Discharging Technique in Practice
(short lecture notes)

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Abstract. This short lecture notes are prepared to formulate and demonstrate so called "discharging technique" in combinatorial proofs. We provide general description of the method, and several easy example applications. Moreover, we refer some more involved applications, among them the unavoidability part of the Four color theorem.

1 The discharging technique

First we need to declare basic graph terms. We consider only finite graphs; however, we allow loops or parallel edges in general. We say that a graph is simple if it contains no loops or parallel edges.

1.1 Motivation

Why do we say "discharging"? Since we use to say that we set *charge* on elements of a graph (vertices, edges, faces, etc), and then we *discharge* it.

And how is this useful? Generaly speaking, we sum the total charge over whole graph twice, once before discharging and once after that. In one summing we use some sort of "generic formula", in the other summing we use specific graph properties. Finally, we derive our conclusion from discrepancies we have found.

The generic formula mentioned above is an (unconditional) equation that holds for our class of graphs. Usually, *Euler's formula* is used for embedded graphs, see below.

The whole discharging technique is just a sophisticated "bridge" between the generic formula and specific graph properties we are interested in.

1.2 Euler's formula

A *surface* is a compact 2-manifold. (However, you do not need to understand much topology to imagine a surface – like a sphere, torus, projective plane, Klein bottle, etc.)

A graph $\boldsymbol{G}$ is *embedded* on a surface $\mathcal{S}$ if its vertices are mapped to distinct points of $\mathcal{S}$, and edges are mapped to simple curves (simple closed curves for loops) in $\mathcal{S}$ connecting its vertex-points. Moreover, no two edge-curves share a point in $\mathcal{S}$ except possibly a common vertex-point in $\boldsymbol{G}$, and no vertex-point belongs to an interior of an edge-curve of G. A *face* of an embedding of $\boldsymbol{G}$ is a connected component of the surface $\mathcal{S}$ after deleting the embedding. A graph is 2-*cell embedded* on $\mathcal{S}$ if all faces are homeomorphic to open discs. (Informally, the embedding "uses" whole surface.) Any plane graph is 2-cell embedded.

Let a graph $\boldsymbol{G}$ be 2-cell embedded on a surface $\mathcal{S}$. Let $V(\boldsymbol{G})$ be the vertex set, $v = |V(\boldsymbol{G})|$, let $E(\boldsymbol{G})$ be the edge set, $e = |E(\boldsymbol{G})|$, and let $F(\boldsymbol{G})$ be the set of faces of the embedding, $f = |F(\boldsymbol{G})|$. (Notice that the graph actually needs to be embedded to speak about Euler's formula!)

Then

$$v + f - e = \Sigma(\mathcal{S}),$$

where $\Sigma(\mathcal{S})$ is a constant depending only on the surface $\mathcal{S}$.
(For example, $\Sigma = 2$ for the plane, $\Sigma = 1$ for the projective plane, and $\Sigma = 0$ for the torus and Klein bottle. The value of Σ is negative for all other surfaces.)

1.3 Formulation

For our purpose, *charge* is just arbitrary real number (value), it may be positive or negative, and it can be arbitrarily divided. In general, the *discharging method* makes the following steps:

"Charging phase"

1. Assign *initial charges* to certain elements of a graph (vertices, edges, faces, etc...).
2. Compute the total charge assigned to the whole graph (usually using Euler's formula).

"Discharging phase"

3. Redistribute charge in the graph according to a set of *discharging rules*. (This is a key step that makes all work. It is important that no additional charge is introduced or lost.)
4. Compute the total charge again (usually using specific properties of the graph), and derive a conclusion.

However, keep in mind that what we presented is just a rough skeleton of the technique. It usually requires a lot of additional ideas to apply the method to our particular problem. (Think about "discharging" as an analogue to "induction" or to other well know mathematical methods.) Also, the above presented steps may look slightly different in practical applications.

2 Examples of discharging

We start with very easy examples to illustrate the discharging technique. The first ones do not even use the discharging phase, they are here to demonstrate how initial charge is assigned, and how total charge can be computed in a graph. (In typical applications we target to "average" initial charges around 0.) The next examples already use both phases, but they are still quite easy to follow and to play with.

2.1 Degree-5 vertex in the plane

Every planar graph without faces of length less than 3 has a vertex of degree at most 5:

We charge vertex of degree d by $6-d$, and face of length l by $2(3-l)$. Total charge is $\sum_{v \in V}(6-d_v) + \sum_{f \in F} 2(3-l_f) = 6|V| - 2|E| + 6|F| - 4|E| = 6(|V| - |E| + |F|) = 6 \cdot 2 = 12$.
So there must be a vertex v such that $6 - d_v > 0$, $d_v \leq 5$.

(Why do we set charges as above? As noted above, we want to average initial charge around 0, and a "typical graph" in this question has faces of length 3 and vertices of degree 6. So our choice "nulls" charge for typical vertices and faces. Of course, the main point is that our choice works nicely in Euler's formula!)

2.2 6-regular graphs on torus

If a 6-regular graph is embedded on torus without faces of length less than 3, then it is a triangulation of torus:

Again, charge vertex of degree d by $6-d$, and face of length l by $2(3-l)$. Total charge is now $\sum_{v \in V}(6-d_v) + \sum_{f \in F} 2(3-l_f) = 6|V| - 2|E| + 6|F| - 4|E| = 0$ on torus.
So if $6 - d_v = 0$ for all v, and $3 - l_f \geq 0$, all faces must be triangles.

(You may see that the choice of initial charge $6-d$ and $2(3-l)$ is pretty typical in applications.)

2.3 Degree + face length in the plane

In any plane graph without vertices of degree less than 3 and without faces of length less than 3, there is a vertex of degree d incident with a face of length l such that $d + l \leq 8$.
(Vertex-face incidence is called a *corner*.)

We charge vertex of degree d by $4-d$, and face of length l by $4-l$. Total charge is $\sum_{v \in V}(4-d_v) + \sum_{f \in F}(4-l_f) = 4|V| - 2|E| + 4|F| - 2|E| = 4 \cdot 2 = 8$.
Then we discharge every vertex and every face equally to all adjacent corners. Precisely, if a vertex of degree d got initial charge $4-d$, it sends charge of $\frac{4-d}{d}$ to

each of d adjacent corners. The result is that this vertex now has no charge, and its former charge (positive or negative) is now in the adjacent corners. Similar rule applies to each face.

Final charge of each vertex or face is 0, so some corner must have positive charge. Let the vertex adjacent to this corner have degree d, and let the face adjacent to this corner have length l. Since we know $d \geq 3$ and $l \geq 3$, we derive:

$$\frac{4-d}{d} + \frac{4-l}{l} > 0 \ .$$

$$4l + 4d - 2dl > 0 \ ,$$

$$l < \frac{2d}{d-2} \leq 6 \ , \quad d < \frac{2l}{l-2} \leq 6 \ ,$$

$$d + l < \frac{l^2}{l-2} \quad \Longrightarrow \quad d + l \leq \max\{8, 7, 8\} \ \text{ for } l = 3, 4, 5.$$

Notice that we actually proved something slightly stronger than was required.

(Why have we chosen different initial charges in this case? In this problem, there is natural vertex-face duality, so vertices and faces should be charged and discharged in the same way. When you try to work that out in Euler's formula, you discover that $4 - d$ and $4 - l$ is a natural choice.
This is the first example that actually uses the discharging phase. Here you may nicely see how the discharging rules build a "bridge" between the graph elements involved in Euler's formula – vertices and faces, and the graph elements we are interested in - corners.)

2.4 Covering graph $\boldsymbol{K_{3,5}}$

Let $\boldsymbol{G}$ be a planar bipartite simple graph with vertex partition $V(\boldsymbol{G}) = V \cup W$, the vertices of V having degrees 5, and the vertices of W having degrees 3. Let $\lambda : V(G) \to \{1, 2, \ldots, 8\}$ be a labelling of vertices of $\boldsymbol{G}$ such that: Each vertex of V has label one of $1, 2, 3$, and its neighbors are labeled by $4, 5, 6, 7, 8$. Each vertex of W has label one of $4, 5, 6, 7, 8$, and its neighbors are labeled by $1, 2, 3$. We show that a graph $\boldsymbol{G}$ of these properties cannot exist. (This ellegant result was discovered independently by several people, among them Fellows and Archdeacon.)

We charge vertex of degree d by $3(4 - d)$, and face of length l by $3(4 - l)$. Total charge is $\sum_{v \in V} 3(4 - d_v) + \sum_{f \in F} 3(4 - l_f) = 3(4|V| - 2|E| + 4|F| - 2|E|) = 12 \cdot 2 = 24$. (We multiply all charge by 3 to avoid fractions later.)

The first discharging rule sends charge of 1 from every vertex of W to each of its three neighbors. (Since a vertex $w \in W$ had initial charge 3, this rule makes charge 0.) The second discharging rule sends charge of 2 from every vertex of V to each adjacent face of length 6 or more. (Of course, this rule applies only to vertices $v \in V$ that *have* some of the five adjacent faces of length 6 or more.)

As was already noted, final charge for every vertex $w \in W$ is 0. If f is a face of length 4, it starts with charge 0, and no discharging rule concerns f. If

f is a face of length $l \geq 6$, then it starts with charge $3(4-l)$, and f receives charge of at most $2(l/2)$ from vertices of V incident with f. So final charge of f is $3(4-l)+l = 12-2l \leq 0$. (Recall that $\boldsymbol{G}$ is bipartite and simple!) Every vertex $v \in V$ starts with charge of -3, then it receives charge of 5 in total by the first discharging rule, and sends charge of 2 to each adjacent face of length at least 6. Therefore, if every vertex $v \in V$ has an adjacent face of length at least 6, then v ends up with final charge of at most 0.

The fact that every vertex $v \in V$ must have an adjacent face of length at least 6 is left to reader. (Just draw a picture and try to assign labels to neighboring vertices.)

So what is the conclusion? The total initial charge assigned to $\boldsymbol{G}$ was 24, which is a positive number. However, the total final charge over whole $\boldsymbol{G}$ is non-positive. We have got a contradiction showing that a graph $\boldsymbol{G}$ of these properties cannot exist.

(This is another illustration of a simple discharging process. It is worth to mention that the discharging rules are all applied at once, there is no process in the formal application. However, we often speak about the rules like if there was some step-by-step process since that makes our ideas easier to follow. Look again at the text above.
Another suggestion is the following: Always design discharging rules to be *local* – i.e. that charge is moved between adjacent graph elements, like from a vertex to its neighbors, to adjacent faces, or to adjacent edges. Otherwise you would have very hard work to compute final charges in the graph. That is why our discharging rules first send "excess" charge from vertices of W to vertices of V, and then from vertices of V to adjacent large faces.)

2.5 Adjacent low-degree vertices

This last example is more involved than the previous ones, and it is designed as an excercise for readers. We want to find in a given planar graph two adjacent vertices such that the sum of their degrees is minimized. (History of this question goes back to Kotzig many year ago...)

Of course, to be able to prove any upper bound, we must restrict our attention to plane graphs $\boldsymbol{G}$ with no vertices of degree less than 3, and no faces of length less than 3. If e is an edge of a graph $\boldsymbol{G}$, we call the sum of degrees of endvertices of e the *weight* of e. It is a good idea to look first at graphs with "heavy" edges. For example, adding a degree-5 vertex to each face of a regular dodecahedron creates a graph with all edges of weight at lest 11. Can you find even "heavier" graphs?

Now try to apply some discharging method to obtain an upper bound (in the sense that one can always find an edge in $\boldsymbol{G}$ of weight at most D for some constant D). First try a method analogous to Section 2.3 – discharge every vertex of $\boldsymbol{G}$ equally to all incident edges, and then look at the edge with positive final charge. Do you get anything useful? Unfortunately not...

Then analyze reasons why the previous simple method did not work. (Hint: It is useless to discharge high-degree vertices to heavy edges, since then such an edge gets very low negative charge – we may say that such an edge is “vasted”.) Try to design more sophisticated discharging rules that take the structure of the neighborhood of a vertex into account. Also, it is useful to consider separately the cases when minimal degree in $\boldsymbol{G}$ is $3, 4$ or 5. Good luck!

3 Applications of the technique

3.1 Planar covers of graphs

This section presents few comments to the paper

`http://www.fields.utoronto.ca/~phlineny/temp/cov2disch.ps.gz`

that was prepared for these notes. So first obtain and print the paper, and look at the related definitions.

Now go back to Section 2.4, and see that the statement presented there is actually a proof that the graph $\boldsymbol{K}_{3,5}$ has no planar cover. The paper presents proofs of nonexistence of a planar cover for another two graphs $\boldsymbol{K}_{4,4} - e$ and $\boldsymbol{\mathcal{E}}_2$. Both proofs use the discharging technique, and both of them have surprisingly simple discharging rules, yet proving rather sophisticated statements.

When reading the proofs, focus on the discharging parts, and notice what all is needed to make the discharging method work. The first proof uses pretty standard discharging method, actually similar (in form) to Section 2.4. On the other hand, the second proof looks completely different from what we have seen so far here, and it respects little from the above written suggestions. Why? In fact, that is hard to say, the only comment we can say is that a “standard discharging technique” does not work well in this particular case.

3.2 Four color theorem

This is probably the best known application of the discharging technique. The history of the four color problem started already in the last century. The question is whether every planar graph (plana map) can be properly colored using just four colors. Nowadays, this is not longer a problem, but a theorem.

The first proof of the Four color theorem by Appel and Haken appeared in 1976. However, the proof was complicated and unclear, and so it was not widely accepted. That is why a new proof by Robertson, Sanders, Seymour and Thomas appeared several years later. Read more about history of the problem and its proofs on

`http://www.math.gatech.edu/~thomas/FC/fourcolor.html`.

The reason to mention Four color theorem here is that one part (“unavoidability”) of both proofs uses discharging technique. We will focus on the newer proof of the theorem. It is best to start by reading the above web page. Then, we do not suggest to read the formal paper that appeared in [N. Robertson, D. P. Sanders, P. D. Seymour and R. Thomas, The four colour theorem, J. Combin.

Theory Ser. B. 70 (1997), 2–44], due to its complexity. It is better to read the survey version

`http://www.math.gatech.edu/~thomas/update.ps`

which appeared in AMS Notices in 1998, since it gives better understanding of the discharging method used in the proof.

Do not be afraid of the length of the proof, the discharging method is quite standard, it only uses (too) many configurations and rules. Notice one thing in the proof – there is no use of charge on faces. The reason is very simple; the problem is formulated on planar triangulations, and so every face would get charge $2(3 - l) = 2(3 - 3) = 0$.

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