H-colorings of Large Degree Graphs *

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Abstract

We consider the generalized colorings (H-colorings) of graphs with all its vertices of large degree. We prove that this problem belongs to P for all cycles H and define the complexity threshold which we determine for an infinite family of graphs of a given chromatic number. Further results and related conjectures are stated.

1 Introduction

Let us recall some basic definitions from graph theory. Given a graph $G = (V, E)$ a *k-coloring* of G is a mapping from V to $\{1, \dots, k\}$ such that no two vertices on the same edge correspond the same color. The *chromatic number* $\chi(G)$ is the minimum number of colors needed to color G. The *clique number* $\omega(G)$ is the maximum k such that K_k is a subgraph of G, where K_k denotes a complete graph on k vertices. The *minimum vertex degree* of G is denoted by $\delta(G)$. A cycle on $2k+1$-vertices is denoted by C_{2k+1}. Given two graphs $G = (V, E)$ en $H = (V', E')$ an *homomorphism* from G to H is an edge preserving mapping from V to V'. Given two graphs G and G' the graph $G \oplus G'$ is formed by taking one copy of G and G' and joint by edges all vertices of G with all vertices of G'. We shall denote the existence of an homomorphism from G to H by $G \to H$. As usual we say a problem is in

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P if there exits a polynomial time algorithm to decide it, and a problem is NP-complete if it belongs to the class NP and every other problem in this class is polynomically reducible to it. Through the paper we shall work under the plausible hypothesis that $\mathsf{P} \neq \mathsf{NP}$.

For any fixed graph H, the *H-coloring problem* consists in: Given a graph G, determine whether there is an homomorphism from G into H.

The homomorphisms of G to a complete graph K_k is just a k-coloring of G and thus the H-coloring problem naturally generalizes decision problems related to the chromatic number. Further examples of H-coloring problems include *circular chromatic number*, [5], *T-colorings* and problems related to the *channel assignments problems*, see e.g. [4].

Given a constant $0 \leq \alpha \leq 1$, a graph $G = (V, E)$ with $|V| = n$ vertices is *α-dense* if $\delta(G) \geq \alpha n$. We are interested in the *H-coloring problem* restricted to α-dense graphs.

For any fixed graph H and a constant $0 \leq \alpha \leq 1$, the *(H, α)-coloring problem* consists in: Given an *α-dense* graph G, determine whether there is an homomorphism from G into H.

From the definition it follows that for any two constants α, β such that $0 \leq \alpha \leq \beta \leq 1$ we get:

- If the (H, α)-coloring problem is in P then also the (H, β)-coloring problem is in P,
- if the (H, β)-coloring problem is NP-complete then also the (H, α)-coloring problem is NP-complete.

These properties motivates the following definition:

Definition 1. *Given a graph H the* complexity threshold $c(H)$ *is defined as the* $\inf\{\alpha \mid (H, \alpha)$*-coloring* $\in \mathsf{P}\}$.

One immediate consequence of the previous definition is the fact that $c(H) = 0$ if and only if for every $0 < \alpha \leq 1$, the (H, α)- coloring problem is in P.

While for every graph H the threshold $c(H)$ exists, and as we shall see in Proposition 1 it can not be 1, a first question of interest is whether $c(H)$ can be defined alternatively as $\sup\{\alpha \mid (H, \alpha)\text{-coloring} \in \mathsf{NP}\text{-complete}\}$, under the assumption of $\mathsf{P} \neq \mathsf{NP}$. A second question is whether there are graphs H for which $c(H) = \min\{\alpha \mid (H, \alpha)\text{-coloring} \in \mathsf{P}\}$. (H, α)-coloring problem is in

P. All non-bipartite examples for which we can compute the exact threshold indicate that $c(H)$ is never attained.

The first question is related and follows from an affirmative solution to the following *dichotomy conjecture*:

> For every H and every α the decision problem of the existence of a H-coloring for α-dense graphs is either NP-complete or P.

A particular case providing a partial affirmative answer to the previous conjecture was given by Edwards who considered the graph family K_k for any $k \geq 3$, where $c(K_k) = (k-3)/(k-2)$ (see Theorem 2.5 of [2]).

In this note we validate the Dichotomy conjecture and we determine the complexity threshold for infinitely many graphs with a given chromatic number. We relate our results to the ones in [2] and in [3]. We also list some interesting open problems.

2 Exact Thresholds

In the following theorem we introduce an infinite family of k-colorable graphs for which $c(H) \leq (k-3)/(k-2)$:

Theorem 1. *Let $k \geq 3$ and let α satisfies $(k-3)/(k-2) < \alpha < 1$. Assume that a graph H satisfies $\chi(H) = k$, and let any subgraph of H isomorphic to K_{k-2} be contained in at most two subgraphs of H isomorphics to K_{k-1}. Then the (H, α)-coloring problem is in the class P.*

Proof. Let G be an α-dense graph. It follows from the extremal graph theory (by an argument similar to the proof of lemma 2.4 in [2]) that there is a set S of $(k-2)$-cliques of V with $|S| = O(\log n)$, such that every $v \in V$ that doesn't belong to a clique of S is completely joined to at least one clique in S, moreover this set S may be found in polynomial time.

Let U be the set of vertices that appear in S, and let $f : U \to H$ be any fixed homomorphism, notice that for any $v \in V \setminus U$ there are at most two possible vertices of H that can extend f to the whole V, let us call these vertices $F(v)$ and let $V \setminus U = \{v_1, \ldots, v_m\}$. Let $\{1, \ldots, r\}$ be the vertex set of H, given $i \in H$ let $L(i)$ be the set of vertices in H that are different from i and are not connected to vertex i. Define the instance of a 2-SAT formula where the set of variables are $x_{i,j}$, $1 \leq i \leq m, 1 \leq j \leq 2k+1$ each of them indicating whether $f(i) = j$, and the clauses are:

1. $\{x_{i,j} | j \in F(i)\}$, for $1 \leq i \leq m$,
2. $\{\overline{x}_{i,j}, \overline{x}_{i,l}\}$ for $1 \leq i \leq m$, $1 \leq j < l \leq r$,
3. $\{\overline{x}_{i,j}, \overline{x}_{h,l}\}$ for $(v_i, v_h) \in E$, $1 \leq j \leq r$ and $l \in L(j)$.

It is easy to see that the homomorphism f can be extended to G if and only if this instance of 2-SAT has a satisfying assignment. As 2-SAT belongs to the class P, then the problem of deciding the extension of f to G is also in P. Notice that the total number of possibilities for the homomorphism f is polynomial, therefore by exhaustive search we can decide in polynomial time if G is H-colorable. ☐

As a corollary we get that the exact threshold for odd cycles is 0.

Corollary 1. *For every $0 < \alpha < 1$ and for every $k \geq 1$, C_{2k+1}- coloring is in the class P, for α-dense graphs.*

Note that also for even cycles and all bipartite graphs H, the threshold is 0, and this is the only known case where the threshold is attained.

Another result we could obtain from the previous theorem give us an exact threshold for infinitely many graphs of a given chromatic number:

Corollary 2. *$c(C_{2l+1} \oplus K_{k-3}) = (k-3)/(k-2)$ for every $k, l \geq 1$.*

Proof. The P part is a consequence of the previous theorem. To prove completeness let us consider the following reduction:
For any fixed constant k, given a graph $G = (V, E)$ construct a graph $G' = (V', E')$ by replicating $k - 3$ blocks of the vertices V, together with a copy of all G, then add edges connecting vertices that are in different blocks and connect the vertices of the copy of G with all the vertices in the blocks, notice $|V'| = (k-2)|V|$. Then G is C_{2l+1}-colorable iff G' is $C_{2l+1} \oplus K_{k-3}$-colorable. Furthermore, every vertex in G' has degree at least $\frac{k-3}{k-2}|V'|$. ☐

3 Threshold Bounds

We have seen that there are graphs for which their threshold is 0, for instance any cycle, in the following result we prove that for every graph its threshold cannot be equal to 1,

Proposition 1. $c(H) < 1$ *for every graph* H.

Proof. Let H be a fixed graph, denote by $\omega = \omega(H)$ the maximal size of a complete graph in H. It follows from the classical result of Turán (see any textbook on Graph Theory, see e.g. [1], p. 108) that a graph G contains $K_{\omega+1}$ providing that $|E(G)| > n^2/2(1 - 1/\omega)$, and therefore it can not be homomorphic to H. It follows that for any graph H, $c(H) \leq 1 - 1/\omega < 1$. $\square$

In the following result we present another family for which we can compute better upper and lower bounds for its threshold,

Proposition 2. *For all* $k > 1$ *there exist a graph* H_k *with* $\chi(H_k) = k + 4$ *such that the* (H_k, α)*-coloring problem is in:*

- NP*-complete, for* $0 \leq \alpha \leq \frac{((2k-2)/k)+1}{((2k+1)/k)+1}$,
- P*, for* $1 > \alpha > \frac{k+1}{k+2}$.

Proof. Let $H_k = K_3 \oplus \overline{C}_{2k+1}$, let us consider the following reduction:
For any fixed constant k, given a graph $G = (V, E)$ such that the number of edges is a multiple of k construct a graph $G' = (V', E')$ by replicating $2k + 1$ blocks of $|V|/k$ vertices, together with a copy of all G, then add edges connecting vertices that are in different blocks according to the $\overline{C}_{2k+1}$ pattern and connect the vertices of the copy of G with all the vertices in the blocks, notice $|V'| = (\frac{2k+1}{k} + 1)|V|$. Then G is 3-colorable iff G' is H_k-colorable. Furthermore, every vertex in G' has degree at least $(\frac{2k-2}{k} + 1)|V|$. Then α for G is less or equal to $\frac{((2k-2)/k)+1}{((2k+1)/k)+1}$.

To prove the P part, notice that for any $k > 1$ clique number of $\overline{C}_{2k+1}$ is k. Furthermore any clique with size $k - 1$ in $\overline{C}_{2k+1}$, is contained in at most two cliques of size k. Therefore for the H_k any clique of size $k+2$ is contained in at most two cliques of size $k + 3$. Using theorem 1 we obtained the second part of the statement. $\square$

It is still open to decide in which class the problem falls when the value of α is between $\frac{((2k-2)/k)+1}{((2k+1)/k)+1}$ and $\frac{k+1}{k+2}$, but notice that for $k = 6$ both values are above 0.8, and differ in 0.1. So for values of $k \geq 6$ it means that the decision problem is NP-complete for most of the dense graphs. The value of the bounds seems to indicate that the complexity threshold for the H_k-graphs will coincide with the threshold for K_{k+4}.

The next result gives a generic condition to guaranty for a complexity threshold of at least 1/2.

Proposition 3. *Let H be a graph which contains a vertex x whose neighborhood $N(x)$ induces a graph that is not bipartite. Then the $(H, \frac{1}{2})$-coloring problem is* NP*-complete.*

Proof. Given a graph H which contains the vertex x, let $H_1, H_2, ..., H_t$ be all neighborhoods of vertices of H such that $\chi(H_i) > 2$ and let $H' = \oplus_{i=1}^{t} H_i$. For a given connected G' let G be a graph with vertices $V(G') \times \{0,1\}$ and all edges of G' on $V(G') \times 0$ together with all edges in between. Then $G \to H$ iff $G' \to H'$ and the later is NP-complete by Theorem 1 in [3]. ☐

These investigations lead to the following interesting problem:

> Characterize graphs H with $c(H) = 0$.

Presently we do not know any 3-chromatic graph H for which the (H, α)-coloring problem is NP-complete for some $\alpha > 0$. A candidate for such a graph H is the particular subdivision of K_4 where we subdivide each edge of a triangle by 2 points. The resulting graph has 10 vertices and it is not C_5-colorable. By the general theorem in [3], H-coloring is NP-complete, and in this particular case it may be seen easily by the reduction from 3NAESAT given in Figure 1. However this reduction and the general reduction in [3] produce graphs with constant minimum degree and thus it does not yield NP-completeness of the (H, α)-coloring problem for any $\alpha > 0$. Perhaps $c(H) = 0$ holds for any 3-chromatic graph H.

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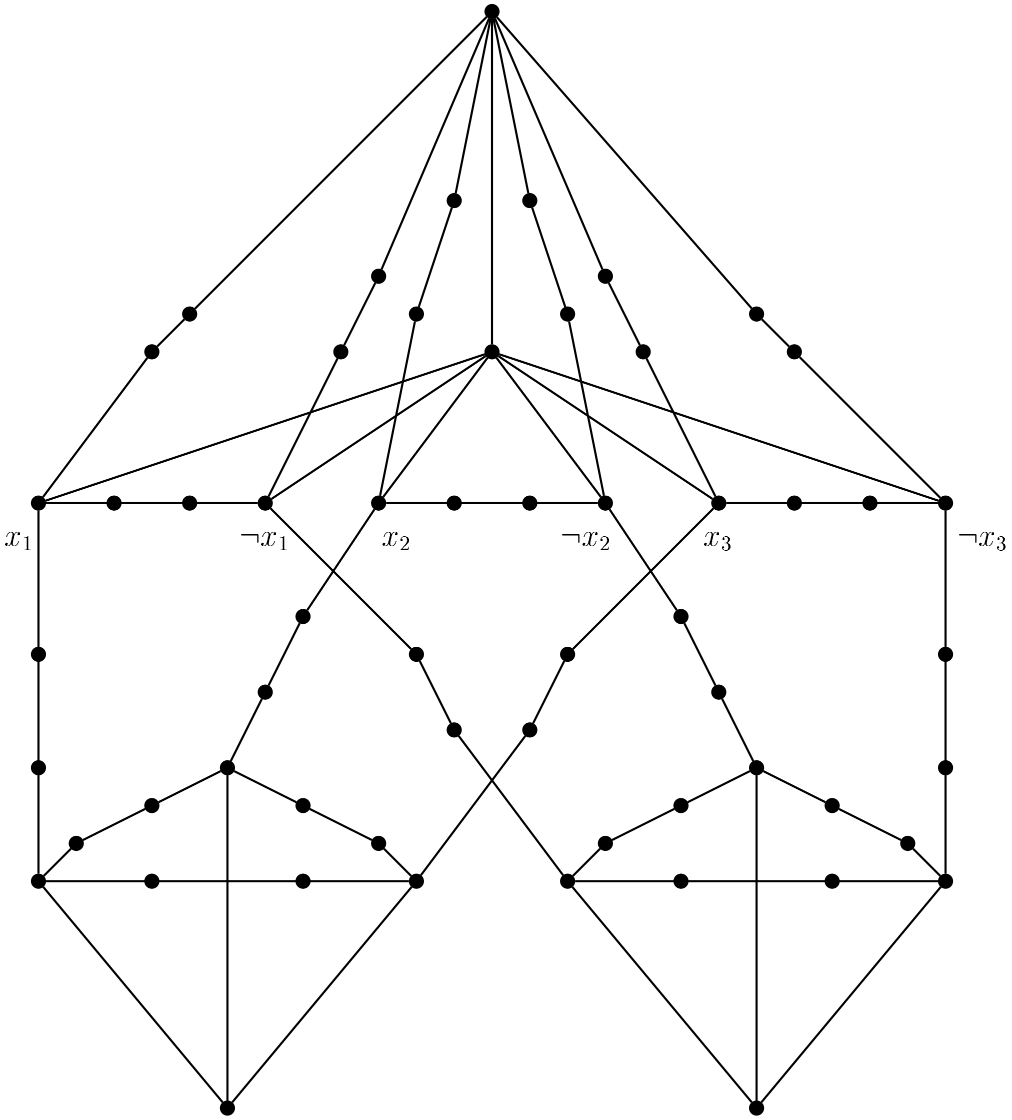

$$F = \{(x_1, x_2, x_3), (\neg x_1, \neg x_2, \neg x_3)\}$$

Figure 1: A sketch of the reduction.

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