On the Oriented Game Chromatic Number*

J. Nešetřil† and E. Sopena‡

March 27, 2000

Abstract

We consider the oriented version of the coloring game introduced by Bodlaender [On the complexity of some coloring games, *Internat. J. Found. Comput. Sci.* **2** (1991), 133–147]. We prove that every oriented path has oriented game chromatic number at most 7 (and this bound is tight), that every oriented tree has oriented game chromatic number at most 19 and that there exists a constant t such that every oriented outerplanar graph has oriented game chromatic number at most t.

1 Introduction

Let G be an undirected graph, with vertex set $V(G)$ and edge set $E(G)$, and X be a set of colors. We consider the two-players game defined as follows. The two players are Alice and Bob and they play alternatively with Alice having the first move. Alice's goal is to provide a proper k-coloring of G and Bob's goal is to prevent her from doing so. A move consists in choosing an uncolored vertex u and assigning it a color from the set X distinct from the colors previously assigned (by either player) to the neighbors of u. If after $|V(G)|$ moves the graph is colored then Alice wins, otherwise Bob wins. In other words, Bob wins whenever an impass is reached before the whole graph is colored, that is if, at some intermediate step, for every uncolored vertex u and every color α, u has some neighbour with color α.

The *game chromatic number* of G, denoted by $gcn(G)$, is then defined as the least cardinality of a color set X for which Alice has a winning strategy. This parameter is well defined since we clearly have $\chi(G) \leq gcn(G) \leq |V(G)|$.

The game chromatic number of a graph was first introduced by Bodlaender in [1] and then studied by Bodlaender and Kratsch [2], Faigle, Kern, Kierstead and Trotter [8],

*This work has been partially supported by the Barrande Grant 97-137 and the NATO Collaborative Research Grant 97-1519.

†Department of Applied Mathematics, Charles University, Prague, Czech Republic.

‡LaBRI, Université Bordeaux 1, 345 cours de la Libération, 33405 Talence Cedex, France.

Kierstead and Trotter [10], Dinski and Zhu [6], Guan and Zhu [9] and Zhu [16, 17]. In particular, Kierstead and Trotter proved in [10] that $gcn(G) \leq 33$ for every planar graph G (a constant, but large, upper bound for the game chromatic number of planar graphs, the existence of which was conjectured by Bodlaender, was previously given by Chen and Shelp in [5]). This bound was later improved by Dinski and Zhu [6] who proved that if a graph G has acyclic chromatic number a then $gcn(G) \leq a(a+1)$. (The *acyclic chromatic number* of a graph G is the minimum number of colors in a proper coloring of G such that every cycle in G uses at least three colors.) Using a result of Borodin [3], which states that every planar graph has acyclic chromatic number at most 5, we get $gcn(G) \leq 30$ for every planar graph G. Using a different technique (not based on acyclic colorings), Zhu recently further reduced this bound to 19 [16]. (In this paper, Zhu mentionned that Kierstead slightly improved this technique to get a bound of 18.) On the other hand, it is known that there exist planar graphs with game chromatic number 8. The edge-coloring version of this game was considered by Cai and Zhu [4] and by Lam, Shiu and Xu [11].

In this paper, we are interested in the oriented version of this game. Let $\vec{G}$ be an *oriented* graph, that is a digraph with no opposite arcs. A mapping c from $V(\vec{G})$ to a set of colors X is an *oriented coloring* of $\vec{G}$ if (i) $c(u) \neq c(v)$ for every arc uv and (ii) $c(u) = c(x)$ implies $c(v) \neq c(w)$ for every two arcs uv and wx (v and w not necessarily distinct). Observe that condition (ii) implies in particular that any two vertices linked by a *directed 2-path*, that is a directed path of length 2, must be assigned distinct colors.

The *oriented chromatic number* of $\vec{G}$, denoted by $\vec{\chi}(\vec{G})$, is then defined as the least cardinality of a set X such that $\vec{G}$ has an oriented coloring on X. Raspaud and Sopena proved in [14] that if an undirected graph G has acyclic chromatic number a then every orientation $\vec{G}$ of G has oriented chromatic number at most $a \times 2^{a-1}$. (Thanks to Borodin's result, this gives in particular that $\vec{\chi}(\vec{G}) \leq 80$ for every oriented planar graph $\vec{G}$.)

Chromatic and oriented chromatic numbers can be equivalently defined in terms of graph homomorphisms. A *homomorphism* φ of a graph G to a graph H, denoted $\varphi : G \longrightarrow H$, is an edge-preserving mapping from $V(G)$ to $V(H)$ ($\varphi(u)\varphi(v)$ is an edge in H whenever uv is an edge in G). Such a mapping φ is called a *H-coloring* of G. The chromatic number of an undirected graph G is then the least k such that G admits a homomorphism to the complete graph K_k on k vertices. Similarly, the oriented chromatic number of an oriented graph $\vec{G}$ is the least k such that there exists an *oriented* target graph $\vec{H}$ on k vertices such that $\vec{G}$ admits a $\vec{H}$-coloring (similarly, homomorphisms of oriented graphs are defined as arc-preserving vertex mappings).

Let $\vec{G}$ be an oriented graph and $\vec{H}$ be a *fixed* oriented target graph whose vertex set will be used as the set of colors. We consider the oriented version of the coloring game defined as follows. Again the two players are Alice and Bob and they play alternatively with Alice having the first move. Alice's goal is to provide a $\vec{H}$-coloring of $\vec{G}$ and Bob's goal is to prevent her from doing so. A move consists in choosing an uncolored vertex u and assigning it a color α from the set $V(\vec{H})$ such that:

1. if v is a vertex with color β and uv (resp. vu) is an arc in $\vec{G}$ then $\alpha\beta$ (resp. $\beta\alpha$) is an arc in $\vec{H}$,

2. if w is a vertex with color γ and there exists a directed 2-path linking u and w (in either direction) then $\alpha \neq \gamma$.

In other words, the (partial) oriented coloring of $\vec{G}$ obtained after such a move can be extended to an oriented coloring of the whole graph $\vec{G}$ (but not necessarily to a $\vec{H}$-coloring of $\vec{G}$). Observe that the undirected game of Bodlaender can also be defined in the same way by taking the complete graph K_k as a target graph and dropping condition (2) above (which comes from the fact that in the oriented case the target graph has no opposite arcs).

In this oriented game, Bob's goal is thus to "surround" some uncolored vertex u in such a way that either there exists no color in $\vec{H}$ which is *compatible* with the already colored neighbours of u, or such colors exist but all of them are already assigned to some vertices in $\vec{G}$ which are linked to u by a directed 2-path.

The *oriented game chromatic number* of $\vec{G}$, denoted by $ogcn(\vec{G})$, is then defined as the least k such that there exists an oriented target graph $\vec{H}$ on k vertices for which Alice has a winning strategy. The fact that this number is well-defined is not as immediate as it is in the undirected case and this will be established in the next section. Since every oriented coloring of an oriented graph is indeed a coloring of the corresponding underlying undirected graph, we get that $gcn(G) \leq ogcn(\vec{G})$ for every orientation $\vec{G}$ of every undirected graph G.

Our main interest is in characterizing families of oriented graphs having bounded oriented game chromatic number. In the next section, we introduce two types of "Go-like" games (by analogy to the Go board-game) and show how these games may help for proving that families of graphs have bounded (oriented or not) game chromatic number. Using that, we prove in Section 3 that the families of oriented paths, oriented cycles or oriented trees have bounded oriented game chromatic number and in Section 4 that the family of oriented outerplanar graphs has bounded oriented game chromatic number. Finally, we discuss in Section 5 another type of oriented coloring game and state some open problems.

2 Go-type games and coloring games

We introduce in this section two new games for undirected graphs and show how these games are related to the coloring games discussed in the previous section. For these games, we shall play with tokens instead of colors (or, equivalently, with only one color) and speak about *marked* or *umarked* vertices.

Let G be an undirected graph whose vertices are either marked or unmarked. We set $V(G) = M(G) \cup U(G)$ where $M(G)$ and $U(G)$ respectively denote the set of marked vertices and the set of unmarked vertices. For every unmarked vertex $u \in U(G)$, we denote by $N_M(u)$ the set of marked neighbours of u and by $s(u)$ the cardinality of $N_M(u)$, called the *score* of u. The score of the graph G, denoted by $s(G)$, is then defined as

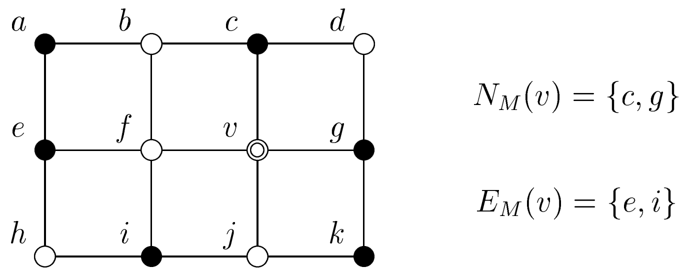

Figure 1: A vertex v with extended score $s^*(v) = 2 + 2 = 4$

$s(G) = Max\{s(u) : u \in U(G)\}$ if $U(G)$ is non-empty and $s(G) = 0$ otherwise.

Let now G be an undirected graph on n vertices all of whose vertices are unmarked ($U(G) = V(G)$) and k be a positive integer. The two-players games *Go(k)* is defined as follows. Alice and Bob play alternatively with Alice having the first move. A move consists in marking any unmarked vertex. The game thus finishes after n moves, when all vertices are marked. Alice wins the game if after each move (of any player) the score of the current graph is at most k.

The *Go-number* of G, denoted by $Go(G)$, is then defined as the least k such that Alice has a winning strategy in the game $Go(k)$. We clearly have $Go(G) \leq \Delta(G)$ for every graph G, where $\Delta(G)$ stands for the maximum degree of G.

It is easy to see that any winning strategy for playing the game $Go(k)$ on an undirected graph G induces a winning strategy for playing the coloring game on G with $k+1$ colors: considering colored vertices as marked, if every uncolored vertex has at most k colored neighbours after each move then every uncolored vertex is colorable provided that we have at least $k+1$ colors available. We have proved the following

Theorem 1 *For every undirected graph G, $gcn(G) \leq Go(G) + 1$.*

The Go-number of a graph was already considered in [16] where it was called the *game coloring number*.

In order to get a similar result for the oriented version of the coloring game, we now introduce a new kind of Go-type game, namely the *extended Go(k)-game*, or *eGo(k)-game* for short. This new game only differs from the previous one when computing the score.

Let G be an undirected graph with $V(G) = M(G) \cup U(G)$. For every unmarked vertex $u \in U(G)$, we denote by $N_M(u)$ the set of marked neighbours of u (as before) and by $E_M(u)$ the set of marked vertices which are at distance 2 from u and which are not linked to u by a 2-path whose middle vertex is marked. (In the following, vertices of $E_M(u)$ will be referred to as *marked 2-neighbours* of u.) The *extended score* of u, denoted $s^*(u)$, is defined as the sum $s^*(u) = |N_M(u)| + |E_M(u)|$ (see Figure 1, where marked vertices are drawn as black vertices). The extended score of the graph G, denoted by $s^*(G)$, is then defined as $s^*(G) = Max\{s^*(u) : u \in U(G)\}$ if $U(G)$ is non-empty and $s^*(G) = 0$ otherwise.

The two-players games *eGo(k)* is defined as follows. Alice and Bob play alternatively with Alice having the first move. A move consists in marking any unmarked vertex. The game thus finishes after n moves, when all vertices are marked. Alice wins the game if after each move (of any player) the extended score of the current graph is at most k.

The *extended Go-number* (or *eGo-number* for short) of G, denoted by $eGo(G)$, is then defined as the least k such that Alice has a winning strategy in the game $eGo(k)$. We clearly have $go(G) \leq \Delta(G) + \Delta(G)(\Delta(G) - 1) = \Delta^2(G)$.

We now introduce a special type of tournaments that we shall use as target graphs for defining a winning strategy for Alice when playing the oriented coloring game on graphs with bounded eGo-number.

Let us call *orientation vector of length* k a k-tuple $\nu = (\nu_1, \nu_2, \dots, \nu_k) \in \{0,1\}^k$. If $\vec{G}$ is an oriented graph and $S = (u_1, u_2, \dots, u_k)$ a sequence of k distinct vertices of $V(\vec{G})$, we say that a vertex $v \in V(\vec{G})$ is a *ν-successor* of S if, for every i, $1 \leq i \leq k$, $u_i v \in E(\vec{G})$ if $\nu_i = 1$ and $v u_i \in E(\vec{G})$ otherwise.

Definition 2 (Property $P(k)$) We say that a tournament $\vec{T}$ satisfies the property $P(k)$ if (i) $\vec{T}$ has at least k vertices and (ii) for every sequence $S = (t_1, t_2, \dots, t_k)$ of k distinct vertices in $V(\vec{T})$ and every orientation vector ν of length k, the sequence S has a ν-successor in $\vec{T}$.

The following observation will be later useful:

Observation 3 *If $\vec{T}$ is a tournament satisfying the property $P(k)$ then for every sequence $S = (t_1, t_2, \dots, t_{k'})$ of $k' \leq k$ distinct vertices in $V(\vec{T})$ and every orientation vector ν of length k', the sequence S has at least $2^{k-k'}$ ν-successors in $\vec{T}$.*

This is obvious since any orientation vector of length k' can be extended to $2^{k-k'}$ disctinct orientation vectors of length k.

Schütte and Erdös (see [7] or [12], pp. 28–31) considered a similar property of tournaments, the so-called property $S(k)$, by only requiring the existence of a ν-successor for the orientation vector $\nu = (1, 1, \dots, 1)$. Erdös proved in [7] that for every k, there exist finite tournaments satisfying the property $S(k)$. In fact, by slightly modifying Erdös's proof, we get the following

Theorem 4 *For every $k > 0$, there exists a finite tournament T_k satisfying the property $P(k)$.*

Proof. Let n be such that $2^k \binom{n}{k} (1 - 2^{-k})^{n-k} < 1$ and consider a random tournament on the set $V = \{1, 2, \dots, n\}$. (By a random tournament we mean here a tournament for which the direction of each arc is determined by the flip of a fair coin.) For every subset S of k distinct vertices and every orientation vector ν of length k, let $A_{S,\nu}$ be the event that there is no ν-successor of S (S is considered here as the sequence obtained by taking the vertices in increasing order).

Since for each fixed vertex $v \in V \setminus S$, the probability that v is not a ν-successor of S is $1 - 2^{-k}$ and all these $n - k$ events corresponding to the possible choices of v are independent, we get $Pr(A_{S,\nu}) = (1 - 2^{-k})^{n-k}$.

It follows that

$$Pr\Big(\cup_{S,\nu} A_{S,\nu}\Big) \leq \sum_{S,\nu} Pr(A_{S,\nu}) = 2^k \binom{n}{k} (1 - 2^{-k})^{n-k} < 1.$$

Therefore, with positive probability, there is a tournament on n vertices which satisfies the property $P(k)$. ∎

For small values of k, optimal tournaments are known. Let q be a prime number such that $q \equiv 3 \pmod 4$; the tournament $\vec{QR}_q$ is then defined by $V(\vec{QR}_q) = \{0, 1, \ldots, q-1\}$ and for every $u, v \in V(\vec{QR}_q)$, $uv \in E(\vec{QR}_q)$ if and only if $v - u \pmod q$ is a non-zero quadratic residue of q. It can be checked that the tournament $\vec{QR}_3$, that is the directed cycle on three vertices, satisfies the property $P(1)$, that the tournament $\vec{QR}_7$ satisfies the property $P(2)$, that the tournament $\vec{QR}_{19}$ satisfies the property $P(3)$ and that these three solutions are optimal (see [15]).

We are now able to prove the following

Theorem 5 *If G is a graph with $eGo(G) \leq k$ then there exists a constant $c(k)$ such that $ogcn(\vec{G}) \leq c(k)$ for every orientation $\vec{G}$ of G.*

Proof. Let $\vec{G}$ be any orientation of G. We know that Alice has a winning strategy for the game $eGo(k)$ on G. Let $\vec{T}$ be a finite tournament satisfying the property $P(k)$ (we know by Theorem 4 that such a tournament exists) and let $c(k)$ denote the order of $\vec{T}$. We claim that if Alice uses the same strategy (for choosing the vertex to be colored) when playing the oriented coloring game on $\vec{G}$ with $\vec{T}$ as a target graph then the whole graph can be $\vec{T}$-colored and Alice wins the game. Since $\vec{T}$ has $c(k)$ vertices, this will prove the desired result.

To see that, recall that playing the $eGo(k)$-game strategy ensures that after each move every uncolored vertex v has at most k_1 colored neighbours and k_2 colored 2-neighbours with $k_1 + k_2 \leq k$. The property $P(k)$ then ensures that such a vertex v can always be $\vec{T}$-colored: if $c_1, c_2, \ldots, c_{k_1}$ denote the colors of the k_1 colored neighbours of v, we know that there exist $2^{k-k_1} > k_2$ colors in $\vec{T}$ available for v according to the direction of the arcs linking v to its k_1 colored neighbours. Since v has at most k_2 colored 2-neighbours, there is still at least one color available for v. ∎

Since for every graph G we have $eGo(G) \leq \Delta^2(G)$, we get that the oriented game chromatic number of every graph G is bounded by some constant only depending on

$\Delta(G)$. The oriented game chromatic number is thus a well-defined parameter. However, compare this with a far less obvious problem discussed in the last section.

3 Paths, cycles and trees

We study in this section the oriented game chromatic number of oriented paths, oriented cycles and oriented trees.

Since every vertex in a path or a cycle has degree at most 2 the eGo-number of paths or cycles is obviously at most 2. Using the tournament $\vec{QR}_7$ introduced in the previous section, which satisfies the property $P(2)$, we get:

Theorem 6 *If $\vec{G}$ is an oriented path or an oriented cycle, then $ogcn(\vec{G}) \leq 7$.*

In fact, this bound is tight, as shown by the following:

Theorem 7 *There exist oriented paths with oriented game chromatic number 7.*

Proof. The idea is to prove that for a specific path, Alice cannot win the game if she uses a target graph which does not satisfy the property $P(2)$. Since no graph having less than 7 vertices satisfies this property, we get the result.

Therefore, suppose that the game is played with a target graph $\vec{H}$ which does not satisfies the property $P(2)$. It means that there exist two vertices in $\vec{H}$, say a and b, and an orientation vector ν of size 2, such that the sequence (a, b) has no ν-successor in $\vec{H}$. We claim that there exists an oriented path $\vec{P}$ such that Bob has a winning strategy when playing the oriented coloring game on $\vec{P}$.

Let $\vec{P} = x_1y_1z_1t_1u_1vx_2y_2z_2t_2u_2$ be the path on 10 vertices oriented in such a way that y_i is a ν-successor in $\vec{P}$ of (z_i, x_i) and t_i is a ν-successor in $\vec{P}$ of (z_i, u_i) for every $i \in \{1, 2\}$. Bob's strategy is the following: after Alice's first move, Bob chooses the vertex z_i, $i \in \{1, 2\}$, such that none of x_i, y_i, z_i, t_i, u_i have been colored, and colors it with a. On his second turn, Bob chooses either the vertex x_i, if none of x_i, y_i have been colored, or the vertex u_i otherwise, and colors it with b. Clearly, the path $\vec{P}$ can no longer be $\vec{H}$-colored. ∎

Faigle, Kern, Kierstead and Trotter proved in [8] that the game chromatic number of every tree is at most 4 (Bodlaender gave in [1] examples of trees with game chromatic number 4). In fact they implicitely deal with the Go-number of trees. Since the proof is short we reproduce it below.

Theorem 8 (Faigle, Kern, Kierstead and Trotter, 1993) *If T is an undirected tree then $Go(T) \leq 3$.*

Proof. We shall give a winning strategy for Alice when playing the game $Go(3)$. During the whole game, Alice will ensure that every subtree T' of T only containing unmarked vertices is adjacent to at most two marked vertices. On her first move, Alice marks any vertex and this condition obviously holds. Suppose now it holds before a move of Bob. After Bob's move, at most one unmarked subtree is adjacent to three marked vertices. Alice then marks the unique vertex which lies on the three paths linking pairs of these marked vertices. After this move, again the condition holds. ∎

In fact, the strategy used in this proof even ensures that the extended score of every intermediate tree is at most 3 and we get:

Theorem 9 *If T is an undirected tree then $eGo(T) \leq 3$.*

From Theorems 5 and 9, we know that the oriented game chromatic number of any oriented tree is bounded by the order of any tournament satisfying the property $P(3)$. Using the tournament $\vec{QR}_{19}$ introduced in the previous section, we get:

Theorem 10 *If $\vec{T}$ is an oriented tree, then $ogcn(\vec{T}) \leq 19$.*

4 Outerplanar graphs

It is not difficult to observe that every outerplanar graph has acyclic chromatic number at most 3 (see e.g. [15]). Using the result of Dinski and Zhu [6] we then get that the game chromatic number of every outerplanar graph is at most 12. In [9], Guan and Zhu reduced this bound to 7. Kierstead and Trotter exhibited in [10] an outerplanar graph with game chromatic number 6. The question whether there exist outerplanar graphs with game chromatic number 7 or not is still an open question.

We shall prove in this section that the oriented game chromatic number of oriented outerplanar graphs is also bounded. Considering the eGo-game on outerplanar graphs we have the following:

Theorem 11 *If G is an undirected outerplanar graph then $eGo(G) \leq 27$.*

Proof. Since any winning strategy on a graph is also a winning strategy on each of its subgraphs we may assume without loss of generality that the graph G is maximal outerplanar. In that case, each face of G except the infinite face is a triangle. We first discuss some structural properties of G.

We consider a linear ordering $v_1, v_2, \ldots, v_n$ of the vertices of G such that v_1v_2 is an edge and every vertex v_i, $i > 2$, has exactly two neighbours v_{i_1} and v_{i_2} such that $i_1 < i_2 < i$. (Such an ordering exists since G is a 2-tree.) For simplicity, we shall write $x < y$ if x and y are two vertices with $x = v_i$, $y = v_j$ and $i < j$.

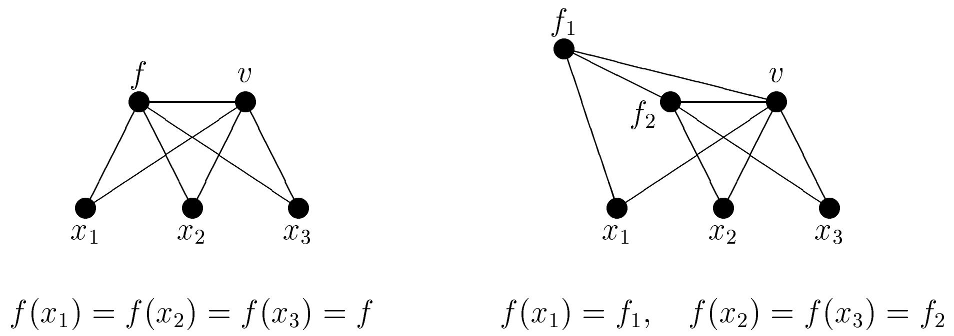

Figure 2: Every vertex v has at most 2 nephews

For every vertex $v \notin \{v_1, v_2\}$, the *father* of v, denoted by $f(v)$, is the smallest neighbour of v and the *uncle* of v, denoted by $u(v)$, is the smallest neighbour of v distinct from $f(v)$ (we thus have $f(v) < u(v) < v$ and v has no other smaller neighbour). Moreover, we say that v_1 is the father of v_2 and that v is a *nephew* of $u(v)$. Since G is maximal outerplanar, we have:

Claim 1. *For every vertex $v \notin \{v_1, v_2\}$, $f(v)u(v)$ is an edge in G.*

Considering the number of nephews that a vertex can have, we have:

Claim 2. *Every vertex v has at most 2 distinct nephews.*

To see that, suppose that v has 3 distinct nephews, say x_1, x_2 and x_3. For every i, $1 \leq i \leq 3$, $f(x_i) < v$ and $f(x_i)v$ is an edge. The set $\{f(x_1), f(x_2), f(x_3)\}$ has thus cardinality 1 or 2. Therefore, G contains $K_{2,3}$ or a subdivision of $K_{2,3}$ as a subgraph, which contradicts the fact that G is outerplanar (see Figure 2).

We call a *strong edge* an edge linking a vertex and its father. An edge which is not strong is a *weak edge*. Then we have:

Claim 3. *The subgraph $T(G)$ induced by the strong edges is a spanning tree of G.*

To see that, observe that $f(v)$ is defined for every vertex $v \neq v_1$ and is such that $f(v) < v$. Therefore, the graph $T(G)$ has $n - 1$ edges and is acyclic.

Suppose now that Alice plays the eGo(27)-game on G as if she were playing on the subgraph $T(G)$, using the strategy defined in the proof of Theorem 8. We claim that this strategy is a winning strategy for the eGo(27)-game on G.

At each step of the game, each unmarked vertex v has at most 3 marked neighbours in $T(G)$ and, by Claim 2, at most 3 marked neighbours not in $T(G)$ (at most one uncle and two nephews). Therefore, the score $s(v)$ of v is at most 6.

Let us now consider the marked 2-neighbours of v. The vertex v has at most one father, one uncle and two nephews, each of them having at most 6 marked neighbours. In order to prove that the number of marked 2-neighbours of v is bounded, we thus only have to consider the marked neighbours of the unmarked sons of v. Due to Alice's strategy, we already know that, alltogether, the unmarked sons of v have at most 3 marked neighbours in $T(G)$. Therefore, we need to bound the number of marked *weak* neighbours of the unmarked sons of v (by a weak neighbour we mean a neighbour linked by a weak edge).

Let s be any unmarked son of v. The uncle $u(s)$ of s has already been considered since it is a neighbour of v. If t is a nephew of s with $f(t) \neq v$, then $f(t) < s$ and thus $f(t) = u(s)$ and $f(t)$ is a neighbour of v. If $f(t)$ is a son of v, then t has already been considered as a 2-neighbour of v in $T(G)$. If $f(t)$ is a nephew of v, then t has also been already considered. Therefore, no unmarked son of v can provide a new unmarked 2-neighbour of v.

Finally, each unmarked vertex v has at most 3 marked neighbours or 2-neighbours in $T(G)$, one father and at most 3 weak neighbours, each of them having at most 6 marked neighbours. The extended score $s^*(v)$ of v is thus at most $3 + 4 \times 6 = 27$. This finishes the proof. ∎

Theorems 5 and 11 lead to the following

Corollary 12 *There exists a constant $t > 0$ such that $ogcn(\vec{G}) \leq t$ for every oriented outerplanar graph $\vec{G}$.*

5 Discussion and open problems

In this paper, we have shown how to use the notion of the eGo-number of a graph in order to derive a bound for the oriented game chromatic number of any of its orientations. In the undirected case, a family of graphs with bounded Go-number has bounded game chromatic number. However, having bounded Go-number is not a necessary condition for a family of graphs for having bounded game chromatic number. For instance, it is easy to see that any bipartite complete graph $K_{n,n}$, $n > 1$, has game chromatic number 3 while its Go-number is n. In the oriented case, it is not so clear whether having bounded eGo-number is a necessary condition for having bounded oriented game chromatic number or not (because all possible orientations of graphs are considered).

It seems not so easy to prove (or disprove) that the eGo-number of planar graphs or of k-trees is bounded. We thus propose the two following problems:

Problem 1 *Does there exist a constant t such that every oriented planar graph has oriented game chromatic number at most t?*

Problem 2 *Let $k > 1$ be an integer. Does there exist a constant t_k such that every oriented k-tree has oriented game chromatic number at most t?*

In our definition of the oriented version of the coloring game, we have used a given oriented graph $\vec{H}$ as a target graph (in other words, Alice's goal is to build a $\vec{H}$-coloring of the graph on which the game is played). Another possibility is to define the oriented coloring game by giving the maximum number of colours that can be used but without choosing a fixed target graph. In that case, the target is built while the game is played, each move having to be "compatible" with the target under construction.

In the undirected case, the coloring game corresponds to this second version. However, fixing the maximum number k of colours that can be used is clearly equivalent to playing with the complete graph K_k as a target graph. The situation is much less clear in the oriented case and we do not know whether families of graphs with bounded oriented game chromatic number (with fixed target) and families of graphs with bounded oriented game chromatic number (with fixed maximum number of colours) coincide or not.

References

[1] H.L. Boadlaender, On the complexity of some coloring games, *Internat. J. Found. Comput. Sci.* **2** (1991), 133–147.

[2] H.L. Bodlaender and D. Kratsch, The complexity of coloring games on perfect graphs, *Theoret. Comput. Sci.* **106** (1992), 309–326.

[3] O. Borodin, On acyclic colorings of planar graphs, *Discrete Math.* **25** (1979), 211-236.

[4] L. Cai and X. Zhu, Game chromatic index of k-degenerate graphs, manuscript (1998).

[5] G. Chen and R.H. Shelp, Graphs with linearly bounded Ramsey numbers, *J. Combin. Theory Ser. B* **57-1** (1993), 138–149.

[6] T. Dinski and X. Zhu, Game chromatic number of graphs, *Discrete Math.* **196** (1999), 109–115.

[7] P. Erdös, On a problem in graph theory, *Math. Gaz.* **47** (1963), 220–223.

[8] U. Faigle, U. Kern, H.A. Kierstead and W.T. Trotter, On the game chromatic number of some classes of graphs, *Ars Combinatoria* **35** (1993), 143–150.

[9] D. Guan and X. Zhu, The game chromatic number of outerplanar graphs, *J. Graph Theory* **30** (1999), 67–70.

[10] H.A. Kierstead and W.T. Trotter, Planar graph coloring with an uncooperative partner, *J. Graph Theory* **18-6** (1994), 569–584.

[11] P.C.B. Lam, W.C. Shiu and B. Xu, Edge game coloring of graphs, *preprint* (1999).

[12] J.W. Moon, *Topics on tournaments*, Holt, Rinehart and Winston, New York (1968).

[13] J. Nešetřil and E. Sopena, On four coloring problems, *KAM-DIMATIA Series* **99-418**, Charles University, Prague.

[14] A. Raspaud and E. Sopena, Good and semi-strong colorings of oriented planar graphs, *Inform. Processing Letters* **51** (1994), 171–174.

[15] E. Sopena, On the chromatic number of oriented graphs, *J. Graph Theory* **25** (1997), 191–205.

[16] X. Zhu, The game coloring number of planar graphs, *J. Combin. Theory Ser. B* **75** (1999), 245–258.

[17] X. Zhu, The game coloring number of pseudo partial k-trees, manuscript (1999).