An Addition to Art Galleries with Interior Walls

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Abstract. Consider an art gallery formed by a polygon on n vertices with m pairs of vertices joined by interior diagonals, the interior walls. Each interior wall has an arbitrarily placed, arbitrarily small doorway. It is shown in [A. Kündgen, *Art Galleries with Interior Walls*, Discrete Comput. Geom. 22 (1999) 248–258] that the minimum number of guards that suffice to guard all art galleries with n vertices and m interior walls is $\min\{\lfloor(2n-3)/3\rfloor, \lfloor(2m+n)/3\rfloor, \lfloor(2n+m-2)/4\rfloor\}$. The proofs for the first two bounds lead to linear time guard placement algorithms, while this is not known for the third case. We present an alternative short proof for the third upper bound $\lfloor(2n+m-2)/4\rfloor$ that also leads to linear time guard placement algorithm.

1 Introduction

The original art gallery problem, posted by Klee and solved by Chvátal [2], is to find the smallest number of guards necessary to cover any simple (not necessarily convex) polygon – the art gallery, on n vertices. Here a covering by g guards means that one can find g points in the interior of the polygon such that every point in the interior is joined with some guard by a straight-line segment not intersecting the polygon. The example in Figure 1 can be generalized to an arbitrary number of vertices divisible by 3, and so $\lfloor n/3 \rfloor$ guards are sometimes necessary. Chvátal showed that $\lfloor n/3 \rfloor$ guards always suffice. For more information on the history of this problem and related problems, see [6].

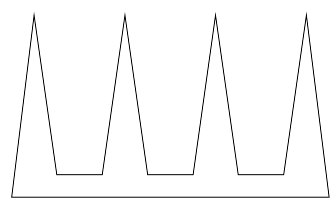

Fig. 1. An art gallery requiring 4 guards.

We will now give Fisk's [3] elegant proof of Chvátal's result, that is an inspiration for our proof: First triangulate the polygon, which results in an outerplanar graph. Then color this graph by 3 colors. Since each triangle in the triangulation must have vertices of all three colors, putting a guard at each vertex in the smallest color class produces a covering by $\lfloor n/3 \rfloor$ guards.

Hutchinson [4] and Griggs generalized the basic art gallery problem by allowing interior walls. Our notation follows [5]. An *art gallery (with interior walls)* is a simple polygon on n vertices with m pairs of vertices joined by non-intersecting interior diagonals – the interior walls. In each interior wall there is an arbitrarily placed, arbitrarily small opening (doorway). Figure 2 is an example of an art gallery on $n = 15$ vertices and $m = 8$ interior walls that requires 9 guards.

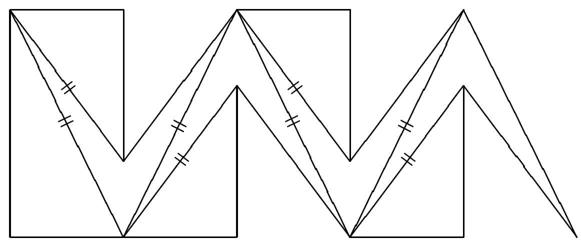

Fig. 2. An art gallery with interior walls requiring 9 guards.

Kündgen [5] found the number of guards that suffice to cover any such art gallery on n vertices and m interior walls.

Theorem 1. (A. Kündgen, 1997) *The minimum number of guards that suffice to cover all art galleries with* n *vertices and* m *interior walls is*

$$g(n,m) = \begin{cases} \lfloor \frac{2n-3}{3} \rfloor & \text{for } m \geq \lfloor \frac{2}{3}n \rfloor - 2, \\ \lfloor \frac{2m+n}{3} \rfloor & \text{for } m < \lfloor \frac{2}{5}n \rfloor, \\ \lfloor \frac{2n+m-2}{4} \rfloor & \text{otherwise.} \end{cases}$$

2 Displacing Checkpoints and Guards

The first two cases of Theorem 1 were proved along the guidelines of the above mentioned Fisk's proof, and they lead to linear-time guard placement algorithms. On the other hand, the third case was proved in [5] by an inductive construction, and it is not known whether this process can be implemented in linear time.

We present a "Fisk-type" proof of the upper bound $g(n,m) \leq \lfloor \frac{2n+m-2}{4} \rfloor$. This bound is valid for all values of m, n, however, it is optimal only in the range specified by Theorem 1. The bound is a straightforward consequence (place guards to the points of the smallest color class) of the following lemma:

Lemma 2. *Let* $\mathcal{P}$ *be an art gallery with* n *vertices and* m *interior walls, and let* $\mathcal{P}'$ *be an outerplanar triangulation of* $\mathcal{P}$ *respecting its interior walls. Then there exists a set* X *of interior points of* $\mathcal{P}$*, and an incidence relation between* X *and the triangular faces of* $\mathcal{P}'$*, satisfying the following:*

1. *Each point of X guards all triangles incident with it.*
2. *Each triangular face of $\mathcal{P}'$ is incident with exactly four points of X.*
3. *The set X can be 4-colored so that no two points incident with the same triangle get the same color.*
4. *The cardinality of X is $2n + m - 2$.*

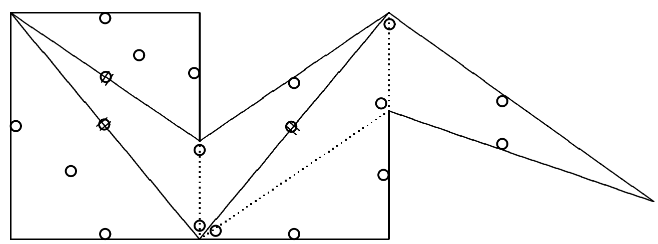

Fig. 3. Placement of 19 checkpoints for a gallery with $n = 9$, $m = 3$.

Proof. Thorough our proof we call the points of X *checkpoints*. We call an edge of $\mathcal{P}'$ a *wall* if it is outer or interior wall of $\mathcal{P}$, and we call it a *non-wall* if it is an edge introduced by the triangulation. A triangular face of $\mathcal{P}'$ is *uniform* if its edges are either all walls, or all non-walls. An angle at vertex v of $\mathcal{P}$ determined by walls incident with v and belonging to the interior of $\mathcal{P}$ is called a *corner* of $\mathcal{P}$. A corner is *divided* if it belongs to more than one face of the triangulation $\mathcal{P}'$.

We introduce one checkpoint per each wall of $\mathcal{P}'$, one checkpoint per each divided corner, and one checkpoint per each uniform triangle of $\mathcal{P}'$. (See an example in Figure 3.) A "wall" checkpoint is incident with one or two triangles adjacent to the wall. A "corner" checkpoint is incident with all triangles spanning the corner. To fulfill the requirement that checkpoints belong to interior of $\mathcal{P}'$, we place the checkpoints as follows: For an interior wall, the checkpoint is placed in the middle of doorway. For an outer wall, the checkpoint is placed at a close distance perpendicularly from the middle of wall. For a divided corner, the checkpoint is put on the line of symmetry of the corner close to the vertex.

It is now routine work to verify validity of properties (1,2). Let $C(T) \subset X$ denote the set of checkpoints incident with triangle T (so $|C(T)| = 4$). To prove (3), notice that the topological dual of $\mathcal{P}'$ is a tree, and hence the triangular faces of $\mathcal{P}'$ can be ordered $T_1, T_2, \ldots, T_{n-2}$ so that T_i is adjacent to just one T_{k_i}, $k_i < i$ for $i = 2, \ldots, n-2$. And since all checkpoints of $C(T_i) \cap \bigcup_{j=1}^{i-1} C(T_j)$ are incident also with T_{k_i}, we can 4-color X by the greedy algorithm in order extending $C(T_1)$, $C(T_2) \setminus C(T_1)$, $\ldots$, $C(T_{n-2}) \setminus \bigcup_{j=1}^{n-3} C(T_j)$.

Surprisingly, the last property (4) is the most difficult one. Clearly there is $m+n$ "wall" checkpoints. Next we show the following claim:

Claim 1. Each room of our gallery (i.e. interior face of $\mathcal{P}$) contains $t+2$ non-divided corners, where t is the number of uniform triangles in this room.

The claim holds if the room is just a triangle. Otherwise, the topological dual of the room in $\mathcal{P}'$ is a tree $\mathcal{T}$ with maximal degree 3. In such case every degree-1 vertex of $\mathcal{T}$ corresponds to one non-divided corner, and every degree-3 vertex of $\mathcal{T}$ corresponds to a uniform triangle. Hence the claim follows from well-known equality $\sum_{v \in V(\mathcal{T})} d(v) = 2|V(\mathcal{T})| - 2$.

Now, our gallery has $m+1$ rooms that have r_j vertices and t_j uniform triangles, $j = 1, \ldots, m+1$. An easy counting gives $\sum_{j=1}^{m+1} r_j = n + 2m$. By the above claim, the total number of divided corners in the gallery is

$$\sum_{j=1}^{m+1} (r_j - t_j - 2) = \sum_{j=1}^{m+1} r_j - \sum_{j=1}^{m+1} t_j - 2(m+1) = n - 2 - \sum_{j=1}^{m+1} t_j .$$

And since there are $\sum_{j=1}^{m+1} t_j$ checkpoints introduced for uniform triangles, the total number of checkpoints displaced in our construction is $m + n + n - 2$. ∎

If we have the triangulation $\mathcal{P}'$ at hand, the proof of Lemma 2 gives a straightforward algorithm to construct X and to find the smallest color class of a 4-coloring of X in linear time. (However, notice that this algorithm just meets the general upper bound, it need not produce optimal guard displacement in every particular case.) The problem to triangulate given polygon can be, at least theoretically, solved also in linear time by the result of Chazelle [1]. For our purpose we need to apply the triangulation algorithm for each room separately, but that still leads to a linear overall computing time.

References

1. B. Chazelle, *Triangulating a simple polygon in linear time*, Discrete Comput. Geom. 6 (1991), 485–524.
2. V. Chvátal, *A combinatorial theorem in plane geometry*, J. Comb. Theory Ser. B 18 (1975), 39–41.
3. S. Fisk, *A short proof of Chvátal's watchman theorem*, J. Combin. Theory Ser. B 24 (1978), 374.
4. J. Hutchinson, *Art Galleries with Walls*, Problem #10478, Amer. Math. Monthly 102 (1995), 746.
5. A. Kündgen, *Art Galleries with Interior Walls*, Discrete Comput. Geom. 22 (1999) 248–258.
6. J. O'Rourke, Art gallery theorems and algorithms, The International Series of Monographs on Computer Science, The Clarendon Press, Oxford University Press, New York, 1987.

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