0$

$$qA_C(z^{-1}) = q + \sum_{i=2}^{q} q(q-1)\ldots(q-i+1)\mathcal{C}_i(G, (z^{-1}, \ldots, z^{-1})) =$$

$$2^{-|E|} z^{-|E|} \sum_{A \subset E} U_q((V,A))(z-1)^{|A|}(z+1)^{|E|-|A|}.$$

If we let $z = (1+(q-1)t)(1-t)^{-1}$ we get for all $t > 0$, $t \neq 1$

$$A_D(t) = q^{-|V|} 2^{-|E|} \sum_{A \subset E} U_q((V,A))(qt)^{|A|}(2+(q-2)t)^{|E|-|A|}.$$

It follows that the leading coefficient of $A_D(t)$, which equals $F(G,q)$, is equal to

$$q^{-|V|} 2^{-|E|} \sum_{A \subset E} U_q((V,A)) q^{|A|} (q-2)^{|E|-|A|}.$$

$\square$

# References


[1] A. Cayley. Sur les determinants gauches. *Crelle's J.*, 38:93–96, 1847.

[2] M.E. Fisher. On the dimer solution of planar Ising models. *Journal of Mathematical Physics 7*, 10, 1966.

[3] R.H. Fowler and G.S. Rushbrooke. *Trans. Faraday Soc.*, 33:1272, 1937.

[4] A. Galluccio and M. Loebl. A theory of pfaffian orientations I: Perfect matchings and permanents. *Electronic Journal of Combinatorics*, 1998.

[5] C. Greene. Weight enumeration and the geometry of linear codes. *Studies in Applied Mathematics*, 55:119–128, 1976.

[6] Lieb Heilmann. Monomers and dimers. *Phys. Rev. Letters*, 24:1412–1414, 1970.

[7] Lieb Heilmann. Theory of monomer dimer systems. *Comm. Math. Phys.*, 25:190–232, 1972.

[8] M.E. Fisher H.N.V. Temperley. *Phil. Mag. Serie 8*, 6:1961, 1961.

[9] M. Kac and J.C. Ward. A combinatorial solution of the two-dimensional Ising model. *Physical Review*, 88, 1952.

[10] P. W. Kasteleyn. The statistics of dimers on a lattice. *Physica*, 27:1209–1225, 1961.

[11] P. W. Kasteleyn. The statistics of dimers on a lattice. *Physica*, 27:1209–1225, 1961.

[12] P. W. Kasteleyn. Dimer statistics and phase transitions. *Jour. Math. Physics*, 4:287–293, 1963.

[13] P.W. Kasteleyn. Graph theory and crystal physics. In *Graph theory and theoretical physics*, New York, 1967. Academic Press.

[14] P.W. Kasteleyn. Graph theory and crystal physics. In *Graph theory and theoretical physics*, New York, 1967. Academic Press.

[15] M.D. Plummer L. Lovasz. *Matching Theory.* Annals of Discrete Mathematics, 1986.

[16] J. MacWilliams. A theorem on the distribution of weights in a systematic code. *Bell Syst. Tech. J.*, 42:654, 1962.

[17] W. McCuaig, N. Robertson, P. D. Seymour, and R. Thomas. Permanents, pfaffian orientations, and even directed circuits. 1996. Preprint.

[18] G. Polya. Aufgabe 424. *Arch. Math. Phys.*, 20:271, 1913.

[19] R.B. Potts and J.C. Ward. The combinatorial method and the two-dimensional Ising model. *Progress of Theoretical Physics*, 13, 1955.

[20] J.K. Roberts. *Proc. Roy. Soc. (London) A*, 161:141, 1935.

[21] G. Szego. Lozung zu 424. *Arch. Math. Phys.*, 21:291–292, 1913.

[22] W.T. Tutte. A ring in graph theory. *Proc. Camb. Phil. Soc.*, 43:26–40, 1947.

[23] W.T. Tutte. A contribution to the theory of chromatic polynomials. *Canad. J. Math.*, 6:80–91, 1954.

[24] L. G. Valiant. The complexity of computing the permanent. *Theoret. Comput. Sci.*, 8:189–201, 1979.

[25] H. Whitney. A logical expansion in mathematics. *Bull. Amer. Math. Soc.*, 38:572–579, 1932.