On the inclusion-exclusion principle.

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April 18, 2000

Abstract

We will be walking for some time where the Inclusion-Exclusion Principle is leading us.

*Supported by GAUK 158/99

1 Preliminaries.

A graph is a pair $G = (V, E)$ where V is a set of *vertices* and E is a set of unordered pairs of elements of V, called *edges*. In this paper we shall consider only graphs with finite number of vertices. If $e = xy$ is an edge then the vertices x, y are called *endvertices* of e. The edges will often have *weight*, i.e. we are often given a function w from E to real numbers. If $A \subset E$ then we let $w(A) = \sum w(e); e \in A$.

A graph $G' = (V', E')$ is called a *subgraph* of a graph $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. A *perfect matching* of a graph is a set of disjoint edges, whose union equals the set of the vertices. A subgraph $G' = (V, E')$ is called *even* if each degree of G' is even.

Let $\{v_1, e_1, v_2, e_2, ..., v_i, e_i, v_{i+1}, ..., e_n, v_{n+1}\}$ be a sequence such that each v_j is a vertex of a graph G, each e_j is an edge of G and $e_j = v_j v_{j+1}$, and $v_i \neq v_j$ for $i < j$ except if $i = 1$ and $j = n + 1$. If also $v_1 \neq v_{n+1}$ then P is called *a path* of G. If $v_1 = v_{n+1}$ then P is called *a cycle* of G. In both cases the *length* of P equals n. When no confusion arises we shall also denote paths by simply listing their edges, namely $P = (e_1, e_2, \ldots, e_n)$.

A graph $G = (V, E)$ is *connected* if it has a path between any pair of vertices, and it is 2-*connected* if the graph $G_v = (V - \{v\}, \{e \in E; v \notin e\})$ is connected for each vertex v of G. Each maximal 2-connected subgraph of G is called a 2-*connected component* of G.

An*edge-cut* of a graph $G = (V, E)$ is a partition of its vertices into two disjoint subsets $V_1, V_2 \subset V$, and the implied set of edges between the two parts: $C(V_1, V_2) = \{\{u, v\} \in E : u \in V_1, v \in V_2\}$

The *generating function of edge-cuts* is a polynomial $\mathcal{C}(G, x)$ which equals the sum of $x^{w(C)}$ over all cuts C of G.

An *even subgraph* of a graph $G = (V, E)$ is a set of edges $U \subset E$ such that each vertex of V is incident with an even number of edges from U.

The *generating function of eulerian subgraphs* is a polynomial $\mathcal{E}(G, x)$ which equals the sum of $x^{w(U)}$ over all eulerian subgraphs U of G.

Let $A \Delta B$ denote the symmetric difference of the sets A and B and let $a \stackrel{2}{=} b$ denote $a = b$ modulo 2.

An *orientation* of a graph $G = (V, E)$ is a *digraph* $D = (V, A)$ obtained from G by fixing an orientation of each edge of G, i.e., by ordering the elements of each edge of G. The elements of A are called *arcs*.

An *embedding* of a graph on a surface is defined in a natural way: the vertices are embedded as points, and each edge is embedded as a continuous non-self-intersecting curve connecting the embeddings of its endvertices. The interiors of the embeddings of the edges are pairwise disjoint and the interiors of the curves embedding edges do not contain points embedding vertices.

A graph is called *planar* if it may be embedded on the plane. A *plane graph* is a planar graph together with its planar embedding. The embedding of a plane graph partitions the plane into connected regions called *faces*. The (unique) unbounded face is called *outer face* and the bounded faces are called *inner faces*.

Let G be a plane graph. A subgraph of G consisting of the vertices and the edges embedded on the boundary of a face will also be called *a face*. If a plane graph is 2-connected then each face is a cycle.

A *matroid* is a pair (X, M) where X is a finite set and $M \subset 2^X$ satisfies some axioms. These notes are concerned with *representable* matroids only. There, set X equals the columns of a matrix A over a field F, and M consists of the linearly independent subsets of X. We say that such a matroid is represented over F by A, and matrix A is its representation over F. The matroids represented over $GF[2]$ are called *binary matroids*. If $S \subset X$ then the rank of S, denoted by $r(S)$, is defined to be equal to the rank of the set of columns of A corresponding to S over the field F.

If A is a matrix representing matroid M then the kernel of A (i.e. $\{x; Ax = 0\}$) is a subspace over F called cyclespace of M. The orthogonal complement of the cyclespace is called *cutspace* of M. It is generated by the rows of matrix A. Note that any subspace may be written as kernel of a matrix. Hence also the cutspace is a kernel of a matrix D, and the matroid represented by D is called *dual matroid of* M and denoted by M^*.

For binary matroids, the elements of the cyclespace and cutspace are 0,1 vectors and so we may view them as characteristic vectors of subsets of the underlying set X. If G is a graph then $calN_G$ will denote the graphic matroid of G, i.e. the matroid $(E(G), N)$ where N consists of all acyclic subsets of edges. Note that the graphic matroid is representable over any field F:

Let O_G be an oriented incidence matrix of G, i.e. an $|V| \times |E|$ matrix obtained from the incidence matrix of G by replacing exactly one '1' of each column by '-1'. The columns of O_G represent $\mathcal{N}_G$ over an arbitrary field F.

For the graphic matrids, the cycle space consists of the incidence vectors of even subgraphs and the cutspace consists of the incidence vectors of edge-cuts.

2 Introduction.

Let us start with the introduction of a paper of Hassler Whitney, which appeared in Annals of Mathematics in August 1932:

"Suppose we have a finite set of objects (for instance books on a table), each of which either has or has not a certain given property A (say of being red). Let n, or $n(1)$, be the total number of objects, $n(A)$ the number with the property A, and $n(\bar{A})$ the number without the property A. Then obviously $n(\bar{A}) = n - n(A)$. Similarly, if $n(A, B)$ denote the number with both properties A and B, nad $n(\bar{A}, \bar{B})$ the number with neither property, then $n(\bar{A}, \bar{B}) = n - n(A) - n(B) + n(AB)$, which is easily seen to be true.

The extension of these formulas to the general case where any number of properties are considered is quite simple, and is well known to logicians. It should be better known to mathematicians also; we give in this paper several applications which show its usefulness."

Indeed, we all know it, under the name 'inclusion-exclusion principle':

if $A_1, ..., A_n$ are finite sets, and if we let $\cap(A_i; i \in J) = A_J$ then

$$| \cup (A_i; i = 1, ..., n) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{J \in \binom{n}{k}} |A_J|.$$

Whitney mentions in his paper three main applications: on the number of primes smaller than x, on 'problem satnarky' and on the number of ways of coloring a graph.

These notes are concerned with the last application. Let us again follow his article for a while:

"Suppose we have a fixed number z of colors at our disposal. Any way of assigning one of these colors to each vertex of the graph in such a way that any two vertices which are joined by an arc are of different colors, will be called admissible coloring, using z or fewer colors. We wish to find the number $M(z)$ of admissible colorings, using z or fewer colors. ... We shall deduce a formula for $M(z)$ due to Birkhoff.

If there are V vertices in the grapg G, then there are $n = z^V$ possible colorings, formed by giving each vertex in succession any one of z colors. Let R be this set of colorings. Let A_{ab} denote those colorings with the property that a and b are of the same color, etc. Then the set of admissible colorings is

$$M(z) = n - [n(A_{ab}) + n(A_{bd}) + ... + n(A_{cf})]$$
$$+[n(A_{ab}A_{bd}) + ...] - ...$$
$$+(-1)^E n(A_{ab}A_{bd}...A_{cf}).$$

With each property A_{ab} is associated an arc ab of G. In the logical expansion, there is a term corresponding to every possible combination of the properties A_{pq}; with this combination we associate the corresponding arcs, forming a subgraph H of G. In particular, the first term corresponds to the subgraph containing no arcs, and the last term corresponds to the whole of G. We let H contain all the vertices of G.

Let us evaluate a typical term $n(A_{ab}A_{ad}...A_{ce})$. This is the number of ways of coloring G in z or fewer colors in such a way that a and b are of the same color, a and d are of the same color, ..., c and e are of the same color. In the corresponding subgraph H, any two vertices that are joined by an arc must be of the same color, and thus all the vertices in a single connected piece in H are of the same color. If there are p connected pieces in H, the value of this term is therefore z^p. If there are s arcs in H, the sign of the term is $(-1)^s$. Thus

$$(-1)^s n(A_{ab}A_{bd}...A_{cf}) = (-1)^s z^p.$$

If there are (p, s) (this is Birkhoff's symbol) subgraphs of s arcs in p connected pieces, the corresponding terms contribute to $M(z)$ an amount $(-1)^s(p, s)z^p$. Therefore, summing over all values of p and s, we find the polynomial in z:

$$M(z) = \sum_{p,s} (-1)^s (p, s) z^p."$$

This is our well known chromatic polynomial.

Let $G = (V, E)$ be a connected graph. For $A \subset E$ let $r(A) = |V| - c(A)$, where $c(A)$ denotes the number of connected components (pieces) of (V, A). Then we can write

$$M(z) = z_{c(E)}(-1)^{r(E)} \sum_{A \subset E} (-z)^{r(E)-r(A)}(-1)^{|A|-r(A)}.$$

This lead Whitney to define the two variable polynomial called Whitney rank generating function $R(G, u, v)$ by

$$R(G, u, v) = \sum_{A \subset E} u^{r(E)-r(A)} v^{|A|-r(A)}.$$

Maybe, in the end of the introduction I should apologise to the readers for possible mistakes, misprints and bad formulations. They were partly caused by time pressure. In particular the references are not complete. Time is also the reason I couldnot include an elegant derivation of critical temperature for the square lattice Ising model.

3 The Tutte Polynomial.

We start considering the *Tutte polynomial*; it has been defined by Tutte ([22]) and it may be expressed as a minor modification of the Whitney rank generating function ([25]).

$$T(G, x, y) = \sum_{A \subset E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

$T(G, x, y)$ is called the *Tutte polynomial* of graph G.

Note that for any connected graph G, $T(G, 1, 1)$ counts the number of spanning trees of G: indeed, the only terms that count are those for which $r(A) = r(E) = |A|$. These are exactly the spanning trees of G. Hence, $T(G, 1, 1)$ may be evaluated using a determinant, as you all know from first year Discrete math lecture. How to get the chromatic polynomial is explained above.

More generally, the Tutte polynomial of a matroid is defined as follows.

Definition 3.1 *Let M be a matroid on set E. For $A \subset E$ let $r(A)$ denote the rank of A in M. Then let*

$$T(M, x, y) = \sum_{A \subset E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

$T(M, x, y)$ is called the Tutte polynomial of matroid M.

For example if G is a graph and $\mathcal{N}_G$ the graphic matroid of G then $T(G, x, y) = T(\mathcal{N}_G, x, y)$.

If M is a matroid and M^* its dual then $r^*(E) - r^*(A) = |A| - r(A)$ where r^* is the rank function of M^* and we immediately get that $T(M, x, y) = T(M^*, y, x)$. This relation is called **duality of the Tutte polynomial**.

3.1 Weight enumerator of a linear code.

Let $V = \mathbf{F}^n$ be a vector space over a field $\mathbf{F}$. Each subspace C of V of dimension k is called a *linear code of length n and dimension k*. The elements of a linear code are called *codewords*. The *weight* of a codeword is the number of its nonzero entries. The *weight distribution* of C is the sequence $A_0, A_1, \ldots, A_n$ where A_i equals the number of codewords of C of weight i, $0 \leq i \leq n$.

The *dual code* of C is denoted by C^* and consists of all those n-tuples $(d_1, \ldots, d_n)$ of $\mathbf{F}^n$ satisfying

$$c_1 d_1 + \ldots + c_n d_n = 0$$

for all codewords $(c_1, \ldots, c_n) \in C$. Hence, C^* is a code of length n and dimension $n - k$.

The *weight enumerator* of C is the polynomial

$$A_C(t) = \sum_{i=0}^{n} A_i t^i.$$

The following theorem was proved by MacWilliams ([16]) and it states a fundamental relation between the weight enumerators of C and of its dual C^*.

Theorem 3.2 *Let C be a linear code of length n and dimension k over $GF[q]$ and $1 + (q-1)t \neq 0$. Then*

$$A_{C^*}(t) = \frac{[1 + (q-1)t]^n}{q^k} A_C(\frac{(1-t)}{1 + (q-1)t}).$$

If the linear code of length n is given as the row space of a $k \times n$ matrix A over a field $\mathbf{F}$, i.e. $C = \{A^T x; x \in \mathbf{F}^k\}$, then we will denote it as $C(\mathbf{F}, A)$. Moreover, if a matroid M is represented by the columns of A, then we let $M = M(\mathbf{F}, C)$ and $C = C(\mathbf{F}, M)$. In this case we have $C^* = \{x \in \mathbf{F}^n; Ax = 0\}$. We have $M(\mathbf{F}, C^*) = M(\mathbf{F}, C)^*$.

The following theorem was proved by Greene ([5]).

Theorem 3.3 *Let C be a linear code of length n and dimension k over $GF[q]$ and $0 \neq t \neq 1$. Then*

$$A_C(t) = (1-t)^k t^{n-k} T(M(GF[q], C), \frac{1 + (q-1)t}{(1-t)}, \frac{1}{t}).$$

Note that Theorem 3.2 follows immediately from Theorem 3.3 and the duality of the Tutte polynomial. As an immediate corollary we get

Corollary 3.4 *Let M be a matroid represented over $GF[q]$. If $(x-1)(y-1) = q$ and $0 \neq y \neq 1$ then*

$$T(M, x, y) = y^n (y-1)^{-k} A_{C(GF[q], M)}(y^{-1}).$$

4 The Dualities.

We explained above the duality of the Tutte polynomial, and a seminal theorem of MacWilliams which follows from it.

Consider now the following example. Let $G = (V, E)$ be a graph and let $\mathcal{N}_G$ be the graphic matroid of G.

Let O_G be an oriented incidence matrix of G, i.e. an $|V| \times |E|$ matrix obtained from the incidence matrix of G by replacing exactly one '1' of each column by '-1'. The columns of O_G represent $\mathcal{N}_G$ over an arbitrary field $\mathbf{F}$.

The set of the characteristic vectors of edge-cuts of a graph G (including the empty cut) equals $C(GF[2], \mathcal{N}_G)$ and the set of the characteristic vectors of eulerian subgraphs of G equals $C(GF[2], \mathcal{N}_G)^*$. Using this terminology, the theorem of MacWilliams expresses the generating function of edge-cuts of **any graph G** using the generating function of even subgraphs of **the same graph G**.

This is exciting: **the geometric duality** of embedded graphs which we know about from studying plane graphs works in a different way! Before saying more on this let us give a direct proof to that special case of MacWilliams' theorem : it was in fact proved by van der Waerden in 1941.

The basic idea of van der Waerden was to substitute hyperbolic functions for the exponential terms:

$$x^y = \cosh(x, y) + \sinh(x, y) = \cosh(x, y)(1 + \tanh(x, y))$$

where

$$\cosh(x, y) = \frac{x^y + x^{-y}}{2}, \qquad \sinh(x, y) = \frac{x^y - x^{-y}}{2}, \qquad \tanh(x, y) = \frac{\sinh(x, y)}{\cosh(x, y)}$$

Thus we obtain the generating function in the following form: let $w = (w_{ij}; \{i, j\} \in E(G))$ and $W = \sum_{\{i,j\} \in E(G)} w_{ij}$. Then

$$\mathcal{C}(G, x) = \frac{1}{2} x^{\frac{W}{2}} \sum_{\sigma} \prod_{\{i,j\} \in E} x^{-\frac{1}{2} w_{ij} \sigma_i \sigma_j} =$$

$$= \frac{1}{2} x^{\frac{W}{2}} \sum_{\sigma} \prod_{\{i,j\} \in E} \cosh(x, -\frac{1}{2} w_{ij} \sigma_i \sigma_j)(1 + \tanh(x, -\frac{1}{2} w_{ij} \sigma_i \sigma_j) =$$

$$= \frac{1}{2} x^{\frac{W}{2}} \prod_{\{i,j\} \in E} \cosh(x, -\frac{1}{2} w_{ij}) \sum_{\sigma} \sum_{U \subset E} \prod_{\{i,j\} \in U} \sigma_i \sigma_j \tanh(x, -\frac{1}{2} w_{ij}) =$$

$$= \frac{1}{2} x^{\frac{W}{2}} \prod_{\{i,j\} \in E} \cosh(x, -\frac{1}{2} w_{ij}) \sum_{U \subset E} (\prod_{\{i,j\} \in U} \tanh(x, -\frac{1}{2} w_{ij}) \sum_{\sigma} \prod_{i \in V} \sigma_i^{d_U(i)})$$

where $d_U(i)$ means the number of edges in U incident with the vertex i. If we consider the sum over all assigments σ for a given $U \subset E$, we can see that whenever there exists a vertex i incident with an odd number of edges from U, the resulting sum will be zero. This is because the terms arising from assigments where $\sigma_i = +1$ will exactly cancel out the corresponding terms with $\sigma_i = -1$. On the other hand, if all vertices have even degrees in U, the sign of all contributing terms will be positive. So if n is the total number of vertices:

$$\sum_{\sigma \in \{\pm 1\}^n} \prod_{i \in V} \sigma_i^{d_U(i)} = 2^n$$

whenever U is eulerian, and zero otherwise. Finally, we get

$$\mathcal{C}(G,x) = 2^{n-1} x^{\frac{W}{2}} \prod_{\{i,j\} \in E} \cosh(x, -\frac{1}{2} w_{ij}) \sum_{U eulerian} \prod_{\{i,j\} \in U} \tanh(x, -\frac{1}{2} w_{ij}) =$$

$$= 2^{n-1} x^{\frac{W}{2}} \prod_{\{i,j\} \in E} \cosh(x, -\frac{1}{2} w_{ij}) \mathcal{E}(G', \tanh(x, -\frac{1}{2} w_{ij})).$$

So the generating function of cuts can be expressed as the generating function of eulerian subgraphs of the same graph, with $x^{w_{ij}}$ replaced by $\tanh(x, -\frac{1}{2} w_{ij})$.

If G is a plane graph then let us denote by G^* its geometric dual. In the geometric duality there is a natural bijection between the edge-cuts of G and the even subgraphs of G^*. Hence the generating function of even subgraphs of G^* equals the generating function of edge-cuts of G. For plane graphs we have two dualities to play with, the duality of the Tutte polynomial and the geometric duality. The game becomes even more interesting when a plane graph is self-dual in the geometric sence. But are there such graphs different from K_4? Well, square grids are almost self-dual, and the interplay of the two dualities for square grids will be looked at in the next section.

5 The Max Cut Problem And The Ising Problem.

The *MAX-CUT problem* is a combinatorial optimization problem which is easy to define, but surprisingly hard to solve:

Given a graph, divide its vertices into two parts so that the number of edges between them is as large as possible. More generally, the edges may have arbitrary weights, and we need to maximize the sum of weights over all the edges between the two sets of vertices.

This combinatorial problem has a history on its own, but what makes it so widely studied is the enormous number of applications it finds in different fields. One of these relevant application comes from the study of the *Ising model*: a theoretical physics model of the nearest-neighbor interactions in a crystal structure.

5.1 The Ising model

Let us consider the *Ising model* of a physical system: the vertices of a graph represent particles and the edges describe interactions between pairs of particles. The most common example is a planar square lattice where each particle interacts only with its neighbors. Often, one adds edges connecting the first and last vertex in each row and column, which represent *periodic boundary conditions* in the model. This makes the graph a *toroidal square lattice*.

Now, we assign a factor J_{ij} to each edge $\{i, j\}$; this factor describes the nature of the interaction between particles i and j. A physical state of the system is an assignment of $\sigma_i \in \{+1, -1\}$ to each vertex i. This describes the two possible spin orientations the particle can take. The *Hamiltonian* (or *energy function*) of the system is then defined as

$$H(\sigma) = -\sum_{\{i,j\}\in E} J_{ij}\sigma_i\sigma_j$$

One of the key questions we may ask about a specific system is:

"What is the lowest possible energy (the *ground state*) of the system?"

Before we seek an answer to this question, we should realize that the physical states (spin assignments) correspond exactly to the cuts of the underlying graph. Let us define:

$$V_1 = \{i \in V; \sigma_i = +1\}$$
$$V_2 = \{i \in V; \sigma_i = -1\}$$

Then this partition of vertices encodes uniquely the assignment of spins to particles. The edges contained in the cut $C(V_1, V_2)$ are those connecting a pair of particles with different spins, and those outside the cut connect pairs with equal spins. This allows us to rewrite the Hamiltonian in the following way:

$$H(\sigma) = \sum_{\{i,j\}\in C} J_{ij} - \sum_{\{i,j\}\in E\setminus C} J_{ij} = 2w(C) - W,$$

where $w(C) = \sum_{\{i,j\}\in C} J_{ij}$ denotes the weight of a cut, and $W = \sum_{\{i,j\}\in E} J_{ij}$ is the sum of all edge weights in the graph.

Clearly, if we find the value of MAX-CUT, we have found the maximum energy of the physical system. Similarly, MIN-CUT (the cut with minimum possible weight) corresponds to the minimum energy of the system. Note that if we allow negative edge weights, we can transform MAX-CUT into MIN-CUT (and vice versa) simply by reversing the value of each edge weight. So the two problems are equivalent (unlike the MIN-CUT problem with positive edge weights arising in the study of network flows, which is significantly easier).

However, the analogy between cuts and physical states goes even deeper than that.

If we assign weight $w_e = J_{ij}$ to each edge $e = \{i, j\}$, the generating function becomes

$$\mathcal{C}(G, x) = \sum_{cutC} x^{w(C)} = \sum_{k} c_k x^k$$

where c_k denotes the number of cuts of weight k. But this is also equal to the number of states with energy $2k - W$. Thus, if we knew the complete generating function, we would also know the distribution of physical states over all possible energy levels. In the language of physics, this information is encapsulated in the *partition function*:

$$Z(\beta) = \sum_{\sigma} e^{-\beta H(\sigma)}$$

Substituting the equalities above and counting over cuts instead of σ assignments yields:

$$Z(\beta) = 2 \sum_{cutC} e^{-\beta(2w(C)-W)} = 2e^{\beta W} \sum_{cutC} e^{-2\beta w(C)} = 2e^{\beta W}\mathcal{C}(G, e^{-2\beta})$$

So we can obtain the value of the partition function simply by substituting $e^{-2\beta}$ into the generating function of cuts. The other way round, multiplying the partition function by $\frac{1}{2}e^{-\beta W}$ and expressing it as a polynomial in $e^{-2\beta}$ yields the generating function of cuts. The difference between the two functions is merely formal; both of them encode the state/energy distribution in a similar way.

Finding the generating function of cuts is generally a harder problem than MAX-CUT, because knowing all the coefficients of the polynomial, we can simply find the highest power $x^{k_{max}}$ with a non-zero coefficient; k_{max} is then the maximum cut value. Similarly, the first non-zero term of the partition function corresponds to the ground state energy, but there is much more information contained in it.

5.2 The Ising Problem.

The lattices of interest for physical application are usually the infinite limits of finite lattice graphs. The Ising problem may be formulated as finding properties of the limit when n goes to infinity of $1/nlogZ(\beta)$, where n is the number of vertices of a finite lattice graph. Which properties do we have in mind? For instance in which β is the thermodynamic limit not continuous, or its derivative is non-continuous etc. This suggests a critical behaviour. There is a nice illustration for the square lattice, where the interplay of the two dualities is used, but unfortunately I do not have time now to write it down.

5.3 How To Solve The Ising Problem.

Well, that is what a lot of people would like to know. We said that a solution is the 'knowledge' of the thermodynamic limit. The most straightforward approach towards this goal is the study of the partition function of finite pieces of the (countably) infinite lattice. Let us describe a discrete approach designed in the beginnig of sixties by Kasteleyn and by Fisher, who reduce the study of the Ising partition function to the study of the generating function of perfect matchings of slightly modified graph. The motivation of Kasteleyn and Fisher and many others in fifties and beginning of sixties was to find other, discrete ways to derive the results of Onsager on the Ising partition function of planar square lattices.

For every finite graph we saw that its Ising partition function is the same thing as the generating function of cuts. This in turn may be obtained from the generating function of the even subgraphs of the same graph using the duality of the Tutte polynomial, as we saw in the theorem of Mac Williams and in the derivation of van der Waerden. Next we show that the generating function of even subgraphs of a graph may be transformed into a generating function of perfect matchings of a slightly modified graph. This may be done in many ways, the transformation given here is based on Fisher's construction described in [2]. It is local in the sense that it only modifies each vertex in a way dependent on its degree. It can also be seen that it preserves the genus of the graph.

A subset A of edges of a graph G is called *perfect matching* if each vertex of G belongs to exactly one element of A.

The *generating function of the perfect matchings* of G is the polynomial $\mathcal{P}(G,x)$ which equals the sum of $x(P)$ over all perfect matchings P of G.

Definition 5.1 *Let $G=(V,E)$ be a graph embedded in an orientable surface of genus g, and $v \in V$ a vertex. Let $e_1, e_2, ..., e_d \in E$ denote the edges incident with v, ordered clockwise as they spread out from v in the embedding. Then the even splitting of v is a graph $G'=(V',E')$ where*

- $V' = V \setminus \{v\} \cup \{v_1, ..., v_d, v'_1, ..., v'_d\}$
- $E' = E \setminus \{e_1, e_2, ..., e_d\} \cup \{e'_1, e'_2, ..., e'_d\} \cup E^A$
- $E^A = \{\{v_i, v'_i\}; i = 1, ..., d\} \cup \{\{v_i, v'_{i-1}\}; i = 2, ..., d\} \cup \{\{v'_i, v'_{i+1}\}; i = 1, ..., d-1\}$

*The edges $e'_i \in E'$ (*image edges*) are obtained from $e_i \in E$ by replacing the vertex v by v_i. The edges E^A will be called* auxiliary.

Lemma 5.2 *The graph obtained by even splitting can be again embedded in the same surface.*

Proof. The transformation replaces a vertex $v \in V$ by a cluster of $2d$ vertices and $3d-2$ edges. The cluster itself is a planar graph which can be embedded in a small neighborhood of the original location of the vertex v. The images of the edges incident with v can be embedded in the same way as they were in the original graph. □

Definition 5.3 *Let $G=(V,E)$ be a graph and $G_s=(V_s,E_s)$ the graph obtained by successive even splitting of all vertices in V. If there are weights w_e assigned to edges $e \in E$, we assign the same weights to their images in E_s: $w_{e'} = w_e$. The auxiliary edges $f \in E_s$ get assigned $w_f = 0$.*

Theorem 5.4 *If G is a graph of genus g, G_s has genus g as well. With the assigment of weights described above, the generating function of perfect matchings of G_s is equal to the generating function of eulerian subgraphs of G:*

$$\mathcal{P}(G_s, x) = \mathcal{E}(G, x).$$

Proof. From the definition of even splitting and Lemma 5.2, it follows that the resulting graph G_s can be embedded again in the same surface.

If M is a perfect matching in G_s, it must cover each of its vertices exactly once. Because the cluster replacing every vertex has an even number of vertices, and any of the auxiliary edges which is in M covers a pair of vertices of the cluster, there remain an even number of vertices to be covered by the image edges incident with the cluster. Therefore, every cluster coincides with an even number of image edges which are in M; in other words, these edges form the image of an eulerian subgraph of G.

Vice versa, the image of any eulerian subgraph of G can be extended (uniquely) by adding some of the auxiliary edges in G_s to make a perfect matching in G_s. Thus, there is a one-to-one correspondence between the perfect matchings of G_s and the eulerian subgraphs of G. As all the auxiliary edges have weights equal to 0, the corresponding terms contributing to either of the generating functions are equal. Consequently, the two generating functions are equal. □

Hence it remains to calculate the generating function of perfect matchings, and it is dealt with in the next subsection.

5.4 A Theory of Pfaffian Orientations.

The theory of Pfaffian orientations of graphs has been introduced by Kasteleyn [13, 12, 10] in early sixties to solve some enumeration problems arising from statistical physics [9, 19]. He proved fundamental results in the planar case and extended his approach to toroidal grids [10, 12, 13]. His method has been recently extended to general graphs by Galluccio and Loebl, and this is what this subsection will be about.

From now on we introduce slightly more general definition of generating functions. We associate with each edge e of G a variable x_e and we let $x = (x_e : e \in E)$. For each $M \subset E$, let $x(M)$ denote the product of the variables of the edges of M.

The *generating function of the perfect matchings* of G is the polynomial $\mathcal{P}(G, x)$ which equals the sum of $x(P)$ over all perfect matchings P of G.

The other generating functions we are dealing with are modified accordingly. Do not worry, a confusion may not occur.

Let M, N be two perfect matchings of a graph G. Then $M \Delta N$ consists of vertex disjoint cycles of even length. These cycles are called *alternating cycles* of M and N.

Let C be a cycle of G of an even length and let D be an orientation of G. C is said to be *clockwise even* in D if it has an even number of edges directed in D in agreement with a chosen direction of traversal. Otherwise C is called *clockwise odd*.

Definition 5.5 *Let G be a graph and let D be an orientation of G. Let M be a perfect matching of G. For each perfect matching P of G let $sgn(D, M\Delta P) = (-1)^n$ where*

n is the number of clockwise even alternating cycles of M and P, and let $\mathcal{P}(D, M)$ be the sum of $sgn(D, M\Delta P)x(P)$ over all perfect matchings P of G.

Definition 5.6 *Let $G = (V, E)$ be a graph with $2n$ vertices and D an orientation of G. Denote by $A(D)$ the skew-symmetric matrix with the rows and the columns indexed by V, where $a_{vw} = x_{vw}$ in case (v, w) is an arc of D, $a_{vw} = -x_{vw}$ in case (w, v) is an arc of D, and $a_{vw} = 0$ otherwise.*

The Pfaffian of the skew-symmetric matrix $A(D)$ is defined as

$$Pf(A(D)) = \sum_P s^*(P) a_{i_1 j_1} \cdots a_{i_n j_n}$$

where $P = \{\{i_1 j_1\}, \cdots, \{i_n j_n\}\}$ is a partition of the set $\{1, \ldots, 2n\}$ into pairs, $i_k < j_k$ for $k = 1, \ldots, n$, and $s^*(P)$ equals the sign of the permutation $i_1 j_1 \ldots i_n j_n$ of $12 \ldots (2n)$.

Each nonzero term of the expansion of the Pfaffian of $A(D)$ equals $x(P)$ or $-x(P)$ where P is a perfect matching of G. If $s(D, P)$ denote the sign of the term $x(P)$, we have that

$$Pf(A(D)) = \sum_P s(D, P)x(P).$$

The following theorem was proved by Kasteleyn [10].

Theorem 5.7 *Let G be a graph and D an orientation of G. Let P, M be two perfect matchings of G. Then*

$$s(D, P) = s(D, M)sgn(D, M\Delta P).$$

Hence,

$$Pf(A(D)) = \sum_P s(D, P)x(P) = s(D, M) \sum_P sgn(D, M\Delta P)x(P) = s(D, M)\mathcal{P}(D, M).$$

The relevance of Pfaffians in our context lies in the fact that, despite their similarity with the permanent, the Pfaffian is a determinant-type function. They are polynomial time computable by a variant of Gaussian elimination. In fact, see [1],[13] for a proof of the following classic theorem.

Theorem 5.8 *Let G be a graph and let D be an orientation of G. Then*

$$Pf^2(A(D)) = det(A(D)).$$

In [10] Kasteleyn introduced the following notion:

Definition 5.9 *A graph G is called Pfaffian if it has a Pfaffian orientation, i.e., an orientation such that all alternating cycles with respect to an arbitrary fixed perfect matching M of G are clockwise odd.*

Hence if a graph G has a Pfaffian orientation D then the signs $s(D,P)$ are equal for all perfect matchings P of G and $\mathcal{P}(G,x)^2 = Pf^2(A(D)) = det(A(D))$.

Kasteleyn [10] observed that the planar graphs have a Pfaffian orientation; more specifically, he proved that

Theorem 5.10 *Every plane graph has a Pfaffian orientation such that all inner faces are clockwise odd.*

Proof. Let G be a plane graph, and let M be its perfect matching. Each alternating cycle of M belongs to a 2-connected component of G.

Observe that G has an orientation so that each inner face of each 2-connected component of G is clockwise odd. Each such face 'encircles' no vertex of the corresponding 2-connected component. Let W be a 2-connected component of G. Observe that the orientation we constructed has the property that a cycle C of W is clockwise odd if and only if C encircles an even number of vertices of W. Let C be an alternating cycle of M and let W be a 2-connected component of G which contains C. Then C encircles an even number of vertices of W and hence it is clockwise odd. □

A lot of attention is given to the problem of characterising graphs which admit a Pfaffian orientation. We will speak more about this in the part devoted to permanents, but let us just say here that the problem goes back to Polya and to 1913, and that a characterisation inducing a polynomial-time algorithm for bipartite graphs has been obtained by McCuaig, Robertson, Seymour and Thomas , and for graphs embeddable on arbitrary orientable surface by Galluccio, Loebl.

A general feeling is that there are few graphs which admit a Pfaffian orientation, perhaps they all may be obtained from planar graphs by some local operations.

Embeddings and Pfaffian orientations

The *genus* g of a graph G is that of the orientable surface $\mathcal{S} \subset I\!R^3$ of minimal genus on which G may be embedded. Any orientable surface of genus g has a *polygonal representation* obtained by cutting the g handles of its space model. In what follows we base our working definition of a surface on this concept.

Definition 5.11 *A surface S_g of genus g consists of a base B_0 and $2g$ bridges B^i_j, $i = 1, ..., g$ and $j = 1, 2$, where*

i) B_0 is a convex $4g$-gon with vertices $a_1, ..., a_{4g}$ numbered clockwise;

ii) B^i_1, $i = 1, \ldots, g$, is a 4-gon with vertices $x^i_1, x^i_2, x^i_3, x^i_4$ numbered clockwise. It is glued with B_0 so that the edge $[x^i_1, x^i_2]$ of B^i_1 is identified with the edge $[a_{4(i-1)+1}, a_{4(i-1)+2}]$ of B_0 and the edge $[x^i_3, x^i_4]$ of B^i_1 is identified with the edge $[a_{4(i-1)+3}, a_{4(i-1)+4}]$ of B_0;

iii) B^i_2, $i = 1, \ldots, g$, is a 4-gon with vertices $y^i_1, y^i_2, y^i_3, y^i_4$ numbered clockwise. It is glued with B_0 so that the edge $[y^i_1, y^i_2]$ of B^i_2 is identified with the edge $[a_{4(i-1)+2}, a_{4(i-1)+3}]$ of B_0 and the edge $[y^i_3, y^i_4]$ of B^i_2 is identified with the edge $[a_{4(i-1)+4}, a_{4(i-1)+5(mod 4g)}]$ of B_0.

Observe that in Definition 5.11 we denote by $[a, b]$ edges of polygons and not edges of graphs. The usual representation in the space of an orientable surface $\mathcal{S}$ of genus g may be then obtained from its polygonal representation S_g by the following operation: for each bridge B, glue together the two segments which B shares with the boundary of B_0, and delete B.

Definition 5.12 *A graph G is called a g-graph if it may be embedded on S_g so that all the vertices belong to the base B_0, and the embedding of each edge uses at most one bridge. The set of the edges embedded entirely on the base will be denoted by E_0 and the set of the edges embedded on the bridge B^i_j will be denoted by E^i_j, $i = 1, \ldots, g$, $j = 1, 2$. If a g-graph G satisfies the following further conditions:*

1. *the outer face of $G_0 = (V, E_0)$ is a cycle, and it is embedded on the boundary of B_0,*
2. *if $e \in E^i_1$ then e is embedded entirely on B^i_1 and one endvertex of e belongs to $[x^i_1, x^i_2]$ and the other one belongs to $[x^i_3, x^i_4]$. Similarly, if $e \in E^i_2$ then e is embedded entirely on B^i_2 and one endvertex of e belongs to $[y^i_1, y^i_2]$ and the other one belongs to $[y^i_3, y^i_4]$.*
3. *each vertex is incident with at most one edge which does not belong to E_0,*
4. *G_0 has a perfect matching,*

then we say that G is a proper g-graph.

Given a proper g-graph G, we denote by C_0 the cycle which forms the outer face of E_0; then, we fix a perfect matching of G_0 and denote it by M_0.

Definition 5.13 *Let G be a proper g-graph and let $G^i_j = (V, E_0 \cup E^i_j)$. If we draw $B_0 \cup B^i_j$ on the plane as follows: B_0 is unchanged, and the edge $[x^i_1, x^i_4]$ ($[y^i_1, y^i_4]$ respectively) of B^i_j is drawn so that it belongs to the external boundary of $B_0 \cup B^i_j$, we obtain a planar embedding of G^i_j. This embedding will be called planar projection of E^i_j outside B_0.*

Definition 5.14 *Let $G = (V, E)$ be a proper g-graph. A Pfaffian orientation D_0 of G_0 such that each inner face of each 2-connected component of G_0 is clockwise odd in D_0 is called a basic orientation of G_0.*

Note that a basic orientation always exists for a planar graph by Theorem 5.10.

Definition 5.15 *Let $G = (V, E)$ be a proper g-graph and D_0 a basic orientation of G_0. We define the orientation D^i_j of each G^i_j as follows: We consider G^i_j embedded on the plane by the planar projection of E^i_j outside B_0 (see Definition 5.13), and complete the basic orientation D_0 of G_0 to an orientation of G^i_j so that each inner face of each 2-connected component of G^i_j is clockwise odd.*
The orientation $-D^i_j$ is defined by reversing the orientation D^i_j of G^i_j.

Observe that after fixing a basic orientation D_0, the orientation D_j^i is uniquelly determined for each i, j.

Definition 5.16 *Let G be a proper g-graph, $g \geq 1$. An orientation D of G which equals the basic orientation D_0 on G_0 and which equals D_j^i or $-D_j^i$ on E_j^i is called relevant. We define its type $r(D) \in \{+1, -1\}^{2g}$ as follows: For $i = 0, \dots, g-1$ and $j = 1, 2$, $r(D)_{2i+j}$ equals $+1$ or -1 according to the sign of D_j^{i+1} in D.*

Definition 5.17 *Let G be a proper g-graph and let A be a subset of its edges. The type of A is a vector $t(A) \in \{0,1\}^{2g}$ defined as follows: For $i = 0, ..., g-1$ and $j = 1, 2$, we let $t(A)_{2i+j}$ equals the number of edges of A which belong to E_j^{i+1}, modulo 2.*

Let $CR(A) \stackrel{2}{=} \sum_{i=0}^{g-1} t(A)_{2i+1} \cdot t(A)_{2i+2}$ denote the number of crossings of the embeddings of the edges of A, after making planar projections of E_j^i for all i, j.

Let $BR(A)$ denote the subset of edges of A which do not belong to E_0. For each $e \in BR(A)$, let $d(e) = 2i + j$ if $e \in E_j^{i+1}$.

We complete the section with a lemma.

Lemma 5.18 *Let G be a proper g-graph. Let $C_1, ..., C_k$ be vertex-disjoint cycles of G and let $\mathcal{C}$ denote their union. Then*

$$CR(\mathcal{C}) \stackrel{2}{=} \sum_{i=1}^{k} CR(C_i).$$

Proof. Let us embed the cycles $C_1, ..., C_k$ using the planar projections of E_j^i outside B_0 by Definition 5.17. Then $CR(\mathcal{C})$ equals the total number of crossings of $\mathcal{C}$ (modulo 2). Now, each cycle C_l, $l = 1, ..., k$ is represented as a closed curve in the plane and each pair of cycles C_i and C_j, $i \neq j$, intersects an even number of times. Hence the sum (modulo 2) of the number of crossings between pairs of cycles C_i and C_j, $i \neq j$, is 0 and does not affect $CR(\mathcal{C})$. Each of the remaining crossings is a crossing of some C_l, $l = 1, ..., k$, with itself and the lemma follows. □

Perfect matchings

Through this section, the graph G will be a proper g-graph embedded on a fixed surface S_g. We also fix a perfect matching M_0 of G_0.

The aim of this section is to prove that, for any perfect matching P, the $sgn(D, M_0 \Delta P)$ depends only on the vectors $t(M_0 \Delta P)$ and $r(D)$.

Given an orientation D of G and an even length cycle C of G, we denote by $l_D(C)$ the number of arcs of C directed in agreement with any of the two possible ways of traversing C, modulo 2. For short, any alternating cycle with respect to M_0 will be simply called an *alternating cycle*. In order to prove our statement, we consider first the case that $M_0 \Delta P$ consists of exactly one alternating cycle.

Theorem 5.19 *Let G be a proper g-graph and let D be a relevant orientation of G. If C is an alternating cycle of G, then*

$$l_D(C) \stackrel{2}{=} |BR(C)| - 1 - CR(C) + \frac{1}{2} \sum_{e \in BR(C)} (r(D)_{d(e)} + 1).$$

Proof. We assume without loss of generality that $G = C \cup C_0 \cup M_0$, where C_0 is the outer face of G_0 and M_0 is the fixed perfect matching of G_0. Let D_0 be the basic orientation of G_0.

Claim 1. *If C intersects at most one of E_1^i, E_2^i, for each $i = 1, ..., g$, then*

$$l_D(C) \stackrel{2}{=} |BR(C)| - 1 + \frac{1}{2} \sum_{e \in BR(C)} (r(D)_{d(e)} + 1).$$

A cycle C satisfying the properties of Claim 1 may be embedded without crossings using the planar projection of each E_j^i outside B_0. Hence $l_D(C) = 1$ if and only if $|\{e \in BR(C) : r(D)_{d(e)} = -1\}| \stackrel{2}{=} 0$. **End of Claim 1.**

The proof is by induction on $|BR(C)|$. The case $|BR(C)| = 0$ is proved by Claim 1. By induction we assume that

$$l_W(C') \stackrel{2}{=} |BR(C')| - 1 - CR(C') + \frac{1}{2} \sum_{e \in BR(C')} (r(W)_{d(e)} + 1)$$

for any alternating cycle C' of a proper g-graph H, with relevant orientation W, such that $|BR(C')| < |BR(C)|$.

We distinguish two cases.

Case 1. *There exists a bridge $B = B_j^i$ containing more than one edge of C.*
Let $e = u_1u_2$ and $f = v_1v_2$ be two edges of $C \cap E_j^i$ which see each other on B, i.e., there is no other edge of C drawn between them on B. Without loss of generality, let e be nearer to the edge $[a_{2(i-1)+j}, a_{2(i-1)+j+3}]$ of $B = B_j^i$ than f and let u_1, v_1 and u_2, v_2 belong to the edge $[a_{2(i-1)+j}, a_{2(i-1)+j+1}]$ and $[a_{2(i-1)+j+2}, a_{2(i-1)+j+3}]$, respectively. Since e, f do not belong to E_0, they are not edges of $M_0 \subset E_0$.

Let R_i be the subpath of C_0 from u_i to v_i, $i = 1, 2$, and let R be the cycle of G consisting of (R_1, f, R_2, e). By the choice of e, f, the cycle R is the boundary of a face of the planar projection of $G_j^i = (V, E_0 \cup E_j^i)$ outside B_0. Observe that $l_W(R) = 1$ for each relevant orientation W of G, since R contains two edges embedded outside B_0.

Let us introduce a new edge h (not belonging to G), between the endvertices of e, f such that one of two cycles $\bar{H}_1, \bar{H}_2$ formed by h and C and containing h is alternating. Without loss of generality, let h have u_1 as an endvertex. Hence we have that $h = u_1v_1$ or $h = u_1v_2$.

We may assume without loss of generality that $\bar{H}_2$ is alternating. Hence $\bar{H}_1$ contains both e, f. Note that $\bar{H}_1$ consists of an even number of edges. We denote by h_1, h_2 the two arcs with the same endvertices as h, directed opositely. Let $D' = D \cup \{h_1, h_2\}$. Let H_i be the subdigraph of D' which is the orientation of $\bar{H}_i$ using h_i, $i = 1, 2$. Observe that $l_D(C) = l_{D'}(H_1) + l_{D'}(H_2)$.

Subcase 1.1: $h_1 = u_1v_1$.

We adjust the boundary of B_0 by replacing $\{R_1\}$ with h_1, h_2. Observe that $CR(C) \stackrel{2}{=} CR(H_1) + CR(H_2)$: attention should be drawn to the question of how crossings of C with itself are manifested as crossings of H_1 or H_2, when all E_j^i are projected outside of B_0 (see Definition 5.13). If two edges of C cross and they are

not separated in C by the endvertices of h_1, then that crossing counts as a crossing with in H_1 or H_2. We must therefore consider the parity of the number of crossings of C where the crossed edges are separated in C by the endvertices of h_1. These crossings are counted as crossings of H_1 with H_2. If the number of such crossings of C is odd, then there must be an additional crossing of H_1 with H_2, since the total number of crossings of H_1 with H_2 must be even. Since h_1 and h_2 do not cross, this additional crossing must occur at an endvertex of h_1. It is easy to see that in the present case there is no such crossing, and so, there are an even number of crossings of C where the crossed edges are separated in C by the ends of h. The required congruence therefore follows in this case.
We construct now two digraphs D_1, D_2 as follows:

- D_1 is obtained from $D-\{e,f\}$ by adding new vertices u_1', v_1' of degree 2, incident with new arcs e', f', h_1'. The arcs e', f', h_1' are obtained from e, f, h_1 by replacing u_1 by u_1' and v_1 by v_1'. We adjust the boundary of B_0 by replacing $\{R_2\}$ with $\{e', f', h_1'\}$. Finally we add h_1' to M_0. Let H_1' be the cycle of D_1 obtained from H_1 by replacing e, f, h_1 by e', f', h_1'. Then $l_{D_1}(H_1') = l_{D'}(H_1)$ and $CR(H_1') \stackrel{2}{=} CR(H_1)$;
- D_2 is obtained from $D-\{e,f\}$ by adding arc h_2. We remind that h_2 is embedded on the adjusted B_0 parallel to R_1. Let $H_2' = H_2$. Then $l_{D_2}(H_2') = l_{D'}(H_2)$ and $CR(H_2') \stackrel{2}{=} CR(H_2)$.

We remind that $l_D(R) = 1$. Hence, exactly one of h_i is oriented so that both cycles it makes with R are clockwise odd. Let it be h_2. Then D_2 is a relevant orientation and D_1 becomes relevant after reversing the orientation of h_1': this digraph, obtained from D_1 by reversing the orientation of h_1', we denote by D_1^*, and its subdigraph corresponding to H_1' we denote by H_1^*. Then, $l_{D_1^*}(H_1^*) \stackrel{2}{=} l_{D_1}(H_1') + 1$.

Note that both D_2 and D_1^* are relevant orientations of proper g-graphs, H_2' is an alternating cycle of D_2, H_1^* is an alternating cycle of D_1^* and $CR(H_1^*) < CR(C)$ and $CR(H_2') < CR(C)$. Hence, by the induction assumption, we have that:

$$l_D(C) \stackrel{2}{=} l_{D'}(H_1) + l_{D'}(H_2) \stackrel{2}{=} l_{D_1}(H_1') + l_{D_2}(H_2') \stackrel{2}{=} l_{D_1^*}(H_1^*) + 1 + l_{D_2}(H_2') \stackrel{2}{=}$$

$$|BR(H_1^*)| - 1 - CR(H_1^*) + \frac{1}{2}\sum_{p\in BR(H_1^*)} (r(D_1^*)_{d(p)} + 1) +$$

$$|BR(H_2')| - 1 - CR(H_2') + \frac{1}{2}\sum_{p\in BR(H_2')} (r(D_2)_{d(p)} + 1) + 1.$$

Now, the theorem follows by observing that $|BR(C)| \stackrel{2}{=} |BR(C-\{e,f\})| \stackrel{2}{=} |BR(H_1^*)| + |BR(H_2')| - 2$, $CR(C) \stackrel{2}{=} CR(H_1^*) + CR(H_2')$ and $r(D_1^*)_{d(p)}$, $r(D_2)_{d(p)}$ and $r(D)_{d(p)}$ coincide for any $p \in BR(C) - \{e,f\}$. Hence,

$$l_D(C) \stackrel{2}{=} |BR(C)| - 1 - CR(C) + \frac{1}{2}\sum_{p\in BR(C)} (r(D)_{d(p)} + 1).$$

(End of Subcase 1.1)

Subcase 1.2: $h_1 = u_1v_2$.

Let h_1 and h_2 be embedded on the bridge B. Observe that $CR(C) \stackrel{2}{=} CR(H_1) + CR(H_2) + 1$: attention again should be drawn to the question of how crossings of C with itself are manifested as crossings of H_1 or H_2, when all E_j^i are projected outside of B_0 (see Definition 5.13). To see this clearly, we introduce some notation. Let A be a subset of arcs of H_1 and B a subset of arcs of H_2. We denote by $CR(A \times B)$ the number of crossings between arcs of A and B, mod 2. We also denote by $cr(i,j)$ the number of crossings of arcs of $H_i \cap C$ with h_j. Hence, we have:

$$CR(H_1) \stackrel{2}{=} CR(H_1 \cap C) + cr(1,1),$$
$$CR(H_2) \stackrel{2}{=} CR(H_2 \cap C) + cr(2,2),$$
$$CR(C) \stackrel{2}{=} CR(H_1 \cap C) + CR(H_2 \cap C) + CR((H_1 \cap C) \times (H_2 \cap C)),$$
$$CR(H_1 \times H_2) \stackrel{2}{=} 0,$$

and

$$\sum_{i,j=1}^{2} cr(i,j) \stackrel{2}{=} 0$$

since each arc which crosses h_1 crosses also h_2.

Hence it remains to show that

$$CR(H_1 \times H_2) \stackrel{2}{=} CR((H_1 \cap C) \times (H_2 \cap C)) + cr(1,2) + cr(2,1) + 1 :$$

this follows since in this case one additional crossing between H_1 and H_2 must occur at an endvertex of h. The required congruence follows.

We construct two digraphs D_1, D_2 as follows:

- D_1 is obtained from $D - \{e,f\}$ by adding a new arc h_1' between v_1 and the endvertex u_2 of e. If $l_{D'}(fh_1e) = 1$ then we let $h_1' = (v_1, u_2)$. If $l_{D'}(fh_1e) = 0$ then we let $h_1' = (u_2, v_1)$.

  We consider h_1' embedded on the bridge B. Let H_1' be obtained from H_1 by replacing $\{f, h_1, e\}$ by h_1'. We have $l_{D'}(H_1) = l_{D_1}(H_1')$ and $CR(H_1) = CR(H_1')$.

- D_2 is obtained from $D - \{e,f\}$ by adding the arc h_2. We consider h_2 embedded on the bridge B. We let $H_2 = H_2'$. Thus again we have $l_{D'}(H_2) = l_{D_2}(H_2')$ and $CR(H_2) = CR(H_2')$.

We remind that $l_D(R) = 1$ and thus exactly one of h_i is oriented so that both cycles it makes with R are clockwise odd. Let it be h_2. Let R_3 be the subpath of C_0 from v_1 to v_2 such that (e, R_1, R_3, R_2) is a cycle. We have $l_{D_1}(h_1', R_3, R_2) \stackrel{2}{=} l_{D'}(e, h_1, f, R_3, R_2) \stackrel{2}{=} l_{D'}(f, R_3) + l_{D'}(e, h_1, R_2)$.

We show now that both D_1 and D_2 are relevant orientations with $r(D_1) = r(D_2) = r(D)$. We only need to show that h_1' and h_2 are correctly oriented in D_1 and D_2. This follows easily for D_2, since both cycles h_2 makes with R are clockwise odd.

For D_1 we distinguish two cases. First, let $r(D)_{2(i-1)+j} = 1$. In this case we have $l_{D'}(f, R_3) = 1$ and $l_{D'}(e, h_2, R_2) = 1$. Hence $l_{D'}(e, h_1, R_2) = 0$. It follows that $l_{D_1}(h'_1, R_3, R_2) = 1$ and D_1 is relevant with $r(D_1) = r(D)$. Secondly, let $r(D)_{2(i-1)+j} = -1$. In this case we have $l_{D'}(f, R_3) = 0$ and $l_{D'}(e, h_2, R_2) = 1$. Hence $l_{D'}(e, h_1, R_2) = 0$. It follows that $l_{D_1}(h'_1, R_3, R_2) = 0$ and D_1 is relevant with $r(D_1) = r(D)$.

Hence, D_i is a relevant orientation of a proper g-graph, H'_i is an alternating cycle of D_i and $|BR(H'_i)| < |BR(C)|$, for $i = 1, 2$, and, by the induction hypothesis, we have that:

$$l_D(C) \stackrel{2}{=} l_{D'}(H_1) + l_{D'}(H_2) \stackrel{2}{=} l_{D_1}(H'_1) + l_{D_2}(H'_2) \stackrel{2}{=}$$

$$|BR(H'_1)| - 1 - CR(H'_1) + \frac{1}{2} \sum_{p \in BR(H'_1)} (r(D_1)_{d(p)} + 1) + \frac{1}{2}(r(D_1)_{d(h_1)} + 1) +$$

$$|BR(H'_2)| - 1 - CR(H'_2) + \frac{1}{2} \sum_{p \in BR(H'_2)} (r(D_2)_{d(p)} + 1) + \frac{1}{2}(r(D_2)_{d(h_2)} + 1).$$

The theorem follows by observing that $|BR(C)| \stackrel{2}{=} |BR(C - \{e, f\})| \stackrel{2}{=} |BR(H'_1)| + |BR(H'_2)| - 2$, $CR(C) + 1 \stackrel{2}{=} CR(H'_1) + CR(H'_2)$ and $r(D_1) = r(D_2) = r(D)$.

(End of Subcase 1.2)

End of Case 1

Case 2. *There exists i such that C contains exactly one edge from both E_1^i and E_2^i.* Let $e \in E_1^i$ and $f \in E_2^i$ and let C_1 and C_2 be two paths such that $C = (C_1, e, C_2, f)$. The endvertices of e, f belong to C_0. Let us assume that along the boundary of B_0 from $a_{4(i-1)+1}$ to a_{4i+1}, the endvertices of e, f appear in the order v_1, u_1, v_2, u_2 where $e = u_1u_2$ and $f = v_1v_2$.

Let R_1, R_2 be the two disjoint subpaths of the segment of C_0 between $a_{4(i-1)+1}$ and a_{4i+1}, which cover the endvertices of e, f. Note that R_1, R_2 contain no other vertex of G incident with an edge out of E_0, by the choice of i. Let R denote the cycle (R_1, e, R_2, f) and let R_3 denote the segment of C_0 between u_1 and v_2.

Let us introduce a new edge h (not belonging to G), between endvertices of e, f such that one of two cycles $\bar{I}_1, \bar{I}_2$ formed by h and C and containing h is alternating.

Without loss of generality let h have u_1 as an endvertex. Hence we have that $h = u_1v_1$ or $h = u_1v_2$. We may also assume without loss of generality that $\bar{I}_2$ is alternating. Hence $\bar{I}_1$ contains both e, f. Note that $\bar{I}_1$ consists of an even number of edges.

We denote by h_1, h_2 the two arcs with the same endvertices as h, directed opositely. Let $D' = D \cup \{h_1, h_2\}$. Let I_i be the subdigraph of D' which is the orientation of $\bar{I}_i$ using h_i, $i = 1, 2$. Observe that $l_D(C) = l_{D'}(I_1) + l_{D'}(I_2)$.

Again we distinguish two subcases.

Subcase 2.1: $h_1 = u_1v_1$.

In this case h forms a cycle with R_1.

As in Subcase 1.1, we extend B_0 along R_1 and consider h_1, h_2 as embedded on the extended B_0.

Observe that $CR(C) \stackrel{2}{=} CR(I_1) + CR(I_2)$: attention should be drawn to the question of how crossings of C with itself are manifested as crossings of I_1 or I_2, when all E^i_j are projected outside of B_0 (see Definition 5.13). In this case, the arguments are identical to those used in the proof of Subcase 1.1, and we omit them.

We construct two digraphs D_1, D_2 as follows:

- D_1 is obtained from $D-\{e,f\}$ by adding new vertices u'_1, v'_1 of degree 2, incident with new arcs e', f', h'_1. The arcs $e'f', h'_1$ are obtained from e, f, h_1 by replacing u_1 by u'_1 and v_1 by v'_1. We extend B_0 along R_2 and we embed the path (e', f', h'_1) on the extended B_0. Finally we add h'_1 to M_0. Let I'_1 be the cycle of D_1 obtained from I_1 by replacing e, f, g_1 by e', f', g'_1. We have $l_{D'}(I_1) = l_{D_1}(I'_1)$ and $CR(I_1) - 1 \stackrel{2}{=} CR(I'_1)$.

- D_2 is obtained from $D-\{e,f\}$ by adding the arc h_2. We consider h_2 embedded on the extended B_0 along R_1. We let $I'_2 = I_2$. Hence, $l_{D'}(I_2) = l_{D_2}(I'_2)$ and $CR(I_2) \stackrel{2}{=} CR(I'_2)$.

Hence, for $i = 1, 2$, D_i is an orientation of a proper g-graph and I'_i is an alternating cycle of D_i. Moreover $|BR(I'_i)| < |BR(C)|$.

Let us assume without loss of generality that h_2 is directed so that the cycle $l_{D'}(R_1, h_2) = 1$. Hence D_2 is a relevant orientation with $r(D_2) = r(D)$.

We show now that D_1 is a relevant orientation with $r(D) = r(D_1)$ if and only if $r(D)_{d(e)} = r(D)_{d(f)}$. We first prove that if $r(D)_{d(e)} = r(D)_{d(f)} = 1$ then D_1 is a relevant orientation.

In this case it suffices to show that $l_{D_1}(R_2, f', h'_1, e') \stackrel{2}{=} 1$. We have $l_{D_1}(h'_1, f', R_3) \stackrel{2}{=} l_{D'}(h_1, f, R_3) \stackrel{2}{=} l_{D'}(h_2, f, R_3) + 1 \stackrel{2}{=} l_{D_2}(h_2, f, R_3) + 1 \stackrel{2}{=} 0$, since $r(D_2)_{d(f)} = r(D)d(f) = 1$, and thus, $l_{D_2}(h_2, f, R_3) = 1$.

Moreover $l_{D_1}(R_2, R_3, e') = l_{D'}(R_2, R_3, e) = 1$, since $r(D)_{d(e)} = 1$ and D is a relevant orientation. Replacing $f'h'_1$ for R_3 gives what we claimed.

Similarly, we can prove that if $r(D)_{d(e)} = r(D)_{d(f)} = -1$ then again $l_{D_1}(R_2, f', h'_1, e') = 1$, and so, D_1 is a relevant orientation.

On the other hand, if $r(D)_{d(e)} \neq r(D)_{d(f)}$ then D_1 is obtained from a relevant orientation by reversing one arc, and so, it is not relevant.

Summarizing, if $r(D)_{d(e)} = r(D)_{d(f)}$ then D_1 is a relevant orientation with $r(D) = r(D_1)$, and if $r(D)_{d(e)} \neq r(D)_{d(f)}$ then D_1 becomes relevant after reversing the orientation of h'_1: this digraph, obtained from D_1 by reversing the orientation of h'_1, we denote by D^*_1, and its subdigraph corresponding to H'_1 we denote by H^*_1. Then $l_{D^*_1}(I'_1) \stackrel{2}{=} l_{D_1}(I'_1) + 1$.

Using the induction assumption of 5.19 for D^*, I^*_1, D_1, I'_1 and D_2, I'_2 we get:

$$l_D(C) \stackrel{2}{=} l_{D'}(I_1) + l_{D'}(I_2) \stackrel{2}{=} l_{D_1}(I'_1) + l_{D_2}(I'_2) \stackrel{2}{=}$$

$$|BR(C)| - 4 - CR(C) + 1 + \frac{1}{2} \sum_{p \in BR(C)-\{e,f\}} (r(D)_{d(p)} + 1) + \frac{1}{2}(r(D)_{d(e)} - 1 + r(D)_{d(f)} - 1) \stackrel{2}{=}$$

$$|BR(C)| - 1 - CR(C) + \frac{1}{2}\sum_{p\in BR(C)} (r(D)_{d(p)} + 1).$$

(End of Subcase 2.1)

Subcase 2.2: $h = u_1v_2$.

In this case h forms a cycle with R_3. We extend B_0 along R_3 and consider h_1, h_2 embedded on the extended B_0.

Observe that $CR(C) \stackrel{2}{=} CR(I_1)+CR(I_2)$:attention should be drawn to the question of how crossings of C with itself are manifested as crossings of I_1 or I_2, when all E^i_j are projected outside of B_0 (see Definition 5.13). In this case, the arguments are identical to those used in the proof of Subcase 1.1 and Subcase 2.1, and we omit them.

We construct two digraphs D_1, D_2 as follows:

- D_1 is obtained from $D-\{e, f\}$ by adding new vertices u'_1, v'_2 of degree 2, incident with new arcs e', f', h'_1. The arcs e', f', h'_1 are obtained from e, f, h_1 by replacing u_1 by u'_1 and v_2 by v'_2. We extend B_0 along $R_1R_3R_2$ and we embed e', f', h'_1 on the extended B_0. Finally we add h'_1 to M_0. Let I'_1 be the cycle of D_1 obtained from I_1 by replacing e, f, h_1 by e', f', h'_1. We have that $l_{D'}(I_1) \stackrel{2}{=} l_{D_1}(I'_1)$ and $CR(I'_1) \stackrel{2}{=} CR(I_1) - 1$.

- D_2 is obtained from $D - \{e, f, \}$ by adding arc h_2. We again extend B_0 along R_3 and consider h_2 embedded on the extended B_0. We let $I'_2 = I_2$. Hence, $l_{D'}(I_2) \stackrel{2}{=} l_{D_2}(I'_2)$ and $CR(I'_2) \stackrel{2}{=} CR(I_2)$.

Hence for $i = 1, 2$, D_i is an orientation of a proper g-graph and I'_i is an alternating cycle of D_i. Moreover $|BR(I'_i)| < |BR(C)|$.

Let us assume without loss of generality that h_2 is directed so that $l(R_3, h_2) = 1$. Hence D_2 is a relevant orientation with $r(D_2) = r(D)$.

As in Subcase 2.1, we shall show that D_1 is a relevant orientation if and only if $r(D)_{d(e)} = r(D)_{d(f)}$: It again suffices to consider the case that $r(D)_{d(e)} = r(D)_{d(f)} = 1$. In this case it suffices to show that $l_{D_1}(R_2, R_3, R_1, f', h'_1, e') = 1$. In fact, we have $l_{D_2}(R_1, f, h_2) = 1$ since $r(D_{d(f)}) = 1$ and D_2 is a relevant orientation. Hence $l_{D_1}(R_1, f', h'_1) = 0$. Moreover $l_{D_1}(R_2, R_3, e') \stackrel{2}{=} l_D(R_2, R_3, e) = 1$ since $r(D)_{d(e)} = 1$. Hence $l_{D_1}(R_2, R_3, R_1, f', h'_1, e') = 1$.

Summarizing, if $r(D)_{d(e)} = r(D)_{d(f)}$ then D_1 is a relevant orientation with $r(D) = r(D_1)$, and if $r(D)_{d(e)} \neq r(D)_{d(f)}$ then D_1 becomes relevant after reversing the orientation of h'_1.

The proof then proceeds analogously as in Subcase 2.1. *(End of Subcase 2.2)*

End of Case 2

It is not difficult to see that the two cases complete the proof. □

Next we show that a statement analougous to that of Theorem 5.19 holds for the set of the alternating cycles of $M_0\Delta P$ as well.

Theorem 5.20 *Let G be a proper g-graph and let D be a relevant orientation of G. Let P be a perfect matching of G. Then*

$$sgn(D, M_0 \Delta P) = (-1)^q,$$

where

$$q \stackrel{2}{=} |BR(M_0\Delta P)| - CR(M_0\Delta P) + \frac{1}{2} \sum_{e \in BR(M_0\Delta P)} (r(D)_{d(e)} + 1).$$

Proof. Let $C_1, ..., C_k$ be the alternating cycles of $M_0\Delta P$. We have that $sgn(D, M_0\Delta P) = (-1)^q$, where $q \stackrel{2}{=} l(C_1) + ... + l(C_k) - k$.

Using Theorem 5.19 for $C_1, ..., C_k$, it remains to show that:

$$CR(M_0\Delta P) \stackrel{2}{=} \sum_{j=1}^{k} CR(C_j),$$

but this holds by Lemma 5.18 and the theorem follows. □

Corollary 5.21 *Let G be a proper 1-graph and D a relevant orientation of G. Let $\mathcal{C}$ be a set of disjoint alternating cycles of M_0. Then:*

1. *If $r(D) = (1, 1)$ then $sgn(D, \mathcal{C}) = 1$ if and only if $t(\mathcal{C}) \in \{(0, 0), (0, 1), (1, 0)\}$.*
2. *If $r(D) = (1, -1)$ then $sgn(D, \mathcal{C}) = 1$ if and only if $t(\mathcal{C}) \in \{(0, 0), (1, 1), (1, 0)\}$.*
3. *If $r(D) = (-1, 1)$ then $sgn(D, \mathcal{C}) = 1$ if and only if $t(\mathcal{C}) \in \{(0, 0), (0, 1), (1, 1)\}$.*
4. *If $r(D) = (-1, -1)$ then $sgn(D, \mathcal{C}) = 1$ if and only if $t(\mathcal{C}) = (0, 0)$.*

Definition 5.22 *Let G be a proper g-graph and D a relevant orientation of G. Let $r(D) = (r_1, ..., r_{2g})$. We let $c(r(D))$ equal to the product of c_i, $i = 0, ..., g-1$, where $c_i = c(r_{2i+1}, r_{2i+2})$ and $c(1, 1) = c(1, -1) = c(-1, 1) = 1/2$ and $c(-1, -1) = -1/2$.*

Observe that $c(r(D)) = (-1)^n 2^{-g}$, where $n = |\{i; r_{2i+1} = r_{2i+2} = -1\}|$.

Corollary 5.23 *Let G be a proper 1-graph. Let D_1, D_2, D_3, D_4 be the relevant orientations of G. Then*

$$\mathcal{P}(G, x) = \sum_{i=1}^{4} c(r(D_i))\mathcal{P}(D_i, M_0).$$

A result analogous to Corollary 5.23 holds for all proper g-graphs, $g > 1$. In order to deduce it we start with another corollary of Theorem 5.20.

Corollary 5.24 *Let G be a proper g-graph and D a relevant orientation of G. Let P be a perfect matching of G. Then $sgn(D, M_0\Delta P)$ is a function of $r(D)$ and $t(M_0\Delta P)$ only. Let us denote this function by $\sigma(r(D), t(M_0\Delta P))$.*

Lemma 5.25 *Let $r = (r_1, \ldots, r_{2g})$ and $t = (t_1, \ldots, t_{2g})$ be $2g$-dimensional vectors. Let $r(j) = (r_{2j+1}, r_{2j+2})$ and $t(j) = (t_{2j+1}, t_{2j+2})$, $j = 0, \ldots, g-1$. Then*

$$\sigma(r, t) = \prod_{j=0}^{g-1} \sigma(r(j), t(j)).$$

Proof. By Corollary 5.24, we have that $sgn(D, \mathcal{C}) = sgn(D', \mathcal{C}')$ if and only if $r(D) = r(D')$ and $t(\mathcal{C}) = t(\mathcal{C}')$. This implies that we can restrict ourselves to consider the following case: $G = C_0 \cup M_0 \cup \mathcal{C}$ is a proper g-graph, D is a relevant orientation of G such that $r(D) = r$ and $\mathcal{C}$ consists of a set of vertex-disjoint cycles $C_1, \ldots, C_k$ satisfying the following properties:

1. each C_i is alternating with respect to the perfect matching M_0,
2. for each i, j $|E^i_j| \leq 1$,
3. for each i there is at most one j such that $|C_j \cap (E^i_1 \cup E^i_2)| \geq 1$,
4. for each C_j there is exactly one i such that C_j intersects $E^i_1 \cup E^i_2$,
5. $t(\mathcal{C}) = t$.

Hence,

$$\sigma(r, t) = sgn(D, \mathcal{C}) = \prod_{i=1}^{k} sgn(D, C_i) = \prod_{i=1}^{k} sgn(D_i, C_i)$$

where D_i is the restriction of D to $C_0 \cup C_i$. Observe that, by Corollary 5.21, $\sigma(z_1, z_2) = 1$ if $z_2 = (0, 0)$. Hence, using Corollary 5.24, we have that $\prod_{i=1}^{k} sgn(D_i, C_i) = \prod_{j=0}^{g-1} \sigma(r^j, t^j)$ as claimed.

□

Theorem 5.26 *Let G be a proper g-graph. Then*

$$\mathcal{P}(G, x) = \mathcal{L}_g(G, x) = \sum_{i=1}^{4^g} c(r(D_i))\mathcal{P}(D_i, M_0)$$

where D_i, $i = 1, \ldots, 4^g$, are the relevant orientations of G.

Proof. Let P be a perfect matching of G. In each term $\mathcal{P}(D_i, M_0)$, the coefficient of $x(P)$ is $sgn(D_i, M_0\Delta P)$. By Corollary 5.24, $sgn(D_i, M_0\Delta P) = \sigma(r(D_i), t(M_0\Delta P))$.

Let

$$\mathcal{K}_g(t(M_0\Delta P)) = \sum_{i=1}^{4^g} c(r(D_i))\sigma(r(D_i), t(M_0\Delta P))$$

denote the coefficient of $x(P)$ in $\mathcal{L}_g(G, x)$.

To prove the theorem it suffices to prove the following claim:

Claim. $\mathcal{K}_g(t(M_0 \Delta P)) = 1$ *for each* $t(M_0 \Delta P)$.

The proof of the claim is by induction on g. The basis of the induction when $g = 1$ is proved in Corollary 5.23.

To prove the inductive step we introduce the following notation: if z is a $2g$-dimensional vector then we let $z = (z(0), \dots, z(g-1))$ where $z(i) = (z_{2i+1}, z_{2i+2})$.

We call two relevant orientations D and D' of G *equivalent* if $(r(D)(1), \dots, r(D)(g-1)) = (r(D')(1), \dots, r(D')(g-1))$. Clearly, the equivalence classes consist of 4 elements; let $\mathcal{R}_1, \dots, \mathcal{R}_{4^{g-1}}$ be the equivalence classes of the relevant orientations of G and let $\mathcal{R}_j = \{D_1^j, D_2^j, D_3^j, D_4^j\}$, $j = 1, \dots, 4^{g-1}$.

Finally let $r(D_i^j)(k) = r_i^j(k)$, $k = 0, \dots, g-1$ and let $t = t(M_0 \Delta P)$. We have that

$$\mathcal{K}_g(t) = \sum_{j=1}^{4^{g-1}} \sum_{i=1}^{4} c(r(D_i^j))\sigma(r(D_i^j), t).$$

Now, by Lemma 5.25, this equals

$$\sum_{j=1}^{4^{g-1}} \sum_{i=1}^{4} c(r_i^j(0))c(r_i^j(1), \dots, r_i^j(g-1)) \prod_{k=0}^{g-1} \sigma(r_i^j(k), t(k)).$$

By the definition of the equivalence classes, $r_1^j(k) = r_2^j(k) = r_3^j(k) = r_4^j(k)$ for $k \geq 1$ and $j = 1, \dots, 4^{g-1}$. Hence, we let $r_i^j(k) = r^j(k)$ and write the above summation as:

$$\sum_{j=1}^{4^{g-1}} c(r^j(1), \dots, r^j(g-1)) \prod_{k=1}^{g-1} \sigma(r^j(k), t(k)) \sum_{i=1}^{4} c(r_i^j(0))\sigma(r_i^j(0), t(0))$$

.

The internal sum equals to 1 for each $j = 1, \dots, 4^{g-1}$ by the basis step of the induction, and hence, using Lemma 5.25 in the external sum, we can write the above summation as

$$\sum_{j=1}^{4^{g-1}} c(r^j(1), \dots, r^j(g-1))\sigma((r^j(1), \dots, r^j(g-1)), (t(1), \dots, t(g-1))) =$$

$$\mathcal{K}_{g-1}(t(1), \dots, t(g-1)) = 1,$$

by the inductive hypothesis for $g-1$. **End of Claim.**

□

As a consequence of Theorem 5.7 and Theorem 5.26, we have:

Corollary 5.27 *Let* G *be a proper* g*-graph. Then* $s(D_i, M_0) = s(D_j, M_0)$ *for each* $i, j \in \{1, \dots, 4^g\}$ *and*

$$\mathcal{P}(G, x) = \mathcal{L}_g(G, x) = s(D_1, M_0) \sum_{i=1}^{4^g} c(r(D_i))Pf(A(D_i))$$

where D_i, $i = 1, \dots, 4^g$, *are the relevant orientations of* G.

Theorem 5.28 *Let G be a graph embeddable on an orientable surface of genus g. Then $\mathcal{P}(G,x)$ may be expressed as a linear combination of 4^g Pfaffians of matrices $A(D)$, where each D is an orientation of G.*

Proof. As observed in the previous section, any orientable surface $\mathcal{S}$ of genus g may be obtained from its polygonal representation S_g as follows: for each bridge B, glue together the two segments in which B intersects the boundary of B_0, and delete B.

If a graph G is embedded on an orientable surface $\mathcal{S}$ of genus g, then without loss of generality no vertex belongs to the boundary of B_0. In this way we get an embedding of G on S_g such that all vertices of G belong to B_0 but the embeddings of some edges may use several bridges.

We construct a graph G' by replacing each edge $e = uv$ which uses k bridges, $k \geq 1$, by a path $P_e = (u, e_1, v_1, \ldots, v_{2k}, e_{2k+1}, v)$. The new vertices $v_1, ..., v_{2k}$ are embedded on the embedding of e so that each new edge uses at most one bridge. Moreover, we let $x'_{e_1} = x_e$ and $x'_{e_i} = 1$ for each $i > 1$. We do a similar construction when G_0 has no perfect matching. In fact, take any perfect matching M of G and replace any edge $e = uv \in M$ embedded on a bridge by a path $u, e_1, y, e_2, z, e_3, v$ and let $x'_{e_1} = x_e$ and $x'_{e_2} = x'_{e_3} = 1$. Then leave the only edge e_2 to be embedded on the bridge B.

Finally, we add edges so that the outer face of the planar part is a cycle and we let $x'_e = 0$ for each such edge e.

It is easy to see that G' is a proper g-graph and that $\mathcal{P}(G', x') = \mathcal{P}(G, x)$.

Now, by Theorem 5.26, $\mathcal{P}(G', x')$ may be written as a linear combination of 4^g terms $Pf(A(D'))$, where each D' is a relevant orientation of G'.

It remains to show that for each relevant orientation D' of G' there is an orientation D of G such that $Pf(A(D')) = Pf(A(D))$ or $Pf(A(D')) = -Pf(A(D))$.

We construct D from D' in two steps:

1. delete the edges e of $G' - G$ with $x'_e = 0$,

2. for each edge e of G which was changed into a path P_e of odd length in the construction of G', orient e in the direction in which an odd number of edges of P_e is directed in D': this is uniquelly determined since P_e has an odd length.

If P is a perfect matching of G then there is a unique perfect matching P' of G' such that $x(P) = x'(P')$.

Observe that $sgn(D, P\Delta Q) = sgn(D', P'\Delta Q')$ for each pair of perfect matchings P, Q of G. The claim now follows from Theorem 5.7.

This finishes the proof of the theorem. □

Hence, here is a method to solve the Ising problem for finite graphs: translate the problem first to the problem of enumeration of the perfect matchings of a slightly modified graph, and then draw that graph on an orientable surface and use the theorem of Galluccio and Loebl. Now, how realistic is this approach? Given the complex nature of the procedure, we doesnot feel obliged to argue against (it is easy, isn't it?). So let us argue for it:

1. This approach lead to a very successful implementation (by Jan Vondrak) of Ising partition function calculation for toroidal graphs. In fact, it even gives the best performance for detecting maximum cuts for these graphs, when the weights are from $\{1, -1\}$. Just imagine it: for toroidal square grids, the best known computer program to detect max cut finds the whole density as well, i.e. the number of max cuts, the number of cuts which are second largest, etc.

2. There is a good chance that it will be applicable in knot theory, where some animals live on surfaces of bounded genus.

3. The lattices important to physics typically do not have bounded genus. Look for instance at the cubic lattice: its genus is proportional to its number of edges. However still we can apply the method to the cubic lattice, in principle. I include my derivation in the subsection below. I really do not know whether the expression it gives will be of any use: but, it is sufficiently crazy and beautiful and anyway, there doesnot seem to be any method with a slightest success towards solving the 3d Ising problem, so perhaps 'less straightforward' methods may be discussed.

5.5 3-dimensional dimer problem.

We present a new expression for the dimer problem, i.e. for the enumeration of dimer arrangements (perfect matchings) of the 3-dimensional cubic lattice. As a consequence this provides a new expression also for the 3-dimensional Ising problem.

The *close-packed dimer problem* of statistical mechanics can be stated as follws. One considers a set of sites and a set of bonds connecting certain pairs of sites. Each bond b can absorb a 'dimer' (which represents a diatomic molecule) with corresponding energy E_b. It is required that each site is occupied exactly once by one of the atoms of a dimer. A state s is an arrangement of dimers which meets this requirement, and its energy $E(s)$ is $\sum E_b$ where the sum is taken over all bonds b which absorb a dimer. Then the partition function may be viewed simply as a density function of energy levels.

The dimer problem was first considered by Roberts ([20]) in 1935, and by Fowler and Rushbrook ([3]). The dimer problem for 2-dimensional lattices appears in calculations of the thermodynamic properties of a system of diatomic molecules-dimers. It was solved by Kasteleyn ([11]) and by Temperley, Fisher ([8]). The same problem for 3-dimensional lattices remains an important open problem of statistical physics (see [14] for references). Many fundamental observations about the dimer and monomer-dimer problem in general graphs are given by Heilmann, Lieb ([6, 7]).

We may reformulate the problem in graph theoretical terms as follows. A graph is a pair $G = (V, E)$ where V is a set of *vertices* and E is a set of unordered pairs of elements of V, called *edges*. A graph with some regularity properties is often called a *lattice*. We associate with each edge e of G a variable x_e and we let $x = (x_e : e \in E)$. For each $M \subset E$, let $x(M)$ denote the product of the variables of the edges of M.

A subset of edges $P \subset E$ is called *perfect matching* or *dimer arrangement* if each vertex belongs to exactly one element of P. The dimer problem may be formulated as the problem to find polynomial $\mathcal{P}(G,x)$ which equals the sum of $x(P)$ over all perfect matchings P of G.

Let m be odd positive integer and k even positive integer. The cubic lattice $Q_{m,m,k}$ is the following graph:

Q_{mmk} has vertices $V_{xyz}(Q_{mmk})$, $x,y = 1,...,m$, $z = 1,...,k$, and the following edges:

1. The edges $v_{xyz}(Q_{mmk}) = \{V_{xyz}(Q_{mmk}), V_{xy(z+1)}(Q_{mmk})\}$, $z = 1,...,k-1$, called *vertical*,
2. The edges $w_{xyz}(Q_{mmk}) = \{V_{xyz}(Q_{mmk}), V_{x(y+1)z}(Q_{mmk})\}$, $y = 1,...,m-1$, called *width*,
3. The edges $h_{xyz}(Q_{mmk}) = \{V_{xyz}(Q_{mmk}), V_{(x+1)yz}(Q_{mmk})\}$, $x = 1,...m-1$, called *horisontal*.

Let us denote the ordered set $(V_{xy1}(Q_{mmk}),...,V_{xyk}(Q_{mmk}))$ by $V_{xy}(Q_{mmk})$. $V_{xy}(Q_{mmk})$ will also stand for the vertical path of Q_{mmk} from $V_{xy1}(Q_{mmk})$ to $V_{xyk}(Q_{mmk})$. Sometimes we let $V_{xy}(Q_{mmk}) = V_{xy}$ if no confusion may arise and we denote by $\bar{V}_{xy}$ the oppositely ordered set $(V_{xyk}(Q_{mmk}),...,V_{xy1}(Q_{mmk}))$

Q_{mmk} is a bipartite graph, which means that its vertices may be partitioned into two sets Z_1, Z_2 such that if e is an edge of Q_{mmk} then $|e \cap Z_1| = |e \cap Z_2| = 1$. Moreover, we have also that $|Z_1| = |Z_2| = mmk/2$. Let $\mathcal{Z}$ be square $(Z_1 \times Z_2)$ matrix defined by $\mathcal{Z}_{ij} = x_{ij}$ if ij is an edge of Q_{mmk} and $\mathcal{Z}_{ij} = 0$ otherwise.

We will consider matrix $\mathcal{Z}$ with its rows and columns ordered in aggreement with the order $(V_{11}(Q_{mmk})V_{12}(Q_{mmk})....V_{1m}(Q_{mmk})V_{21}(Q_{mmk}).....V_{mm}(Q_{mmk}))$ and we will assume that $V_{111}(Q_{mmk}) \in Z_1$.

Note that $\mathcal{P}(Q_{mmk},x)$ equals the permanent of $\mathcal{Z}$.

A seminal observation of Heilmann and Lieb [6, 7] asserts that $\mathcal{P}(Q_{mmk},x)$ equals the average of $(det(Z))^2$ over ALL signings Z of $\mathcal{Z}$, where a *signing* of a matrix is obtained by multiplying some of its entries by -1.A proof of this observation included below is taken from the monograph [15]. We say that signing Z of $\mathcal{Z}$ corresponds to orientation D of Q_{mmk} if $Z_{ij} = x_{ij}$ if $(ji) \in E(D)$, $Z_{ij} = -x_{ij}$ if $(ij) \in E(D)$, and $Z_{ij} = 0$ otherwise.

Theorem 5.29 *$\mathcal{P}(Q_{mmk},x)$ equals the average of $(det(Z))^2$ over ALL signings Z of $\mathcal{Z}$.*

Proof. If D is an orientation of Q_{mmk} then let $A(D)$ denote the skew-symmetric adjacency matrix of D, i.e. matrix consisting of 4 blocks where both blocks on the main diagonal are 0-matrices and the remaining two blocks equal Z and $-Z$, where Z is the signing of $\mathcal{Z}$ corresponding to D. Clearly $det(A(D)) = (det(Z))^2$, hence we need to show that $\mathcal{P}(Q_{mmk},x)$ equals the expectation of $det(A(D))$ over all orientations D of Q_{mmk}. For the expectation we have

$$E(det(A(D))) = \sum sgn(\pi)E(a_{1\pi(1)}...a_{n\pi(n)})$$

where $n = mmk$ and $A(D) = (a_{ij})$. If π is a permutation having a fix point or such that i and $\pi(i)$ are non-adjacent for some $i \leq n$ then the term corresponding to π equals 0. If there is i such that $\pi(\pi(i)) \neq i$ then the random variable $a_{i\pi(i)}$ occurs in the term corresponding to π but the random variable $a_{\pi(i)i}$ doesnot. Hence $E(a_{1\pi(1)}...a_{n\pi(n)}) = E(a_{i\pi(i)})E(a_{1\pi(1)}...a_{(i-1)\pi(i-1)}a_{(i+1)\pi(i+1)}...a_{n\pi(n)}) = 0$. So we are left with the terms corresponding to those permutations which have no fix point, for which i and $\pi(i)$ are adjacent and $(\pi)^2$ is the identity. Such permutations uniquelly correspond to perfect matchings of Q_{mmk} and the signs turn out correct. □

Here we show that $\mathcal{P}(Q_{mmk}, x)$ may be computed from the average of determinants of CERTAIN signings of $\mathcal{Z}$.

A difference of our expression with the result of Heilmann and Lieb is that we replace the average of a multiquadratic function by the average of a multilinear function, with a restricted range though. This may help to solve the dimer problem and the Ising problem in three dimensions.

The proof of our result is involved: we embed the three-dimensional cubic lattice to a 2-dimensional orientable surface, use a Theorem of Galluccio and Loebl ([4]) to express $\mathcal{P}(Q_{mmk}, x)$ as a linear combination of Pfaffians of matrices associated with orientations of Q_{mmk} and finally characterize the coefficients of this linear combination.

Statement of the main result.

If $D = (V, A)$ is an orientation of Q_{mmk} and $e \in A$ then we say that e is oriented according to the *natural ordering* if its orientation is in aggreement with the ordering $(V_{11}(Q_{mmk})V_{12}(Q_{mmk})....V_{1m}(Q_{mmk})V_{21}(Q_{mmk}).....V_{mm}(Q_{mmk}))$ of the vertices of Q_{mmk}.

An orientation D is called *stable* if the edges of each $V_{xy}(Q_{mmk})$ are oriented in D in aggreement with the natural ordering.

Let D be a stable orientation. We define orientation $\bar{D}$ as follows:

1. For each x, y, z such that $y < m$ odd do the following: let $n(xyz)$ be the number of arcs w_{xab}, $a \leq y$ odd and $b = z$ or $b = z+1$, oriented in D against the natural ordering. If $n(xyz)$ odd then we orient w_{xyz} in $\bar{D}$ against the natural ordering, else according to the natural ordering.

2. For each x, y, z such that $x < m$ odd do the following: let $n(xyz)$ be the number of arcs h_{abc} oriented in D against the natural ordering. Here (abc) are the triples of indices satisfying $a \leq x$ odd and $(b, c) = (y, z)$ or $(b, c) = (y', z')$ where (y', z') is such that vertex $V_{x,y',z'}$ is immediate successor of vertex $V_{x,y,z}$ in the following ordering of vertices of Q_{mmk}: $(V_{11}, \bar{V}_{12}, V_{13}, ...V_{1m}, \bar{V}_{2m}, V_{2(m-1)}, ..., \bar{V}_{21}, V_{31}, ..., V_{mm})$.

   If $n(xyz)$ odd then we orient h_{xyz} in $\bar{D}$ against the natural ordering, else according to the natural ordering.

3. All the remaining arcs orient in $\bar{D}$ in the same way as in D.

Let D be a stable orientation. Then we let

$$sgn(D)) = (-1)^{h+\sum_{x=1}^{m} w(x)},$$

where $w(x) = |\{(yz); y$ even and both $w_{xyz}, w_{x,(y-1),z}$ are oriented against the natural ordering in $\bar{D}\}|$; $h = |\{(xyz); x$ even and both $w_{xyz}, w_{(x-1),y,z}$ are oriented against the natural ordering in $\bar{D}\}|$.

Let D be an orientation of Q_{mmk}. Let us associate a signing $Z(D)$ of $\mathcal{Z}$ with it such that $Z(D)_{ij} = x_{ij}$ if $(ji) \in E(D)$, $Z(D)_{ij} = -x_{ij}$ if $(ij) \in E(D)$, and $Z(D)_{ij} = 0$ otherwise.

Now we state the main results of the paper. Let $\alpha = \prod_{e \in M} x_e$ where M is unique perfect matching of the collection of paths $V_{11}(Q_{mmk}) \cup ... \cup V_{mm}(Q_{mmk})$.

Theorem 5.30 *$\mathcal{P}(Q_{mmk}, x) = -2^{C_r}\alpha + \beta(2^{C_r} + 1)$, where β equals the average of $det(Z(D))$, D stable orientation of Q_{mmk} with $sgn(D) = +1$, and $C_r = km(m-1)$.*

Some basic notions and facts.

Definition 5.31 *Let G be a graph and let G' be an even subdivision of G. Let D' be an orientation of G'. An orientation D of G induced by D is constructed as follows: for each edge e of G which was changed into a path P_e in the construction of G', orient e in the direction in which an odd number of edges of P_e is directed in D': this is uniquelly determined since P_e has an odd length.*

Let G be a graph and let x be a vector of variables associated with the edges of G. Let G' be an even subdivision of G and let x' be a vector of variables associated with the edges of G' induced by x. If P is a perfect matching of G then observe there is a unique perfect matching P' of G' such that $x(P) = x'(P')$.

Observe that $sgn(D, P\Delta Q) = sgn(D', P'\Delta Q')$ for each pair of perfect matchings P, Q of G. Hence the following theorem follows from Theorem 5.7.

Theorem 5.32 *Let G be a graph and let x be a vector of variables associated with the edges of G. Let G' be an even subdivision of G and let x' be a vector of variables associated with the edges of G' induced by x. Let D' be an orientation of G' and let D be the orientation of G induced by D'. Let M be an arbitrary perfect matching of G. Then $s(D, M)s(D', M')Pf(A(D')) = Pf(A(D))$.*

We remind that the following theorem is proved in Galluccio, Loebl [4].

Theorem 5.33 *Let G be a g-graph with a perfect matching $M_0 \subset E_0$. If we order the vertices of G so that $s(D_0, M_0)$ is positive then*

$$\mathcal{P}(G, x) = \sum_{i=1}^{4^g} c(r(D_i))Pf(A(D_i))$$

where D_i, $i = 1, \ldots, 4^g$, are the relevant orientations of G.

We need a generalisation of the notion of a g-graph.

Definition 5.34 *Any graph G obtained by the following construction will be called generalised g-graph.*

1. *Let $g = g_1 + ... + g_n$ be a partition of g into positive integers.*

2. *Let S_{g_i} be a surface of genus g_i, $i = 1, ..., n$. Let us denote the basis and the bridges of S_{g_i} by B_0^i and $B_{j,k}^i$, $i = 1, ..., n$, $j = 1, ..., g_i$ and $k = 1, 2$.*

3. *For $i = 1, ..., n$ let H_i be a g_i-graph with the property that the subgraph of H_i embedded on B_0^i is a cycle, embedded on the boundary of B_0^i. Let us denote it by C^i.*

4. *Let G_0 be a 2-connected plane graph and let $F_1, ..., F_n$ be a subset of faces of G_0. Let K^i be the cycle boundering F_i, $i = 1, ..., n$. Let each K^i be isomorphic to C^i.*

5. *Then G is obtained by glueing the H_i's into G_0 so that each K^i is identified with C_i.*

For each generalised g-graph G we can define 4^g relevant orientations $D_1, ..., D_{4^g}$ with respect to a fixed basic orientation of G_0, and coefficients $c(r(D_i))$, $i = 1, ..., n$ in the same way as for a g-graph. The following theorem can be proved in the same way as Theorem 5.26.

Theorem 5.35 *Let G be a generalised g-graph with a perfect matching M_0 of G_0. Let D_0 be a basic orientation of G_0. If we order the vertices of G so that $s(D_0, M_0)$ is positive then*

$$\mathcal{P}(G, x) = \sum_{i=1}^{4^g} c(r(D_i)) Pf(A(D_i))$$

where D_i, $i = 1, \ldots, 4^g$, are the relevant orientations of G.

Cubic lattices as generalised g-graphs.

In this section we will describe how to draw 3−dimensional cubic lattices as generalised g-graphs.

The Cubic lattice.

Let m, n be odd positive integers such that $k = (n-1)/2$ is even. The cubic lattice $Q = Q_{m,m,n}$ is the following graph: It has vertices $V_{xyz}(Q) = V_{xyz}$, $x, y = 1, ..., m$, $z = 1, ..., n$, and the following edges:

1. The edges $\{V_{xyz}, V_{xy(z+1)}\} = v_{xyz}(Q) = v_{xyz}$, $z = 1, ..., n-1$, called *vertical*,

2. The edges $\{V_{xyz}, V_{x(y+1)z}\} = w_{xyz}(Q) = w_{xyz}$, $y = 1, ..., m-1$, called *width*,

3. The edges $\{V_{xyz}, V_{(x+1)yz}\} = h_{xyz}(Q) = h_{xyz}$, $x = 1, ...m-1$, called *horisontal*.

Let us denote vertical subpath $(V_{xy1},...,V_{xyn})$ by $V_{xy}(Q) = V_{xy}$ and let $\bar{V}_{xy}$ denote V_{xy} traversed in the oposite direction.

Let $H_x(Q) = H_x = \{h_{xyz}; z = 1,...,n, y = 1,...,m\}$ and $W_{xy}(Q) = W_{xy} = \{w_{xyz}; z = 1,...n\}$.

Next we describe a drawing of Q on the plane.

First draw the paths V_{xy} along a cycle in the following order:

$$V_{11}, \bar{V}_{12}, V_{13}, ...V_{1m}, \bar{V}_{2m}, V_{2(m-1)}, ..., \bar{V}_{21}, V_{31}, ..., V_{mm}.$$

Next, draw the horisontal edges inside this cycle, and the width edges outside of this cycle as depicted in Fig. 1 below.

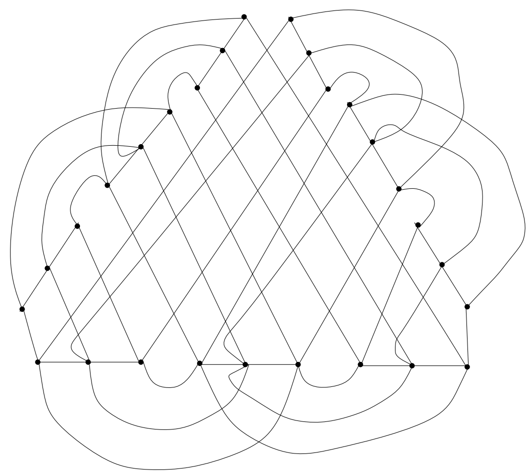

Figure 1

For each $x = 1,...,m-1$ the curves representing the edges of H_x are pairwise disjoint and for $x = 2,...,m-2$ the curves representing the edges of H_x intersect the curves representing the edges of H_{x-1} and H_{x+1}. We keep the following rule: the interiors of the curves representing h_{xyz} and $h_{(x+1)yz}$ intersect if and only if z is even.

For each $x = 1,...,m$ and $y = 1,...,(m-1)$ the curves representing the edges of W_{xy} are pairwise disjoint and for $y = 2,...,m-2$ the curves representing the edges of W_{xy} intersect the curves representing the edges of $W_{x(y-1)}$ and $W_{x(y+1)}$. We again keep the rule that the interiors of the curves representing w_{xyz} and $w_{x(y+1)z}$ intersect if and only if z is even. The curve representing an edge e will be denoted by $C(e)$.

Now we modify Q into a generalised g-graph Q'.

First we describe the modification for W_x, $x = 1,...,m$. The modification is illustrated in Fig. 2 where the construction is illustrated on edges among $V_{x(y-1)}, V_{xy}$ and $V_{x(y+1)}$ for $y < m-1$ even.

For each $x = 1,...,m$ perform the following construction:

1. For each y even let $Aux_1 = \{w_{xyz}; z \text{ odd }\}$. For each edge e of Aux_1 introduce a new vertex to each intersection of $C(e)$ with the curves representing the edges of $W_{x(y-1)} \cup W$, where $W = W_{x(y+1)}$ in case $y < m-1$ and $W = \emptyset$ otherwise.

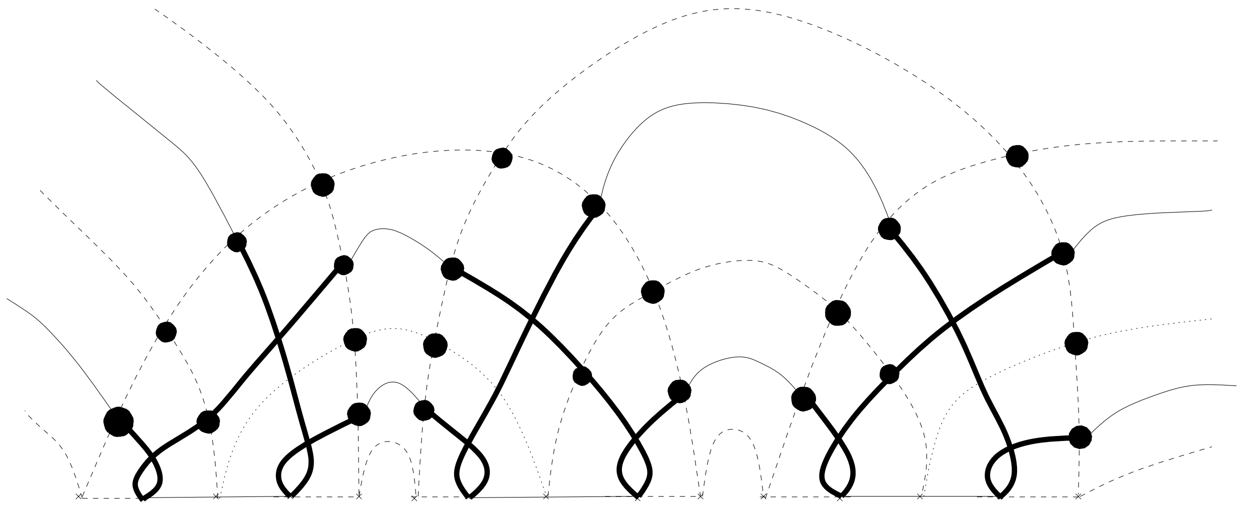

Figure 2

By this operation, each $e \in Aux_1$ is replaced by a path. Call each edge of this path *auxiliary*.

2. For each y even let $Aux_2 = \{w_{x(y-1)1}, w_{x(y-1)n}\} \cup A$, where $A = \{w_{x,(y+1)1}, w_{x(y+1)n}\}$ in case $y < m-1$ and $A = \emptyset$ otherwise. For each edge e of Aux_2 introduce a new vertex to each intersection of $C(e)$ with the curves representing the edges of W_{xy}. Hence each $e \in Aux_2$ is replaced by a path. Call each edge of this path *auxiliary*.

   For each y even the edges $v_{xy1}, v_{xy(n-1)}$ and also $v_{x(y+1)1}, v_{x(y+1)(n-1)}$ will also be called *auxiliary*.

   In Figure 2, the auxiliary edges are represented by dashed lines.

3. We introduce a new variable a and let $x_e = a$ for each auxiliary edge e.

4. The edges w_{xyz}, y even and z even will be called *relevant* for Q. If $y < m-1$ then the relevant edges are subdivided by two vertices (added in 2.) into three edges of Q'. The middle one will be called *special* and the other two *long*.

   If $y = m-1$ then the relevant edge w_{xyz} is subdivided by one vertex into two edges of Q'. The one incident to V_{xm} will be called *special* and the other one *long*.

   If e is a relevant edge of Q, then we choose a corresponding long edge f and we let $x_e = x_f$. We let the variable of the special edge and of the remaining long edge be equal to 1.

5. The edges of $W_{x(y-1)} \cup W$ also got subdivided by new vertices introduced in step 2 and step 3.

6. We delete all edges of the paths obtained from $w_{x(y-1)z}$ and $w_{x(y+1)z}$, $1 < z < n$ odd.

In Figure 2, the deleted edges are represented by dotted lines.

7. Each edge $w \in \{w_{x(y-1)z}, w_{x(y+1)z}; z \text{ even }\}$, is subdivided by new vertices introduced in 2. into a path. We let the variables assigned to the edges of the path equal to 1 except of one initial edge whose variable is let equal x_w. The edge e of this path such that the interior of $C(e)$ doesnot intersect interior of any curve representing a long edge will also be *special.*

8. All vertical edges which are not auxiliary (see 2.) will be called *special.*

   In Figure 2, the special edges are represented by normal lines.

This finishes the construction for the width edges. In Figure 2, the edges which are neither auxiliary nor special nor deleted are represented by fat lines.

Now we perform an analogous construction with the horisontal edges of Q.

1. For each x even let $Aux_3 = \{h_{xyz}; z \text{ odd }\}$. For each edge e of Aux_3 introduce a new vertex to each intersection of $C(e)$ with the curves representing the edges of $H_{x-1} \cup K$, where $K = H_{x+1}$ in case $x < m-1$ and $K = \emptyset$ otherwise. By this operation, each $e \in Aux_3$ is replaced by a path. Call each edge of this path *auxiliary.*

2. For each x even let $Aux_4 = \{h_{(x-1)11}, h_{(x-1)nn}\} \cup B$, where $B = \{h_{(x+1)11}, h_{(x+1)nn}\}$ in case $x < m-1$ and $B = \emptyset$ otherwise. For each edge e of Aux_4 introduce a new vertex to each intersection of $C(e)$ with the curves representing the edges of H_x. Hence each $e \in Aux_4$ is replaced by a path. Call each edge of this path *auxiliary.*

3. We let $x_e = a$ for each auxiliary edge e.

4. The edges h_{xyz}, x even and z even will be called *relevant* for Q. If $x < m-1$ then the relevant edges are subdivided by two vertices (added in 2.) into three edges of Q'. The middle one will be called *special* and the other two *long.*

   If $x = m-1$ then the relevant edge h_{xyz} is subdivided by one vertex into two edges of Q'. The one incident to V_m . will be called *special* and the other one *long.*

   If e is a relevant edge of Q, then we choose a corresponding long edge f and we let $x_e = x_f$. We let the variable of the special edge and of the remaining long edge equal 1.

5. The edges of $H_{x-1} \cup K$ also got subdivided by new vertices introduced in step 2 and step 3.

6. We delete all edges of the paths obtained from $h_{(x-1)yz}$ and $h_{(x+1)yz}$, $1 < z < n$ odd.

7. Each edge $h \in \{h_{(x-1)yz}, h_{(x+1)yz};\ z \text{ even }\}$, is subdivided by new vertices introduced in 2. into a path. We let the variables assigned to the edges of the path equal to 1 except of one initial edge whose variable is let equal x_h. Each edge e of this path such that the interior of $C(e)$ doesnot intersect interior of any curve representing a long edge will also be *special*.

Finally, let Aux denote the set of all auxiliary edges. Then $Q' - Aux$ is a subdivision of Q_{mmk}. We conclude the construction of Q' by subdividing some special edges so that the graph $\mathcal{Q} = Q' - Aux$ is an even subdivision of Q_{mmk}.

This finishes the construction of Q'.

Some properties of Q'.

1. Each edge e of Q' such that $C(e)$ doesnot intersect any curve representing other edge in its interior is auxiliary or special. Let us denote the plane subgraph of Q' formed by the auxiliary and special edges by Q^p.

2. Any other edge of Q' is drawn on a face of Q^p. Moreover, the edges drawn on a face of Q^p may be drawn onto a pair of bridges above this face, where one of the bridges contains one long edge, and the other bridge contains the remaining edges. Hence, we may view Q' as a generalised g-graph with the planar part equaled to Q^p.

3 The special edges form an acyclic subgraph of Q^p (see Fig. 2). Hence any orientation of the special edges may be extended into a basic orientation of Q^p. We will choose basic orientation D^p of Q^p with the following properties:

   1. D^p on special edges is in aggreement with the ordering $(V_{11}V_{12}....V_{1m}V_{21}.....V_{mm})$,
   2. Q^p has a perfect matching M whose sign in $Pf(A(D^p))$ is positive,
   3. The orientation of edges on a bridge has positive sign if and only if it is in aggreement with the ordering $(V_{11}V_{12}....V_{1m}V_{21}.....V_{mm})$.

4. We constructed Q' so that $\mathcal{Q} = Q' - Aux$ is an even subdivision of Q_{mmk}. If x is a vector of variables associated with Q_{mmk} then let x' be the vector of variables associated with $\mathcal{Q}$ and induced by x and let x'' be a vector of variables associated with Q' which equals x' on the edges of $\mathcal{Q}$ and $x''_e = a$ for each special edge of Q'. If we let $a = 0$ then $\mathcal{P}(Q', x'') = \mathcal{P}(Q_{mmk}, x)$.

We have described how to view Q_{mmk}, m odd and k even, as a generalised g-graph Q'. Now we can use Theorem 5.35 for Q' to compute $\mathcal{P}(Q_{mmk}, x)$.

The relevant orientations of Q'.

Each relevant edge of Q corresponds to unique edge of Q_{mmk}; this unique edge will also be called *relevant* in Q_{mmk}. Hence the relevant edges of Q_{mmk} are: $w_{xyz}(Q_{mmk})$, $x = 1, ..., m$, y even and and $h_{xyz}(Q_{mmk})$, x even. Hence there are $1/2km(m-1)$ relevant width edges and $1/2(m-1)mk$ relevant horisontal edges in Q_{mmk}.

We let $\mathcal{R}$ be the set of relevant edges of Q_{mmk} and $C_r = |\mathcal{R}| = km(m-1)$ denote the number of relevant edges of Q_{mmk}.

The set $\mathcal{S}$ of the edges of $V_{ij}(Q_{mmk})$, $i,j = 1, ...m$, corresponds to a subset of special edges of Q'. The orientation D^p induces orientation $\mathcal{S}^d$ of $\mathcal{S}$ which is in aggreement with the ordering $(V_{11}(Q_{mmk})V_{12}(Q_{mmk})....V_{1m}(Q_{mmk})V_{21}(Q_{mmk}).....V_{mm}(Q_{mmk}))$.

Each relevant orientation D' of Q' is determined by the fixed basic orientation D^p of Q^p, and by a pair of signs for each pair of bridges. Each pair of bridges is associated with a long edge of Q'. Hence these signs may be given by specifying $(d^1_{D'}(e), d^2_{D'}(e)) \in \{+-\}^2$, for each long edge e, where $d^1_{D'}(e)$ denotes the sign of the bridge containing e, and $d^2_{D'}(e)$ denotes the sign of the other bridge.

The long edges of Q' are associated with relevant edges of Q, and hence also with relevant edges of Q_{mmk}.

The relevant edges $w_{x(m-1)z}(Q_{mmk})$ and $h_{(m-1)yz}(Q_{mmk})$ are associated with only one long edge of Q'. If e is such relevant edge of Q_{mmk}, we will call e *border edge* and we denote by e_1 the corresponding long edge. We let $d_{D'}(e) = (d^1_{D'}(e_1), d^2_{D'}(e_1), +, +)$.

Let $C_b = 2mk$ denote the number of border edges.

Each relevant non-border edge e of Q_{mmk} has two long edges e_1, e_2 associated with it. We let $d_{D'}(e) = (d^1_{D'}(e_1), d^2_{D'}(e_1), d^1_{D'}(e_2), d^2_{D'}(e_2)))$.

A *relevant vector* is any element r of $[\{+,-\}^4]^{\mathcal{R}}$ such that $r(e)_3 = r(e)_4 = 1$ for each relevant border edge e of Q_{mmk}. Hence there are $4^{2C_r - C_b}$ relevant vectors.

There is a bijection between relevant orientations of Q' and relevant vectors. If s is a relevant vector, then let $D'(s)$ denote the corresponding relevant orientation of Q' and let $sgn(s)$ equals 1 or -1 according to the sign of $c(r(D'(s)))$. The sign $sgn(s)$ of a relevant vector s may be calculated according to Theorem 5.35 as follows: $sgn(s) = (-1)^{|\{(e,i); i=0,1, s(e)_{2i+1} = s(e)_{2i+2} = -1\}|}$.

If $D'(s)$ is a relevant orientation of Q' then let $D(s)$ denote the orientation of Q_{mmk} induced by $D'(s)$ (see 5.31). An orientation of Q_{mmk} will be called *relevant* if it equals $D(s)$ for some relevant vector s.

Hence using Theorem 5.35 and Theorem 5.32 we have the following.

Theorem 5.36

$$\mathcal{P}(Q_{mmk}) = 2^{-2C_r + C_b} \sum sgn(s) Pf(A(D(s)))$$

where the sum is over all relevant vectors.

Note that possibly $D(r) = D(r')$ for relevant vectors $r \neq r'$. Next we clarify this.

Definition 5.37 *We define an equivalence $*$ on the relevant vectors as follows. $r * s$ if the following holds: there is exactly one relevant non-border edge e such that $r(e) \neq s(e)$ and $r(f) = s(f)$ for each $f \neq e$. Moreover, $r(e)_1 \neq s(e)_1$, $r(e)_3 \neq s(e)_3$, $r(e)_2 = s(e)_2 \neq r(e)_4 = s(e)_4$.*

Proposition 5.38 *If $r * s$ then $D(r) = D(s)$ and $sgn(r) \neq sgn(s)$.*

Proof. If $r * s$ then $D(r) = D(s)$ by the definition of '*'. Moreover $sgn(r) \neq sgn(s)$ since $r(e)_2 = s(e)_2 \neq r(e)_4 = s(e)_4$ where e is the only relevant edge for which $r(e) \neq s(e)$. □

Definition 5.39 *A relevant vector is called stable if it forms a one-element class w.r.t. equivalence* $*$.

Corollary 5.40

$$\mathcal{P}(Q_{mmk}, x) = 2^{-2C_r+C_b} \sum sgn(r)Pf(A(D(r)))$$

where the sum is over all stable vectors r.

Definition 5.41 *If* r, s *are stable vectors we write* $r ** s$ *if* $D(r) = D(s)$.

Proposition 5.42 1. *Each equivalence class of '**' has* $2^{C_r-C_b}$ *elements.*

2. *If* $r ** s$ *then* $sgn(r) = sgn(s)$.

Proof. Let r be a stable vector. Then $D(r)$ determines uniquelly $r(e)_2$ and $r(e)_4$ for each relevant edge e and also $r(f)_1$ and $r(f)_3$ for each relevant border edge f. Hence $D(r)$ determines uniquelly $r(f)$ for each relevant border edge f. Moreover $D(r)$ determines uniquelly the product $r(e)_1 \times r(e)_3$ for each relevant non-border edge e. Since there are $C_r - C_b$ relevant non-border edges, each equivalence class of '**' has $2^{C_r-C_b}$ elements.

Let r, s be stable and let $r ** s$. Then $r(e)_2 = s(e)_2 = s(e)_4 = r(e)_4$ for each relevant non-border edge e and $r(e)_2 = s(e)_2$ and $s(e)_1 = r(e)_1$ for each relevant border edge. This implies that $sgn(r) = sgn(s)$. □

Proposition 5.43 *If* D *is an orientation of* Q_{mmk} *that extends* $\mathcal{S}^d$ *then there is uniquelly determined class* C *of equivalence* $**$ *such that* $D = D(r)$ *for each* $r \in C$.

Hence, given an orientation D of Q_{mmk} that extends $\mathcal{S}^d$, let us call it *stable orientation* and let us define its sign $sgn(D)$ to be equaled to $sgn(r)$ for any stable vector r such that $D = D(r)$. This is a correct definition by Proposition 5.42.

Now we can formulate the main result of the paper.

Theorem 5.44

$$\mathcal{P}(Q_{mmk}, x) = 2^{-2C_r+C_b+C_r-C_b} \sum sgn(D)Pf(A(D))$$

over all stable orientations D.

We continue by characterising $sgn(D)$.

As we noticed before, the sign $sgn(r)$ of a relevant vector r may be calculated according to Theorem 5.35 as follows: $sgn(r) = (-1)^{|\{(e,i); i=0,1, r(e)_{2i+1}=r(e)_{2i+2}=-1\}|}$. If r is a stable vector then $r(e)_2 = r(e)_4$ for each relevant edge e and we get the following observation.

Proposition 5.45 *Let* r *be a stable vector. Then* $sgn(r) = (-1)^{|\{e; r(e)_1 \times r(e)_3=-1, r(e)_2=-1\}|}$.

Definition 5.46 *Let D be a stable orientation. We define orientation $\bar{D}$ as follows:*

1. *For each x, y, z such that $y < m$ odd do the following: let $n(xyz)$ be the number of arcs w_{xab}, $a \leq y$ odd and $b = z$ or $b = z + 1$, oriented in D against the natural ordering. If $n(xyz)$ odd then we orient w_{xyz} in $\bar{D}$ against the natural ordering, else according to the natural ordering.*

2. *For each x, y, z such that $x < m$ odd do the following: let $n(xyz)$ be the number of arcs h_{abc} oriented in D against the natural ordering. Here (abc) are the triples of indices satisfying $a \leq x$ odd and $(b, c) = (y, z)$ or $(b, c) = (y', z')$ where (y', z') is such that vertex $V_{x,y',z'}$ is immediate successor of vertex $V_{x,y,z}$ in the following ordering of vertices of Q_{mmk}: $(V_{11}, \bar{V}_{12}, V_{13}, ...V_{1m}, \bar{V}_{2m}, V_{2(m-1)}, ..., \bar{V}_{21}, V_{31}, ..., V_{mm})$.*

   If $n(xyz)$ odd then we orient h_{xyz} in $\bar{D}$ against the natural ordering, else according to the natural ordering.

3. *All the remaining arcs orient in $\bar{D}$ in the same way as in D.*

Note that relevant edges are oriented in the same way in both D and $\bar{D}$.

Proposition 5.47 *Let D be a stable orientation and let r be a stable vector such that $D = D(r)$. Let e be a relevant edge of Q_{mmk}. Then*

1. *$r(e)_1 \times r(e)_3 = -1$ if and only if e is oriented in D (and hence also in $\bar{D}$ against the natural ordering.*

2. *If $e = w_{xyz}$, y even then $r(e)_2 = -1$ if and only if $w_{x(y-1)z}$ is oriented in $\bar{D}$ against the natural ordering. If $e = h_{xyz}$, x even then $r(e)_2 = -1$ if and only if $h_{(x-1)yz}$ is oriented in $\bar{D}$ against the natural ordering.*

Proof. $r(e)_1 \times r(e)_3 = -1$ if and only if exactly one long edge of Q' corresponding to e is oriented in D' (we remind that D is induced by orientation D' of Q') against the natural ordering. Since D is induced by D', this happens if and only if e is oriented in both D and $\bar{D}$ against the natural ordering.

It remains to prove 2. We will show the case $e = w_{xyz}$ since the other case is completely analogous. If f is an edge of Q_{mmk} then we let $f(D) = 1$ if f is oriented in D according to the natural ordering, and we let $f(D) = -1$ otherwise. We proceed by induction on $(y, k - z)$.

Firstly assume $y = 2$ and $z = k$. In this simplest case $r(e)_2 = w_{x1k}(D)$ (see Fig. 2). Moreover the orientation of w_{x1k} is the same in both D and $\bar{D}$.

Secondly let $y = 2$ and $z < k$. Then $w_{x1z}(D) = w_{x1(z+1)}(D) \times r(e)_2$ (see Fig. 2). It follows from Definition 5.46 that $r(e)_2 = -1$ if and only if w_{x1z} is oriented in $\bar{D}$ against the natural ordering.

Thirdly, let $y = 4$ and $z = k$. Then $w_{x3k}(D) = w_{x1k}(D) \times r(e)_2$ (see Fig. 2). It follows from Definition 5.46 that $r(e)_2 = -1$ if and only if w_{x3k} is oriented in $\bar{D}$ against the natural ordering.

Forthly, let $y = 4$ and $z < k$. Then $w_{x3z}(D) = w_{x3(z+1)}(D) \times r(e)_2 \times r(w_{x2z})_2 = w_{x3(z+1)}(D) \times r(e)_2 \times w_{x1z}(D) \times w_{x1(z+1)}(D)$ (see Fig. 2). It follows from Definition 5.46 that $r(e)_2 = -1$ if and only if w_{x3z} is oriented in $\bar{D}$ against the natural ordering.

In general if $e = w_{xyz}$, y even and $z = k$ then $w_{x(y-1)k}(D) = r(w_{x(y-2)k})_2 \times r(e)_2 = r(e)_2 \times \prod(w_{xy'k}(D); y' < y-1 \text{ odd })$. It follows from Definition 5.46 that $r(e)_2 = -1$ if and only if $w_{x(y-1)k}$ is oriented in $\bar{D}$ against the natural ordering.

Finally if $e = w_{xyz}$, y even and $z < k$ then $w_{x(y-1)z}(D) = w_{x(y-1)(z+1)}(D) \times r(e)_2 \times r(w_{x(y-2)z})_2 = w_{x(y-1)(z+1)}(D) \times r(e)_2 \times \prod(w_{xy'z}(D); y' < y-1 \text{ odd }) \times \prod(w_{xy'(z+1)}(D); y' < y-1 \text{ odd })$. It follows from Definition 5.46 that $r(e)_2 = -1$ if and only if w_{x3z} is oriented in $\bar{D}$ against the natural ordering. □

This proposition has the following corollary.

Corollary 5.48 *Let D be a stable orientation. Then $sgn(D)) = (-1)^{h+\sum_{x=1}^{m} w(x)}$, where $w(x) = |\{(yz); y$ even and both $w_{xyz}, w_{x,(y-1),z}$ are oriented against the natural ordering in $\bar{D}\}|$; $h = |\{(xyz); x$ even and both $w_{xyz}, w_{(x-1),y,z}$ are oriented against the natural ordering in $\bar{D}\}|$.*

Proposition 5.49 *There are 2^{2C_r} stable orientations. There are $2^{C_r-1}(2^{C_r}+1)$ stable orientations with positive sign.*

Proof. The first statement follows directly from the definition of a stable orientation. For Q_{132} there are 16 stable orientations, from which 10 have positive sign. For Q_{152} there are 4^4 stable orientations from which $10 \times 10 + 6 \times 6$ have positive sign and $2(6 \times 10) = 120$ have negative sign. For $Q_{1(2a+1)2}$ there are 4^{2a} stable orientations, and the difference between the number of stable orientations of positive sign and stable orientations of negative sign is 2^{2a}. For Q_{mmk} the difference between the number of stable orientations of positive sign and those of negative sign equals $2^{(m-1)km} = 2^{C_r}$. From this Proposition follows. □

From Pfaffians to determinants.

In the introduction we let $\mathcal{Z}$ be square $(Z_1 \times Z_2)$ matrix defined by $\mathcal{Z}_{ij} = x_{ij}$ if ij is an edge of Q_{mmk} and $\mathcal{Z}_{ij} = 0$ otherwise.

Let D be an orientation of Q_{mmk}. In the introduction we associate a signing $Z(D)$ of $\mathcal{Z}$ with it such that $Z(D)_{ij} = x_{ij}$ if $(ji) \in E(D)$, $Z(D)_{ij} = -x_{ij}$ if $(ij) \in E(D)$, and $Z(D)_{ij} = 0$ otherwise.

Note that $Pf(A(D)) = det(Z(D))$. Hence we can reformulate Theorem 5.44:

Corollary 5.50 *$\mathcal{P}(Q_{mmk}, x) = 2^{-C_r} \sum sgn(D)det(Z(D))$ where the sum is over all stable orientations D of Q_{mmk}.*

Let $\alpha = \prod_{e \in M} x_e$ where M is unique perfect matching of the collection of paths $V_{11}(Q_{mmk}) \cup ... \cup V_{mm}(Q_{mmk})$.

Proposition 5.51 *The average of $det(Z(D))$, D stable, equals α.*

Hence finally we can formulate Theorem 5.44 as follows.

Theorem 5.52 *$\mathcal{P}(Q_{mmk}, x) = -2^{C_r}\alpha + \beta(2^{C_r} + 1)$, where β equals the average of $det(Z(D))$, D stable orientation with $sgn(D) = +1$, and $C_r = km(m-1)$.*

Proof. By Corollary 5.50 and Proposition 5.52 we have that

$\mathcal{P}(Q_{mmk}, x) = 2^{-C_r}[-2^{2C_r}\alpha + 2\beta(2^{C_r-1}(2^{C_r}+1)] = -2^{-C_r+2C_r}\alpha + \beta(2^{-C_r+1+C_r-1})(2^{C_r}+1).$

□

Well, this may be a good end of the section on how to solve the Ising problem. In the next short section let us turn our attention to calculation of permanents.

6 On permanents.

If $A = (a_{ij})$ is a square $(n \times n)$ matrix then *permanent* of A is defined by

$$per(A) = \sum_{\pi} \prod_{i=1}^{n} a_{i\pi(i)}$$

where π ranges over the permutations of $1, ..., n$. In other words, permanent is defined in a 'similar' way as determinant, one only 'forgets' the signs of the terms. The signs make the two notions completely different from the computational point of view (and most of other views as well).

There is an efficient algorithm to compute the determinant, while Valiant proved that the problem of computing the permanent of a $(0,1)$-matrix is $\#P$-complete [24].

In 1913, Pólya [18] suggested computing the permanent of a matrix A by changing the signs of some entries of A so that the determinant of the resulting matrix equals the permanent of A. A matrix obtained from A by changing the signs of some entries will be called *signing of* A. Let us call a matrix A *convertible* if such a change is possible.

Szegö [21] pointed out in the same year that not all matrices are convertible.

The computational problem of recognising the convertible matrices had been studied extensively and it was proved recently to admit a polynomial algorithm in a seminal work by McCuaig, Robertson, Seymour and Thomas [17]. The problem of recognizing convertible matrices is known to be equivalent to many computational problems. Let us mention just one of them here, the one which gave it a name. It is the *Even Cycle Problem*: given a directed graph, decide whether it contains a directed cycle of even length.

Let A be a square matrix. Denote by $G(A)$ the bipartite graph whose two bipartition classes are indexed by the rows and the columns of A, and for each edge ij, $a_{ij} = x_{ij}$. Then $per(A) = \mathcal{P}(G(A), x)$.

Let A be a matrix. A seminal observation of Heilmann and Lieb [6, 7], theorem5.29 asserts that $per(A) = \mathcal{P}(G(A), x)$ equals the average of $(det(Z))^2$ over ALL signings Z of A.

It was proved by Galluccio and Loebl [4] that the method of Pólya may be completed as follows:

Theorem 6.1 *Let A be a square matrix. Then $per(A)$ may be expressed as a linear combination of terms of the form $det(A^i)$, $i = 1, ..., 4^g$, where each A^i is a signing of A and g is the genus of $G(A)$.*

This theorem immediately follows from the considerations of the previous section.

Again, let us note that a difference of this expression with the result of Heilmann and Lieb is that the average of a multiquadratic function is replaced by a linear combination of multilinear functions.

7 Back to binary codes.

In this last section let us return back to the linear codes.

Definition 7.1 *Let $k \geq 1$ and let $G = (V, E)$ be a graph. A pair $(\{V_1, ..., V_k\}, E')$ is called k-cut if $\{V_1, ..., V_k\}$ is a partition of V into k non-empty disjoint subsets and E' is the set of all edges with the endvertices in different parts V_i, $i = 1, ..., k$.*

Definition 7.2 *The generating function of the k-cuts is the polynomial $\mathcal{C}_k(G, x)$ which equals the sum of $x(C)$ over all k-cuts $(\{V_1, ..., V_k\}, C)$ of G.*

First let us extend the considerations of van der Waerden from the section of dualities:

We use the following notation:

$$sinh(z, x) = \frac{z^x - z^{-x}}{2}, \quad cosh(z, x) = \frac{z^x + z^{-x}}{2}, \quad th(z, x) = \frac{sinh(z, x)}{cosh(z, x)}.$$

Note that $sinh(x) = sinh(e, x)$ and $cosh(x) = cosh(e, x)$.

Given a graph $G = (V, E)$, we denote by $\sigma \in \{1, \ldots, k\}^V$ a $|V|$-dimensional vector whose components σ_i, $i = 1, \ldots, |V|$, take values in the set $\{1, \ldots, k\}$. Clearly, any such vector identifies a partition of V into $i \leq k$ disjoint sets and, consequently, an i-cut of G.

Let us denote by δ the vector indexed by the edges of G whose component $\delta_{ij} = \delta(\sigma_i \sigma_j)$, $ij \in E$, equals 1 if $\sigma_i = \sigma_j$ and -1 otherwise.

Moreover, for any $A \subset E$ we let

$$U_k((V, A)) = \sum_{\sigma \in \{1, \ldots, k\}^V} \prod_{ij \in A} \delta(\sigma_i \sigma_j).$$

Theorem 7.3 *Let $G = (V, E)$ be a graph, z a variable and $k > 1$. Then*

$$z^{\sum_{f \in E} x_f}[k + \sum_{i=2}^{k} i! \binom{k}{i} \mathcal{C}_i(G, (z^{-2x_f} : f \in E))] =$$

$$(\prod_{f \in E} cosh(z, x_f)) \sum_{A \subseteq E} U_k((V, A)) \prod_{f \in A} th(z, x_f).$$

Proof. Using the identity

$$z^{x\delta(\sigma_i\sigma_j)} = cosh(z,x) + \delta(\sigma_i\sigma_j)sinh(z,x)$$

the result follows after some algebraic manipulations. In fact,

$$z^{\sum_{f\in E} x_f}[k + \sum_{i=2}^{k} i!\binom{k}{i}\mathcal{C}_i(G,(z^{-2x_f} : f \in E))] = \sum_{\sigma\in\{1,\dots,k\}^V} (\prod_{ij\in E} z^{\delta(\sigma_i\sigma_j)x_{ij}}) =$$

$$\sum_{\sigma\in\{1,\dots,k\}^V} (\prod_{ij\in E} (cosh(z,x_{ij}) + \delta(\sigma_i\sigma_j)sinh(z,x_{ij}))) =$$

$$(\prod_{f\in E} cosh(z,x_f)) \sum_{\sigma\in\{1,\dots,k\}^V} (\prod_{ij\in E} (1 + \delta(\sigma_i\sigma_j)th(z,x_{ij}))) =$$

$$(\prod_{f\in E} cosh(z,x_f)) \sum_{\sigma\in\{1,\dots,k\}^V} \sum_{A\subseteq E} (\prod_{ij\in A} \delta(\sigma_i\sigma_j)th(z,x_{ij})) =$$

$$(\prod_{f\in E} cosh(z,x_f)) \sum_{A\subseteq E} U_k((V,A)) \prod_{f\in A} th(z,x_f)$$

where $\prod_{ij\in A} \delta(\sigma_i\sigma_j)th(z,x_{ij}) = 1$ if $A = \emptyset$. Hence, the theorem follows. □

Specializing the above result to the case of edge-cuts, i.e. $k = 2$, we obtain the following result:

Theorem 7.4 *Let G be a graph, z a variable and let $\mathcal{C}'_2(G,x) = \mathcal{C}_2(G,x) + 1$. Then*

$$2z^{\sum_{f\in E} x_f}\mathcal{C}'_2(G,(z^{-2x_f} : f \in E)) = (\prod_{f\in E} cosh(z,x_f))2^{|V|}\mathcal{E}(G,(th(z,x_f) : f \in E)).$$

Proof. We have, from Theorem 7.3, that

$$2z^{\sum_{f\in E} x_f}\mathcal{C}'_2(G,(z^{-2x_f} : f \in E)) = (\prod_{f\in E} cosh(z,x_f)) \sum_{A\subseteq E} U_2((V,A)) \prod_{f\in A} th(z,x_f).$$

Now observe that if $A \subset E$ is a cycle and $\sigma \in \{1,2\}^V$ arbitrary then $\prod_{ij\in A} \delta(\sigma_i\sigma_j) = 1$. Hence $U_2((V,A)) = 2^{|V|}$ when (V,A) is an eulerian subgraph. Moreover, if (V,A) is not an eulerian subgraph, then observe that $U_2((V,A)) = 0$.

□

Theorem 7.5 *Let $G = (V,E)$ be a connected graph. Then*

$$A_{C(GF[q],\mathcal{N}_G)}(t) = 1 + \sum_{i=2}^{q}(q-1)\dots(q-i+1)\mathcal{C}_i(G,(t,\dots,t)).$$

Proof. Let $C = C(GF[q], \mathcal{N}_G)$. We have $C = \{O_G^T x; x \in GF[q]^V\}$ and $A_C(t)$ is the weight enumerator of C. Let us define an equivalence on $GF[q]^V$ by $x \equiv y$ if $O_G^T x = O_G^T y$. Observe that each equivalence class consists of q elements since $O_G^T x = O_G^T y$ if and only if $x - y$ is a constant vector, i.e. $(x-y)_i = (x-y)_j$ for each $i, j \in \{1, ..., |V|\}$. Let C^+ be the system (in contrast with *a set*, some elements of a system may appear several times) defined by $C^+ = (O_G^T x; x \in GF[q]^V)$. Let $A_{C^+}(t) = \sum_{i=0}^{|E|} A_i^+ t^i$, where A_i^+ equals the number of vectors of C^+ with i non-zero components. Since each equivalence class of $\equiv$ consists of q elements we have $A_{C^+}(t) = qA_C(t)$.

If $x = (x_1, ..., x_{|V|}) \in GF[q]^V$ then let l_x be the number of different values of x_i, $i = 1, ..., |V|$ and let $V_1^x, ..., V_{l_x}^x$ be the partition of V into l_x non-empty classes such that the components of x are equal in each class.

Let $cut(x)$ be the subset of E formed by those edges having endvertices in different sets V_i^x, $i = 1, ..., l_x$ and let $Cut(x) = (\{V_1, ..., V_{l_x}\}, cut(x))$.

Each $Cut(x)$ is an l_x-cut of G and the weight of the codeword $O_G^T x$ equals $|cut(x)|$. Let C^{++} be the system defined by $C^{++} = (cut(x); x \in GF[q]^V)$. Let $A_{C^{++}}(t) = \sum_{W \in C^{++}} t^{|W|}$. We have $A_{C^+}(t) = A_{C^{++}}(t)$.

For $i = 1, ..., q$ let $X_i = \{x \in GF[q]^V; l_x = i\}$. Define an equivalence on X_i by $x \equiv^* y$ if $Cut(x) = Cut(y)$. Observe that each equivalence class of $\equiv^*$ consists of $q(q-1)...(q-i+1)$ elements. Hence

$$qA_C(t) = A_{C^+}(t) = A_{C^{++}}(t) = q + \sum_{i=2}^{q} q(q-1)...(q-i+1)\mathcal{C}_i(G, (t, ..., t)).$$

This proves the Theorem.

□

By Corollary 3.4 and Theorem 7.5 we have

Corollary 7.6 *Let $G = (V, E)$ be a connected graph and let $\mathcal{N}_G$ be the graphic matroid of G. If $(x-1)(y-1) = 2$ and $0 \neq y \neq 1$ then*

$$T(G, x, y) = T(\mathcal{N}_G, x, y) = y^{|E|}(y-1)^{1-|V|}[1 + \mathcal{C}_2(G, (y^{-1}, ..., y^{-1})].$$

It follows that the Tutte polynomial of a graph of genus g may be expressed along the hyperbola $(x-1)(y-1) = 2$ as a linear combination of 4^g Pfaffians, and hence it may be determined efficiently for the graphs embeddable on an arbitrary fixed orientable surface.

It is natural to ask whether there is an analogy of this statement for binary matroids.

7.1 Flow polynomial of graphs.

The dual notion to coloring is nowhere-zero flow. There are very nice facts known about it, and interestingly this notion seem to lead naturally to applications in knot theory and in statistical physics. This will not be touched here.

We have $C(GF[q],\mathcal{N}_G)^* = \{z \in GF[q]^E; O_G z = 0\}$. The elements of $C(GF[q],\mathcal{N}_G)^*$ are *flows* on G with values in $GF[q]$. An element of $C(GF[q],\mathcal{N}_G)^*$ is called a *nowhere-zero flow* if its weight equals $|E|$. Let $F'(G,q)$ be the subset of $C(GF[q],\mathcal{N}_G)^*$ consisting of nowhere-zero flows. $F(G,q) = |F'(G,q)|$ is called the *flow polynomial* of G.

Theorems 3.2, 7.5 express a duality between flows and cuts of a graph. It is a duality of the Tutte polynomial.

In fact, there was a paper partly on this subject for Jarni skola last year: J. Nesetril, A. Raspaud, Duality, Nowhere-Zero Flows, and Cycle Covers.

The following theorem was proved by Tutte ([23]).

Theorem 7.7 *Let $G = (V,E)$ be a graph and let q be a power of a prime. Then*

$$F(G,q) = (-1)^{|V|-|E|-c(G)} T(G,0,1-q).$$

We give an interesting expression for $F(G,q)$ which is new as far as we know.

Theorem 7.8 *Let $G = (V,E)$ be a graph. Then*

$$F(G,q) = q^{-|V|} 2^{-|E|} \sum_{A \subset E} U_q((V,A)) q^{|A|} (q-2)^{|E|-|A|}.$$

Proof. Let $C = C(GF[q],\mathcal{N}_G)$ and let $D = C(GF[q],\mathcal{N}_G)^*$. By Theorem 3.2 we have

$$A_D(t) = q^{1-|V|}[1+(q-1)t]^{|E|} A_C((1-t)(1+(q-1)t)^{-1}).$$

From Theorem 7.5 and Theorem 7.3 we get for $z > 0$

$$qA_C(z^{-1}) = q + \sum_{i=2}^{q} q(q-1)...(q-i+1)\mathcal{C}_i(G,(z^{-1},...,z^{-1})) =$$

$$2^{-|E|} z^{-|E|} \sum_{A \subset E} U_q((V,A))(z-1)^{|A|}(z+1)^{|E|-|A|}.$$

If we let $z = (1+(q-1)t)(1-t)^{-1}$ we get for all $t > 0$, $t \neq 1$

$$A_D(t) = q^{-|V|} 2^{-|E|} \sum_{A \subset E} U_q((V,A))(qt)^{|A|}(2+(q-2)t)^{|E|-|A|}.$$

It follows that the leading coefficient of $A_D(t)$, which equals $F(G,q)$, is equal to

$$q^{-|V|} 2^{-|E|} \sum_{A \subset E} U_q((V,A)) q^{|A|} (q-2)^{|E|-|A|}.$$

□

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