Cycles in Graphs and Digraphs

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Abstract. I will present a few elementary observations and constructions on cycles in graphs and digraphs, which I have found useful. This text is based on a survey paper with Martin Matamala, which appeared in KAM series already last year(and will appear in Discrete Mathematics).

1 Introduction

A graph G is *k-edge-connected*, $k \geq 2$, if there exist k edge-disjoint paths connecting any pair of vertices of G. A graph G is *k-connected* if there exist k vertex-disjoint paths connecting any pair of vertices of G.

A digraph D is *strongly connected* if there is a directed path from any vertex to any other vertex of D.

It was proved by Robbins in [6] that a graph G is 2-edge-connected if and only if G has a strongly connected orientation. This theorem is first of many other 'orientation theorems'. Andras Frank from Budapest is a person who knows them all. One motivation for writing this note is an algebraic version of this theorem which I proved several years ago with Peter Greenberg. I like it and so I remind it to myself and to others from time to time.

Given an undirected graph $G = (V, E)$, the *cycle space* of G is the subspace of $GF[2]^{|E|}$ generated by the incidence vectors of the cycles of G. The cycle space of a directed graph D is the cycle space of the underlying undirected graph. A *cycle basis* is a basis of the cycle space of G, equivalently a minimal set of elements of the cycle space such that any cycle of G is a modulo 2 sum of some of them. Let D be a digraph and let $\mathcal{D}$ be a set of subgraphs of D which are orientations of cycles. Then $\mathcal{D}$ is called *directed cycle basis* if it is linearly independent and any incidence vector of a cycle of the underlying graph of D is a modulo 2 sum of incidence vectors of some directed circuits of $\mathcal{D}$.

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The lattice generated by a set A of vectors is the set of all integer linear combinations of vectors of A. It is a well-known fact (see e.g. [7]) that each lattice generated by a finite set of rational vectors has a *basis*, i.e. a set of linearly independent vectors (over rationals) such that any other element of the lattice is an integer linear combination of them.

Throughout the paper we denote by $\chi_C \in \{0,1\}^{|E|}$ the incidence vector of the set $C \subseteq E$. For the sake of simplicity we shall write χ_e to indicate the incidence vector of the set $\{e\}$. The *degree* of a vertex in a graph is the number of the edges incident with the vertex. A graph is called *eulerian* if its vertices have even degrees. Each eulerian graph is a union of edge-disjoint cycles.

A *subdivision* of an edge (arc) of a graph (directed graph) consists of replacing the edge (arc) by a path (directed path) whose endvertices coincide with the endvertices of the edge (arc), and whose intermediate vertices do not belong to the graph (directed graph).

A *subdivision* of a graph (directed graph) is obtained by subdividing some of the edges (arcs) of the graph (directed graph).

2 Cycles in Digraphs.

In this section we first describe a result of Galluccio, Loebl ([1]). We show that the directed cycle bases naturally defined from an ear decomposition of a digraph are bases of the lattice generated by the directed cycles as well.

2.1 Directed Cycle Bases.

A digraph is strongly connected if and only if it may be built up from a vertex by sequentially adding arcs (loops are allowed) and by subdividing arcs. This property leads to the concept of *ear decomposition.* An ear decomposition of D is a sequence $D_0, \ldots, D_t = D$ of subdigraphs of D such that D_0 consists of a single vertex and no arc, and each D_i arises from D_{i-1} by adding a directed path P_i whose endvertices (not necessarily distinct) belong to D_{i-1} while the *arcs and intermediate vertices* of P_i do not. The paths P_i are called *ears* and the endvertices of P_i are called *initial vertices* of the ear.

A digraph is strongly connected if and only if it has an ear decomposition. If $D_0, ..., D_t = D$ is an ear decomposition of a strongly connected digraph D then each D_i, $i = 1, ..., t$ is strongly connected as well.

It is well known that from each ear decomposition of a strongly connected digraph it is possible to obtain a directed cycle basis by simply completing each new ear to a directed cycle using a directed path in the already built subdigraph. Such directed cycle bases will be called *directed ear-bases.*

Let us state now a basic result of [1] concerning the lattice generated by the directed cycles of a digraph D.

Definition 2.1 *Let x be an integer vector indexed by the arcs of a digraph D. The indegree of a vertex (of D) in x is the sum of the entries of x corresponding to the arcs entering that vertex. The outdegree of a vertex (of D) in x is the sum of the entries of x corresponding to the arcs leaving that vertex. An integer vector x is eulerian if each vertex of D has its indegree equal to its outdegree. We denote $\mathcal{E}(D)$ the set of eulerian vectors.*

Theorem 2.2 *Let D be a strongly connected digraph. Any directed ear-basis of D is a basis of the lattice $L(D)$ generated by the directed cycles of D. Moreover $L(D) = \mathcal{E}(D)$.*

Proof. Let $\mathcal{B} = \{\chi_{C_1}, \ldots, \chi_{C_m}\}$ denote a directed ear-basis of D, i.e. a set of incidence vectors of the directed cycles C_i obtained from an ear-decomposition $D_0, D_1, \ldots, D_m = D$ of D by completing the directed path P_i into a directed cycle of D_i. Let $\mathcal{B}_i = \{\chi_{C_1}, \ldots, \chi_{C_i}\}$.

In order to prove the first part of the theorem, we need to show that the vectors of $\mathcal{B}$ are linearly independent over the rationals and that the characteristic vectors of directed cycles of D are integer linear combinations of them.

The linear independence follows from the construction of the directed ear-basis: for each $j < i \leq m$ C_j contains no arc of P_i while C_i contains P_i.

Define $\mathcal{E}_i$ to be the set of vectors of $\mathcal{E}(D)$ having nonzero components only on arcs of D_i. Hence $\mathcal{E}(D) = \mathcal{E}_m$.

We will prove by induction on i that each element of $\mathcal{E}_i$ is an integer linear combination of elements of $\mathcal{B}_i$. This finishes the proof of the theorem since the incidence vector of each directed cycle of D is eulerian.

Let ξ be any vector in $\mathcal{E}_i - \mathcal{E}_{i-1}$. Since ξ is eulerian, the components of ξ corresponding to the arcs of P_i are equal, say p. Hence, the vector $\xi - p\chi_{C_i}$ belongs to $\mathcal{E}_{i-1}$, and the result follows from the induction hypothesis. □

To conclude this subsection let us remark that the lattice generated by directed cycles of a strongly connected digraph was considered also in [5] where several algebraic properties were derived.

2.2 2-chains.

For a digraph D and two vertices s and t we define a *directed path decomposition* of D from s to t like an ear decomposition with two differences. First, D_0 is a directed path from s to t. Second, D_i is obtained from D_{i-1} by adding directed paths $P_{i,1}, ..., P_{i,r_i}$ such that there exists a directed path P_i from s to t and $P_{i,j}$, $j = 1, ..., r_i$ are the parts of P_i not in D_{i-1}.

Let s, t be vertices a digraph D. D is called *distribution digraph from s to t* if any arc of D belongs to a directed path from s to t.

Theorem 2.3 *A digraph D is a distribution digraph from s to t if and only if it admits an directed path decomposition from s to t.*

Proof. Let D be a distribution digraph from s to t. First we show that it admits a directed path decomposition from s to t. We take D_0 any directed path from s to t. If it does not exist then D has no arc and the decomposition follows. Suppose that we have already built D_i. Let a be an arc not in D_i. Since D is a distribution digraph a belongs to a directed path P_i from s to t. Let $P_{i,1}, ..., P_{i,r_i}$ be the parts of P_i not in D_i. We add all these parts to D_{i-1} to get D_i. This process may continue until finally we obtain $D_m = D$.

On the other hand if $D_0, ..., D_m$ is a directed path decomposition from s to t, where $D = D_m$, we will prove that D is a distribution digraph. In fact we will prove that for each i the subdigraph D_i is a distribution digraph. Clearly D_0 is a distribution digraph. Assume that D_{i-1} is a distribution digraph. Let a be an arc in $D_i \setminus D_{i-1}$. Then a belongs by definition to a directed path from s to t and hence D_i is a distribution digraph. □

An orientation of a cycle is called *2-chain* if it consists of two directed paths with the same origin and the same destination. From a directed path decomposition of D from s to t it is easy to construct a directed cycle basis consisting of 2-chains. Let us call such basis a *2-chain basis.* Hence previous theorem shows that for distribution digraphs there exists a 2-chain basis.

3 Cycles in Graphs.

In this section we turn our attention to undirected graphs. Let G be a 2-edge-connected undirected graph. Is there a natural set of cycles which form a basis of the lattice of cycles of G? This question was answered affirmatively by Galluccio, Loebl in [1]. We will describe the result below.

A *binary code* is a subspace of $GF[2]^m$. The characteristic vectors of cycles of a graph, and in general the characteristic vectors of cycles of a binary matroid, form a binary code. The lattices generated by the cycles of binary matroids were studied by Lovasz, Seres in [3].

The result of Galluccio and Loebl led Hochstaettler and Loebl ([2]) to formulate the following conjecture: 'The lattice generated by a binary code always has a basis of codewords.' The conjecture was proved to be true for instance for regular matroids and at present the best result towards proving the conjecture is obtained by Fleiner, Hochstaettler, Laurent, Loebl in ([8]).

Each cycle of an undirected graph is contained in its 2-connected component and these components are edge-disjoint. Hence we may restrict ourselves to 2-connected graphs when studying the cycles of undirected graphs.

A graph is called *eulerian* if all of its vertices have even degree.

An *ear decomposition* of a (2-connected) graph G is a sequence $G_1, \ldots, G_t = G$ of subgraphs of G such that G_1 is a cycle and each G_i, $i > 1$, arises from G_{i-1} by adding a path P_i whose endvertices are distinct and belong to G_{i-1} while the edges and intermediate vertices of P_i do not. The paths P_i are called *ears* and the endvertices of P_i are called *initial vertices* of the ear.

A graph is 2-connected if and only if it has an ear decomposition; from an ear decomposition we may obtain a cycle basis, i.e., a basis of the vector space over $GF[2]$ generated by the incidence vectors of the circuits, by completing each new ear to a circuit using a path in the already built subgraph. Such cycle bases are called *ear-bases.*

An ear decomposition $G_1, \ldots, G_t = G$ of G will be called *correct ear decomposition* if each G_i, $i = 2, \ldots, t$, is a subdivision of a 3-edge-connected graph (possibly with parallel edges).

Theorem 3.1 *Let G be a subdivision of a 3-edge-connected graph. Then G has a correct ear-decomposition.*

Proof. Let G_i, $i \geq 2$, be a subdivision of a 3-edge-connected graph H and let G_i be a subgraph of G. Call an ear P_{i+1} *correct* if the initial vertices of P_{i+1} are not subdividing vertices of the same edge of H. Observe that if P_{i+1} is correct then G_{i+1} is a subdivision of a 3-edge-connected graph.

If a correct ear P_{i+1} does not exist then let S be an edge of H with a subdividing vertex connected by an edge to a vertex of $G - G_i$. The terminal edges of the subdivision of S in G_i must form a 2-edge-cut of G, which is a contradiction. □

Definition 3.2 *Let G be a subdivision of a 3-edge-connected graph. Let $G_1, \ldots, G_t = G$ be a correct ear decomposition of G.*
An improved ear-basis $\mathcal{A}(G) = \mathcal{A}(G_t)$ is recursively defined as follows:

1. *$\mathcal{A}(G_2)$ consists of all three cycles of G_2.*

2. *Let $i > 2$ and G_i be obtained from G_{i-1} by adding the ear P_i.*

   We distinguish three cases:

   (i) if the endvertices of P_i have degree greater than 2 in G_{i-1} then $\mathcal{A}(G_i)$ is obtained from $\mathcal{A}(G_{i-1})$ by adding an arbitrary circuit C_i^1 of G_i containing P_i;

   (ii) if one endvertex of P_i have degree 2 in G_{i-1} then let e_1, e_2 be two edges of G_{i-1} incident with that vertex. Then $\mathcal{A}(G_i)$ is obtained from $\mathcal{A}(G_{i-1})$ by adding two circuits C_i^1, C_i^2 of G_i, C_i^1 containing P_i and e_1 and C_i^2 containing P_i and e_2;

   (iii) if both endvertices of P_i have degree 2 in G_{i-1} then let e_1, e_2 and f_1, f_2 be two pairs of edges of G_{i-1} incident with each endvertex of P_i. Since the ear decomposition is correct, e_1, e_2, f_1, f_2 do not belong to a subdivision of the same edge in G_{i-1}.

   Then $\mathcal{A}(G_i)$ is obtained from $\mathcal{A}(G_{i-1})$ by adding three circuits C_i^1, C_i^2, C_i^3 of G_i, C_i^1 containing P_i and e_1, f_1, C_i^2 containing P_i and e_2, f_1, and C_i^3 containing P_i and e_1, f_2.

The following Theorem is proved in [1].

Theorem 3.3 *Let G be a subdivision of a 3-edge-connected graph H. Any improved ear-basis of G is a basis of the lattice generated by the circuits of G. Moreover this lattice contains all vectors of form $2F$, where F is the set of the edges of the path of G obtained by subdividing an edge of H.*

4 About Robbins' Theorem.

As mentioned in the introduction, it was proved by Robbins in [6] that a graph G is 2-edge-connected if and only if G has a strongly connected orientation. It was observed by Greenberg and Loebl ([4]) that this theorem has a linear algebra generalisation.

Theorem 4.1 *Let $L \subset R^d$ be a lattice and let $A \subset L$. There exists $s \in \{1,-1\}^A$ such that each element of L is a non-negative integer linear combination of $\{s(a)a; a \in A\}$ if and only if the following two conditions are satisfied:*

1. *Each element of L is an integer linear combination of A,*

2. *For each $z \in A$, $0 = \sum_{a\in A} b_z(a)a$ where $b_z(a)$ is integer for each a and $b_z(z) \neq 0$.*

Proof. Condition 1 is clearly necessary. To show that condition 2 is necessary let s exist and assume that for $a \in A$, $s(a) = 1$. Since $-a \in L$, we have that $-a = \sum_{b\in A} s'(b)b$ where $s'(b) = 0$ or $s'(b)$ has the same sign as $s(b)$ for each $b \in A$. Adding a to both sides, we get condition 2 for a.

Let us prove that the two conditions are sufficient. In fact, we will prove a stronger statement:

Claim. Let us assume that $A \cup \{-a; a \in A\}$ generates L, and let $A' \subset A$ and $s' \in \{1,-1\}^{A'}$ is given with the following property **P**: For each $z \in A$, $0 = \sum_{a\in A} b_z(a)a$ where $b_z(a)$ is integer for each a, $b_z(z) \neq 0$ and for $a' \in A'$, if $b_z(a') \neq 0$ then it has the same sign as $s'(a')$.

Let $b \in A - A'$. Then s' may be extended to $s'' \in \{1,-1\}^{A'\cup\{b\}}$ so that **P** is valid for s''.

Proof of Claim. For a contradiction assume that s' cannot be extended to $A' \cup \{b\}$. Hence if we let $s''(b) = 1$ then **P** is violated for some $x \in A$ and if we let $s''(b) = -1$ then **P** is violated for some $y \in A$. Since **P** holds for s' we have that $x \neq y$ and

$0 = \sum_{a\in A} b_x(a)a$ where $b_x(x) \neq 0$,$b_x(a') = 0$ or it has the same sign as $s'(a')$ for each $a' \in A'$ and $b_x(b)$ is negative; by the choice of y we also have $b_x(y) = 0$.

$0 = \sum_{a\in A} b_y(a)a$ where $b_y(y) \neq 0$,$b_y(a') = 0$ or it has the same sign as $s'(a')$ for each $a' \in A'$ and $b_y(b)$ is positive; by the choice of x we also have $b_x(y) = 0$.

Without loss of generality assume that $-b_x(b) \geq b_y(b)$.

Then $0 = \sum_{a\in A}(b_x(a) + b_y(a))a$, and if we let $b'_y(a) = b_x(a) + b_y(a)$ for each $a \in A$ then $b'_y(y) \neq 0$, $b'_y(a') = 0$ or it has the same sign as $s'(a')$ for each $a' \in A'$ and $b'_y(b)$ is negative of equals zero. When we let $s''(b) = -1$ property **P** is satisfied for y and s'', which contradicts the choice of y.

□

Remark 1. Theorem 4.1 indeed generalises the Robbins' theorem: Let $G = (V, E)$ be a graph with vertices $v_1, ..., v_n$. For any pair of vertices v_i, v_j of G, $i < j$, let $x(i,j) \in \{0, 1, -1\}^n$ be a vector whose components are all equal to zero except $x(i,j)_i = 1$ and $x(i,j)_j = -1$. Let L be the lattice generated by all the vectors $x(i,j)$, and let $A = \{x(i,j); \{v_i, v_j\} \in E\}$. Then G has a strongly connected orientation if and only if s from Theorem 4.1 exists for A. Moreover, the two conditions of 4.1 are equivalent to G being connected (condition 1) and each edge belonging to a cycle (condition 2). This is equivalent to G being 2-edge-connected.

Theorem 4.1 has an interesting consequence.

Corollary 4.2 *Let L be a lattice and $A \subset L$ such that each element of L is an integer linear combination of $A - \{a\}$, for any $a \in A$. Then s from Theorem 4.1 exists.*

Remark 2. It may be observed that subdivisions of 3-connected graphs satisfy condition 4.2 and thus s from Theorem 4.1 always exists. It may be interesting to investigate which properties does the set of all such s have, perhaps in relation with the double cycle cover conjecture.

References

[1] A. Galluccio and M. Loebl. (p,q)-odd digraphs. *Jour. of Graph Theory*, 23:175–184, 1996.

[2] W. Hochstaettler and M. Loebl. On bases of circuit lattices. Technical Report CORR 96-02, University of Waterloo, January 1996.

[3] L. Lovász and A. Seres. The cocycle lattice of binary matroids. *Europ. J. Comb.*, 14:241–250, 1993.

[4] M. Loebl P.Greenberg. Strong connectivity of polyhedral complexes. *J. of Algebraic Combinatorics*, 5:117–125, 1996.

[5] T. Nagnibeda R. Bacher, P. de la Harpe. The lattice of intergral flows and the latice of integral cuts in a finite graph. *Bull. Soc. math. France*, 125:167–198, 1997.

[6] H.E. Robbins. A theorem on graphs, with an application to a problem of traffic control. *Americ. Mathem. Monthly*, 46:281–283, 1939.

[7] A. Schrijver. *Theory of Linear Programming and Integer Programming.* John Wiley & Sons, New York, 1986.

[8] M. Laurent M. Loebl T. Fleiner, W. Hochstaettler. Cycle bases for lattices of binary matroidswith no fano dual minor and their one-element extensions. *J. Comb. Theory B*, 77:25–38, 1999.