On directional convexity*

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Abstract

Motivated by problems from calculus of variations and partial differential equations, we investigate geometric properties of D-convexity. A function $f: \mathbf{R}^d \to \mathbf{R}$ is called *D-convex*, where D is a set of vectors in $\mathbf{R}^d$, if its restriction to each line parallel to a nonzero $v \in D$ is convex. The *D-convex hull* of a compact set $A \subset \mathbf{R}^d$, denoted by $\mathrm{co}^D(A)$, is the intersection of the zero sets of all nonnegative D-convex functions that are 0 on A. We provide a little more explicit description of $\mathrm{co}^D(A)$, namely as the zero set of the D-convex envelope of the distance function of A. We give an example of an n-point set $A \subset \mathbf{R}^2$ where the D-convex envelope of the distance function is exponentially close to 0 at points lying relatively far from $\mathrm{co}^D(A)$, showing that the definition of the D-convex hull can be very non-robust. For separate convexity in $\mathbf{R}^3$ (where D is the orthonormal basis of $\mathbf{R}^3$), we construct arbitrarily large finite sets A with $\mathrm{co}^D(A) \neq A$ whose all proper subsets are equal to their D-convex hull. This implies the existence of analogous sets for rank-one convexity and for quasiconvexity on 3×3 (or larger) matrices.

1 Introduction

Let X be a finite-dimensional real vector space (which can be identified with some $\mathbf{R}^d$), and let $D \subseteq X$ be a set of vectors, which are thought of as directions. A function $f: X \to \mathbf{R}$ is called *D-convex* if the restriction of f to each line parallel to a nonzero vector in D is a convex function. The *D-convex hull* of a compact set $A \subset X$, denoted by $\mathrm{co}^D(A)$, is defined as the intersection of the zero sets of all nonnegative D-convex functions $f: X \to [0, \infty)$ that are 0 on A. (Later in Section 3, we give a more direct characterization of the D-convex hull. Also, let us remark that this D-convex hull is called the *functional D-convex hull* in [MP98], in order to distinguish it from the *set-theoretical D-convex hull*. The latter will not be considered in the present paper.)

The "usual" notion of convexity is obtained for $D = X$. Our investigation is mainly motivated by *rank-one convexity*, which is a special case of D-convexity, where X is the space of real $n \times n$ matrices and D is the set of $n \times n$ matrices of rank 1. In the sequel, this D will be denoted by rc. Rank-one convex hull,

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as an inner approximation to the so-called *quasiconvex hull*, is important in the theory of partial differential equations and in calculus of variations and it was studied in a number of papers, among which we mention only a few: [Mor52], [Šve92], [BFJK94], [DKMŠ99], [MŠ99a], [MŠ99b]. The lecture notes [Mül98] can serve as a nice and up-to-date introduction to this area.

Another significant special case of D-convexity is that with D being the standard orthonormal basis of $\mathbf{R}^d$: the *separate convexity* (this D will be denoted by sc). This arises by restricting the rank-one convexity on the subspace of diagonal matrices, and has been considered in this connection [Tar93], but it seems natural and interesting in its own right and was independently studied e.g. in probability theory [AH86].

New results. In the first part of this paper, we concentrate on separate convexity in $\mathbf{R}^d$. For the usual convexity, well-known *Carathéodory's theorem* holds: if $A \subseteq \mathbf{R}^d$ and x lies in the convex hull of A, then x is in the convex hull of some at most $(d+1)$-point subset of A; we say that the *Carathéodory number* for convexity in $\mathbf{R}^d$ is $d+1$. In [MP98], it was proved that the Carathéodory number for separate convexity in $\mathbf{R}^2$ is 5. Here we show that the Carathéodory number for separate convexity in dimensions 3 and higher is infinite. As a consequence, the Carathéodory number for rank-one convexity and for quasiconvexity on 3×3 matrices is infinite as well. We also report other results concerning minimal nontrivial configurations for separate convexity, and mention outcomes of computer experiments with separate convexity [Let99].

In Section 3, we give a somewhat more direct description of the D-convex hull of a set A. While the usual definition takes into account all nonnegative D-convex functions vanishing on A, we show that $\mathrm{co}^D(A)$ is actually the zero set of the D-convex envelope of the distance function of A. (The D-convex envelope of a function $f\colon X \to \mathbf{R}$, denoted by $C_D f$, is defined as the pointwise supremum of all D-convex functions g satisfying $g \leq f$ on X.)

This suggests an algorithmic approach to computing the D-convex hull, via D-convexification of the distance function. In Section 4, we show that this approach may be quite problematic in some cases: we exhibit an n-point set $A \subset \mathbf{R}^2$ and a point x lying relatively far from $\mathrm{co}^{sc}(A)$ but such that the separately convex envelope of the distance function has value exponentially close to 0 at x. Computational experiments indicate bad behavior in this respect (although not as drastic as in the example just mentioned) even for random n-point subsets A of the $n \times n \times n$ grid in $\mathbf{R}^3$.

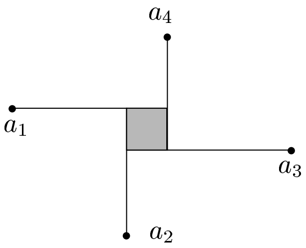
In the remaining sections, we establish some general properties of D-convexity. In Section 5 we show that the D-convex envelope of a 1-Lipschitz function is again 1-Lipschitz; we present an argument due to Kirchheim, which is similar to our original proof but simpler. Let us remark that Kirchheim, Kristensen, and Ball [KKB99] recently proved strong results, somewhat related to the Lipschitz condition, concerning the differentiability of D-convex envelopes (as well as quasiconvex envelopes) of differentiable functions. Essentially, they show that if a differentiable function satisfies suitable growth conditions at the infinity, then the envelopes are differentiable as well, while the differentiability may fail without the growth condition.

In [MP98], a result on local behavior of the D-convex hull was stated (Corollary 2.9), but, as was pointed out by Kirchheim, it was not sufficiently substantiated. We prove it in Section 6; an independent proof of a somewhat stronger result was recently given by Kirchheim [Kir99].

2 New configurations for separate convexity

Let us call a (finite) set $A \subset X$ *nontrivial* (for some fixed D) if $\mathrm{co}^D(A) \neq A$, and *trivial* otherwise. One simple reason for the nontriviality of A is that A contains two points x, y such that the vector $x - y$ is parallel to a direction in D (we say that *A has a D-connection*); for separate convexity in $\mathbf{R}^d$, this means that x and y share $d - 1$ coordinates.

As was independently discovered by several authors ([Sch74], [AH86], [Tar93], [CT93]), a set can be nontrivial without possessing a D-connection. In the plane, the four-point configuration $T_4 = \{a_1, a_2, a_3, a_4\}$ below has separately convex hull as indicated in the picture (the shaded square and four segments):



Note that this nontrivial configuration is *generic nontrivial*, meaning that any sufficiently small perturbation of its points again gives a nontrivial set. For separate convexity in $\mathbf{R}^d$, a nontrivial configuration in which no two points share the value of any coordinate is necessarily generic (because the combinatorial structure of the separately convex hull is determined by the orderings of the points along the coordinate axes; see Section 2.1 below).

For separate convexity in $\mathbf{R}^2$, the situation is relatively simple: any nontrivial set without a *sc*-connection contains a copy of the configuration T_4 or of its mirror reflection [MP98]. Moreover, as was mentioned in the introduction, the Carathéodory number is 5, meaning that any point in the separately convex hull of A is in the hull of some at most 5 points of A. As we will see in this section, there is no such simple description of nontrivial configurations for separate convexity in higher dimensions. Inclusion-minimal generic nontrivial configuration can be arbitrarily large (and, consequently, the Carathéodory number is infinite), and the number of small configurations is astronomic.

In Sections 2.1 through 2.4, we will always consider separate convexity unless explicitly stated otherwise.

2.1 Preliminaries

Let $B \subseteq \mathbf{R}^d$ be a set. A point $x \in B$ is called *sc-extremal in B* if x is contained in no open segment $s \subseteq B$ parallel to one of the coordinate axes.

For reader's convenience, we briefly review an algorithmic description of the separately convex hull of a finite set $A \subset \mathbf{R}^d$ derived in [MP98]. For $i = 1, 2, \ldots, d$, let $x_i(A) = \{x_i(a) : a \in A\}$, where $x_i(a)$ denotes the ith coordinate of a, and let $\mathrm{grid}(A) = x_1(A) \times x_2(A) \times \cdots \times x_d(A)$. For a point $x \in G = \mathrm{grid}(A)$, let a^{i+} (resp. a^{i-}) denote the point of G whose all coordinates but the ith coincide with those of a, and whose ith coordinate is the successor (resp. predecessor) of $x_i(a)$ in $x_i(G)$ (thus, a^{i+} or a^{i-} need not exist for "border" points of G). Let $B \subseteq G$ and $a \in B$; we call a point $a \in \mathrm{grid}(A)$ *grid-extremal* in B if, for each $i = 1, 2, \ldots, d$, at least one of a^{i+}, a^{i-} either does not exist or does not belong to B; intuitively, a is a "local corner" of B.

Given a finite $A \subset \mathbf{R}^d$, put $B_0 = \mathrm{grid}(A)$, and for $j = 0, 1, 2, \ldots$, if B_j contains a grid-extremal point $b \notin A$, set $B_{j+1} = B_j \setminus \{b\}$ and continue with the next j. This procedure terminates with a set B_{j_0} with all grid-extremal points lying in A, and this set describes the separately convex hull of A. Namely, let an *elementary box* for $\mathrm{grid}(A)$ be a Cartesian product of the form $I_1 \times I_2 \times \ldots \times I_d$, where each I_i is either $\{x_i\}$ for some $x_i \in x_i(A)$ or $[x_i(a), x_i(a^{i+})]$ for an $a \in \mathrm{grid}(A)$. The *box complex* of $B \subseteq \mathrm{grid}(A)$ consists of the elementary boxes whose all corners lie in B. Then, as shown in [MP98], $\mathrm{co}^{sc}(A)$ is the union of the box complex of the set B_{j_0} obtained by the above algorithm.

As a consequence of this algorithmic description, we get that if $B = \mathrm{co}^{sc}(A)$ for A finite, then all *sc*-extremal points of B belong to A and B is the separately convex hull of its extremal points.

2.2 Generic nontrivial configurations in all dimensions

The existence of generic nontrivial configurations for separate convexity in $\mathbf{R}^3$ was established in [MP98]; a 20-point configuration was exhibited. Its nontriviality was verified by applying the above algorithm. Here we generalize the idea of that construction and we present a systematic inductive construction in any dimension.

Theorem 2.1 *For any $d \geq 2$, there exists a finite generic nontrivial configuration A_d for separate convexity in $\mathbf{R}^d$.*

Proof. We proceed by induction on the dimension d. We need a slightly stronger statement; to state the additional condition, we use the following definition. Let $a \in A$ be an *sc*-extremal point of $\mathrm{co}^{sc}(A)$, and let $u \in \{e_1, -e_1, e_2, -e_2, \ldots, e_d, -e_d\}$ be a direction of some coordinate semiaxis. We call u an *inward direction at a* if an open neighborhood U of a exists such that for any $x \in U$, the ray $\{x + tu : t \geq 0\}$ intersects $\mathrm{co}^{sc}(A)$.

Our inductive hypothesis is the claim of the theorem with the additional conditions that the origin 0 lie in the interior of $\mathrm{co}^{sc}(A_d)$ and that every point $a \in A_d$ have an inward direction. The basis of the induction is provided by the configuration T_4 in the plane.

Suppose that the claim has been proved for d, and we want to construct the configuration $A_{d+1} \subset \mathbf{R}^{d+1}$. Let us refer to the direction of the x_{d+1}-axis as "vertical", and to hyperplanes perpendicular to the x_{d+1}-axis as "horizontal".

As a first step, we place copies of A_d into the horizontal hyperplanes $x_{d+1} = 1$ and $x_{d+1} = -1$, and we perturb the points of each copy vertically. Formally, for $a = (a_1, a_2, \ldots, a_d) \in A_d$, we put $a_+ = (a_1, a_2, \ldots, a_d, 1 + z_a)$ and $a_- = (a_1, a_2, \ldots, a_d, -1 - z_a)$, where the z_a's are pairwise distinct positive real numbers, and we set $A_+ = \{a_+ : a \in A_d\}$ and similarly for A_-.

Next, let us choose pairwise distinct numbers $t_a \in (-1, 1)$ for $a \in A_d$. Let $\varepsilon > 0$ be a sufficiently small parameter. Let $P_a = ([0, \varepsilon]^d + a) \times \{t_a\}$ be a small horizontal "plate" lying at height t_a with corner on the vertical segment connecting a^+ and a^- (see Fig. 1). Moreover, let $A'_+ = A_+ + (\varepsilon, \varepsilon, \ldots, \varepsilon, 0)$ be a horizontal translate of A_+, and put

$$B = A'_+ \cup A_- \cup \bigcup_{a \in A_d} P_a .$$

We observe that for any horizontal hyperplane $h = \{x_{d+1} = z\}$ with $-1 \leq z \leq 1$, $h \cap \mathrm{co}^{sc}(B)$ contains an approximate copy $A(h)$ of A_d with each point perturbed by at most ε. By the stability of the combinatorial structure of $\mathrm{co}^{sc}(A_d)$ under sufficiently small perturbations, we see that the combinatorial structure of $\mathrm{co}^{sc}(A(h))$ is the same for all h (if ε is sufficiently small), and an inward direction u_a at a point $a \in A_d$ remains an inward direction for the corresponding point in each $A(h)$ (u_a is a direction in $\mathbf{R}^d$, but from now on we interpret it in $\mathbf{R}^{d+1}$ by appending a zero x_{d+1}-coordinate).

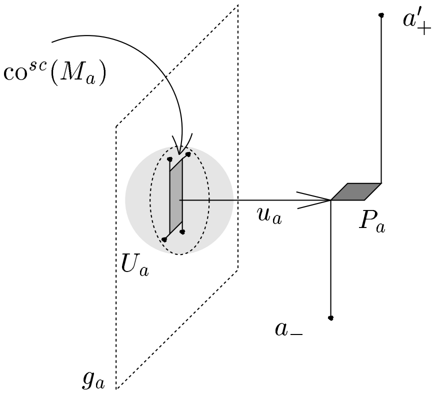


Figure 1: Illustration to the construction of A_{d+1}.

For each $a \in A_d$, let $c_a = (a_1, a_2, \ldots, a_d, t_a) - u_a$ be the point reached from the corner of the horizontal plate P_a by going one unit against the direction u_a (in the horizontal hyperplane $x_{d+1} = t_a$). Previous considerations show that there exists an open ball U_a centered at c_a, whose radius is *independent of* ε (provided that $\varepsilon > 0$ is sufficiently small), such that all rays $\{x + tu_a : t \geq 0\}$ for $x \in U_a$ intersect $\mathrm{co}^{sc}(B)$.

Let g_a be the hyperplane perpendicular to u_a and containing the point c_a. Identify $\mathbf{R}^d$ with g_a so that 0 is placed into c_a and axes directions in $\mathbf{R}^d$ remain parallel to axes directions in $\mathbf{R}^{d+1}$, and let M_a be a copy of A_d in g_a scaled by the factor of $\sqrt{\varepsilon}$. In this way, we may assume that M_a is contained in the ball U_a and, moreover, that the union of rays emanating from points of $\mathrm{co}^{sc}(M_a)$ in the direction u_a contains the whole plate P_a in its interior. As a final step of the construction, shift each point $b \in M_a$ by $-y_b u_a$, where the y_b's are pairwise distinct positive real numbers. This yields a set $\tilde{M}_a$.

Set

$$C = B \cup \bigcup_{a \in A_d} \tilde{M}_a \,,$$

and define A_{d+1} as the set of all sc-extremal points of $\mathrm{co}^{sc}(C)$. By the remark at the end of Section 2.1, we have $\mathrm{co}^{sc}(C) = \mathrm{co}^{sc}(A_{d+1})$, and A_{d+1} consists of points of $A_- \cup A'_+ \cup \bigcup_{a \in A_d} \tilde{M}_a$ plus possibly of sc-extremal points of the P_a's. But since $\mathrm{co}^{sc}(C)$ contains each P_a in its interior, we get $A_{d+1} \subseteq A_- \cup A'_+ \cup \bigcup_{a \in A_d} \tilde{M}_a$. So A_{d+1} is finite, nontrivial with 0 lying in the interior of $\mathrm{co}^{sc}(A_{d+1})$, and it is easily checked from the construction that no two of its points share a common coordinate hyperplane, and hence A_{d+1} is also generic. Finally, for the points of A_- and A'_+, inward directions are $(0, 0, \ldots, 0, 1)$ and $(0, 0, \ldots, 0, -1)$, respectively (because an open neighborhood of each P_a is contained in $\mathrm{co}^{sc}(A_{d+1})$), and for each $b \in \tilde{M}_a$, u_a can be chosen as an inward direction. This finishes the proof of Theorem 2.1. □

2.3 Computer experiments in dimension 3

The algorithm for 3-dimensional separately convex hull reviewed in Section 2.1 was fine-tuned and implemented by Boris Letocha in his M.Sc. thesis [Let99]. The correctness of the implementation was checked by comparison of many results with an earlier, slower implementation by J. Matoušek. Letocha noticed that the generic nontrivial configuration constructed in [MP98] is not inclusion-minimal, and an 18-point generic nontrivial configuration can be obtained from it by removing suitable 2 points. This is also a smallest generic nontrivial configuration known so far.

Letocha conducted extensive computer search for inclusion-minimal generic nontrivial configurations. At each experiment, independent random permutations π_1 and π_2 of $\{1, 2, \ldots, n\}$ were generated, and the (generic) set

$$A = \{(i, \pi_1(i), \pi_2(i)) : i = 1, 2, \ldots, n\} \qquad (1)$$

was considered. (For example, for $n = 100$, the computation of the separately convex hull for such a set took about 0.2s on a Pentium II, 300MHz machine.) If A turned out to be nontrivial (successful experiment), it was checked for inclusion-minimality, and as soon as a point $a \in A$ with $A \setminus \{a\}$ nontrivial was found, it was removed. This was repeated until an inclusion-minimal nontrivial set was obtained. As noted in [MP98], the existence of a single generic nontrivial configuration implies that the probability of success in this experiment tends to 1 as $n \to \infty$. However, it turned out that the probability of success is quite large even for fairly small n; for $n = 65$ it is (estimated to be) slightly over 0.5, and for $n = 78$ it exceeds 0.9.

Minimal configurations with sizes between 18 and 28 were discovered by this method. The 18-point configurations were most frequent; for $n = 66$, with about 55% of successful experiments, about 36% of the successful experiments led to 18-point configurations, 33% to 19-point ones, and 19% to 20-point ones. From such data, one can estimate from below the number of distinct minimal nontrivial configurations of the "canonical" form (1). For example, for $n = 40$, an 18-point configuration was observed in about 0.12% of the cases (in 10^6 experiments). Thus, up to the small statistical uncertainty, we get that the probability $P(40, 18)$ of a random set (1) containing a minimal nontrivial 18-point configuration as a subset is at least 0.0012. A random 40-point set (1) contains $\binom{40}{18}$ 18-point subsets, each of which can be regarded as a random 18-point set of the form (1). These random subsets are not independent, but certainly we have $P(40, 18) \leq \binom{40}{18}P(18)$, where $P(18)$ is the probability of a random 18-point set (1) being minimal nontrivial. Since the total number of 18-point sets (1) is $(18!)^2$, we can conclude that the number of distinct 18-point minimal generic nontrivial configurations is at least about $(18!)^2 P(40, 18)/\binom{40}{18} \geq 4 \times 10^{17}$.

Generic nontrivial configurations of 17 or fewer points were never encountered in the experiments, and they must be much more rare than those with 18 points, if they exist at all.

2.4 Arbitrarily large minimal configurations

First we exhibit an arbitrarily large inclusion-minimal nontrivial set in $\mathbf{R}^3$ which is not generic. For $n = 1, 2, \ldots$, the set A_n has $3n+3$ points d, e, f and a_i, b_i, c_i, $i = 1, 2, \ldots, n$. The construction for $n = 3$ is drawn in Fig. 2. The cube drawn by thin line is included solely for better visualization; the points are drawn by dots and the set $B_3 = \mathrm{co}^{sc}(A_3)$ by thick lines. For other n, the construction is analogous, but the "stairs" produced by a_i, b_i, c_i are made smaller and n of them are put in. Using the algorithm in Section 2.1, it is not difficult to check that $B_n = \mathrm{co}^{sc}(A_n)$. (The inclusion $B_n \subseteq \mathrm{co}^{sc}(A_n)$ is especially easy, since all sc-extreme points of B_n lie in A_n.) If we remove, for example, the point c_3

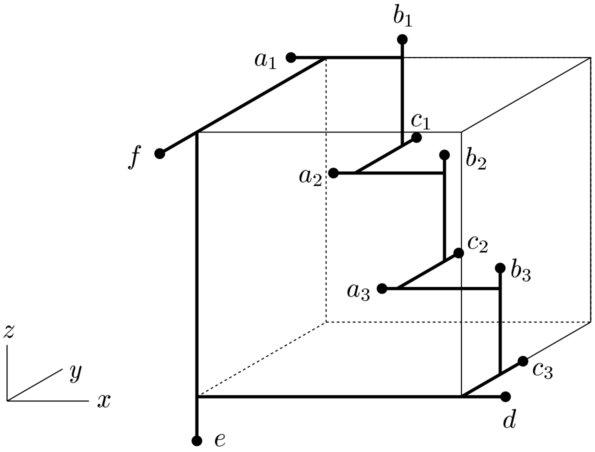


Figure 2: The minimal nontrivial set A_3 and its sc-hull.

from A_3, the algorithm allows us to remove the segment of B_3 ending in c_3, and then, successively, the segments ending in b_3, a_3, c_2, b_2,…,e, and d. The set $A_3 \setminus \{c_3\}$ is trivial, and the situation with removing any other point is entirely analogous.

Proposition 2.2 *The Carathéodory number for the separate convexity in* $\mathbf{R}^d$*, for* $d \geq 3$*, is infinite. In fact, arbitrarily large (finite) inclusion-minimal nontrivial configurations exist.*

Recently, Müller proved that if $\mathbf{R}^d$ is identified with the space $Diag_d$ of $d \times d$ diagonal matrices in the obvious manner ($(x_1, x_2, \ldots, x_d)$ becomes the matrix with $x_1, x_2, \ldots, x_d$ on the diagonal and 0's elsewhere), and if $f\colon Diag_d \to \mathbf{R}$ is any separately convex function, then for any $\varepsilon > 0$ and a compact set $K \subset Diag_d$, a quasiconvex function g on the space $M^{d\times d}$ of all $d \times d$ matrices exists with $|f(x) - g(x)| < \varepsilon$ for all $x \in K$ ([Mül99] deals with the case $d = 2$ and announces the result for an arbitrary d). Consequently, for compact $A \subseteq Diag_d$, the separately convex hull of A within $Diag_d$ equals the quasiconvex hull, and also the rank-one convex hull, of A in $M^{d\times d}$. (Let us remark that the just mentioned result is not obvious even for rank-one convexity.) Therefore, we get

Corollary 2.3 *The Carathéodory number for quasiconvexity, as well as for rank-one convexity, on* $d \times d$ *matrices,* $d \geq 3$*, is infinite, and arbitrarily large (finite) inclusion-minimal nontrivial configurations exist.*

The configuration A_n constructed above is not generic, but an arbitrarily large minimal generic nontrivial configuration for separate convexity can be obtained as well (for rank-one convexity or quasiconvexity, the existence of such configurations is open at present). The idea is to replace each point of A_n by a small (perturbed) copy of the planar configuration T_4. We first note that if the horizontal rectangle R as in Fig. 3 lies in the hull of some set, and the four points $a_1, \ldots, a_4$ are in the set, then also the rectangle R_1 lies in the hull. Then, such rectangles (and the corresponding fourtuples) can be arranged cyclically, similar to the segments in Fig. 2, as depicted in Fig. 4 (the position

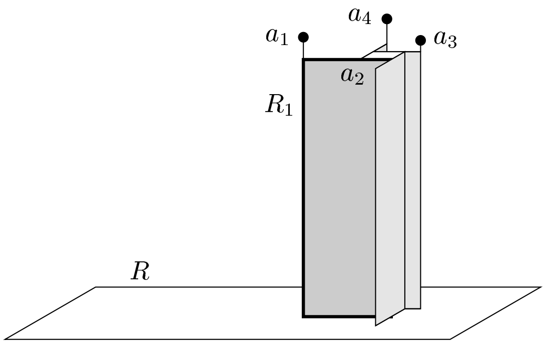


Figure 3: Suitable 4 points and R generate the rectangle R_1.

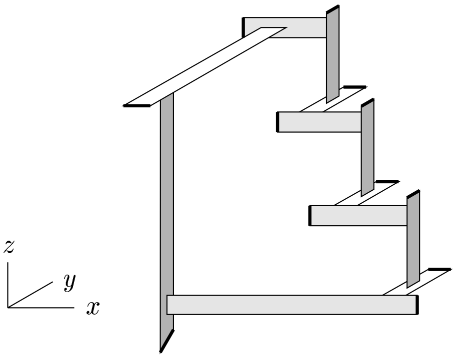


Figure 4: Constructing arbitrarily large generic nontrivial configuration.

of the 4-tuples is indicated schematically by thick segments). The resulting configuration is generic nontrivial. It is not minimal (it turns out that 2 points suffice at each turn of the "stairs"; Fig. 5 shows a minimal subconfiguration obtained for $n = 3$), but if we delete any of the 4-tuples, we get a trivial configuration, and hence any minimal nontrivial subset has at least $3(n + 1)$ points.

Let us remark that by applying this construction with $n = 1$ and selecting a minimal nontrivial subset, we arrive at the nicely symmetric 18-point configuration (the smallest known size) shown in Fig. 6. The picture displays the separately convex hull as the appropriate box complex; the 2-dimensional elementary boxes are shown semi-transparent. It is also remarkable that, unlike to the usual convex hulls, the separately convex hulls in dimension 3 need not be contractible.

3 Envelope of the distance function defines the hull

Theorem 3.1 *Let $D \subseteq \mathbf{R}^d$ be a set of directions containing a basis of $\mathbf{R}^d$. Let $A \subset \mathbf{R}^d$ be a compact set. Let δ_A be the function giving distance from A; that is, $\delta_A(x) = \inf_{y \in A} \|x - y\|$. Let Z be the zero set of $C_D \delta_A$. Then $Z = \mathrm{co}^D(A)$.*

Proof. The inclusion $\mathrm{co}^D(A) \subseteq Z$ is clear. To prove the opposite inclusion, consider a point $x_0 \notin \mathrm{co}^D(A)$. This means that a nonnegative D-convex function f exists with $f(x_0) > 0$ and $f(A) = 0$. Our goal is to produce a D-convex function g with $g(x_0) > 0$ and satisfying $g \leq \delta_A$ everywhere. Then we will also have that the D-convex envelope of δ_A majorizes g, and in particular it cannot be 0 at x_0.

Choose the coordinate system so that $0 \in A$. Let $B(0, R)$, $R \geq 1$, be a (closed) ball containing both x_0 and A. Set

$$\eta = \tfrac{1}{4} \inf \left\{ \frac{\delta_A(x)}{\max(\delta_A(x), f(x))} : x \in B(0, 2R) \setminus A \right\}.$$

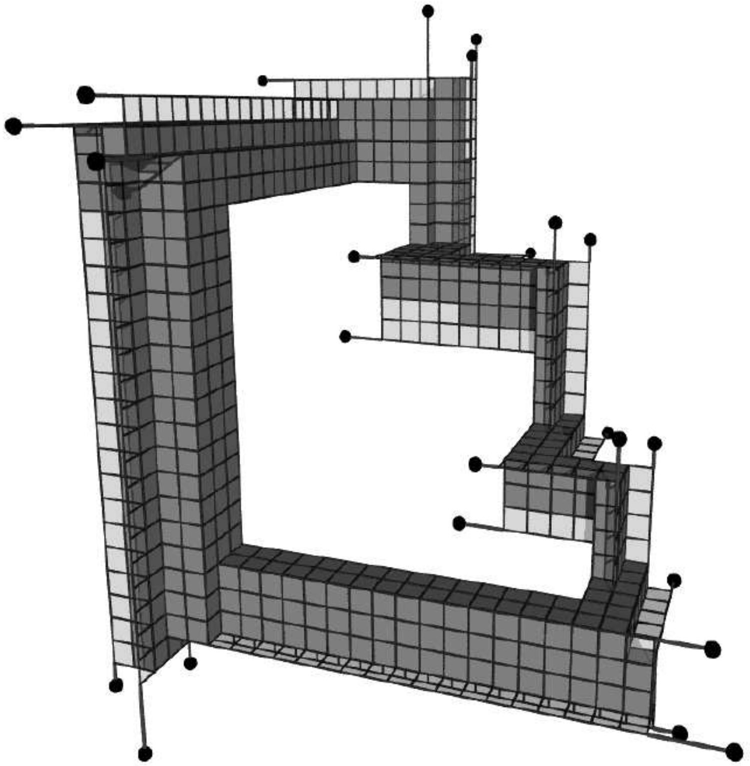


Figure 5: A minimal 30-point generic nontrivial configuration and its *sc*-hull.

We want to prove that $\eta > 0$. We recall that the function f, being D-convex, is locally Lipschitz ([MP98], Observation 2.3), and hence Lipschitz on any compact set. Let C be the Lipschitz constant of f on $B(0, 2R)$. If $x \in B(0, 2R)$ lies at distance $t > 0$ from A, then there is a point $a \in A$ at distance t from x, and since $f(a) = 0$ we have $f(x) \le Ct$. From this we get $\eta \ge \frac{1}{4C}$.

Now we know that ηf is a D-convex function nonzero at x_0 satisfying $\eta f \le \delta_A$ everywhere on the ball $B(0, R)$; we still need to extend it on the whole $\mathbf{R}^d$ so that it remains below δ_A everywhere. To this end, we use a standard trick from convex analysis for extending a convex function defined on a ball.

Define a function g by setting

$$g(x) = \begin{cases} \max(\eta f(x), \|x\| - R) & \text{for } \|x\| \le 2R \\ \|x\| - R & \text{for } \|x\| > 2R. \end{cases}$$

First we note than on $B(0, R)$, g coincides with ηf, and hence $g(x_0) > 0$. We also have $g \le \delta_A$ everywhere (because $\eta f \le \delta_A$ on $B(0, 2R)$ and $\|x\| - R \le \delta_A(x)$). It remains to show that g is D-convex. Clearly it is D-convex on $B(0, 2R)$, being a maximum of two D-convex functions there. We note that for all x with $\frac{3}{2}R \le \|x\| \le 2R$ we have

$$\eta f(x) \le \tfrac{1}{4}\, \delta_A(x) \le \tfrac{1}{4}\|x\| \le \tfrac{1}{2}\, R \le \|x\| - R\,.$$

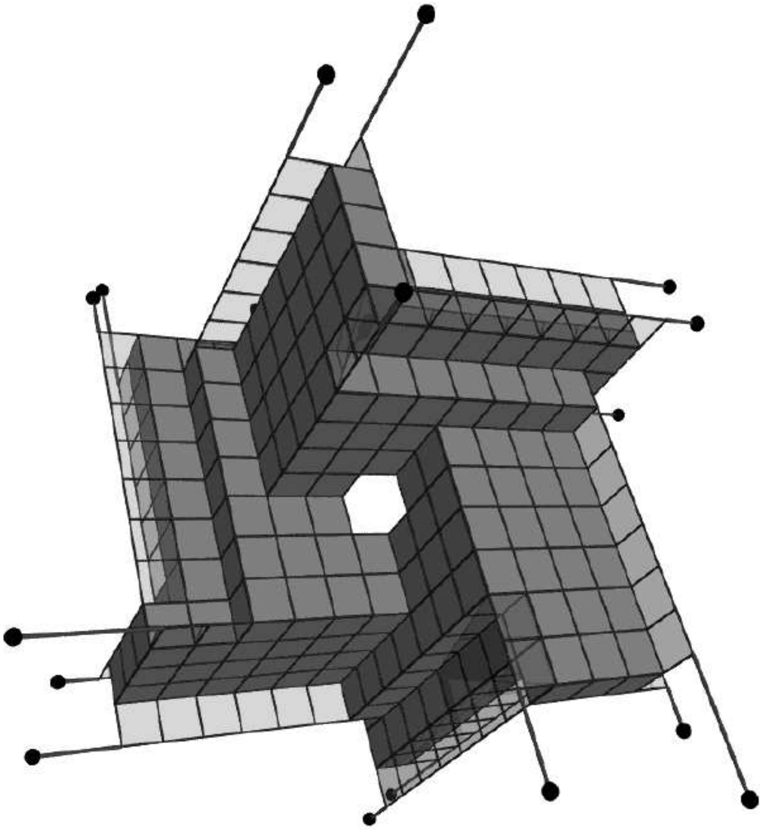


Figure 6: An 18-point generic configuration with its sc-hull.

This means that in the annulus $\frac{3}{2}R \leq \|x\| \leq 2R$, $g(x)$ coincides with $\|x\| - R$, and from this it is routine to check the D-convexity of g on the whole $\mathbf{R}^d$. $\square$

4 Non-robustness of the D-convex hull

For separate convexity, there is a simple exact algorithm for computing the sc-hulls of finite sets, but for rank-one convexity (or for any D with more than d directions in $\mathbf{R}^d$), no such algorithm is known so far. In view of Theorem 3.1, it seems natural to try to approximate the D-convex hull by approximately computing the D-convex envelope of the distance function and taking the "near-zero" set. But even for separate convexity in the plane, this is generally unrealistic, because enormous accuracy would be required.

Proposition 4.1 *For each $n \geq 1$, there exist: an $(n+1)$-point set $A \subset \mathbf{R}^2$ contained in the $m \times m$ integer grid with $m = O(n)$ and with $\mathrm{co}^{sc}(A) = A$, and two points b_0 and b_n, both at distance 1 from A, such that any separately convex function f that is 0 on A satisfies $f(b_n) \leq n^{-n} f(b_0)$.*

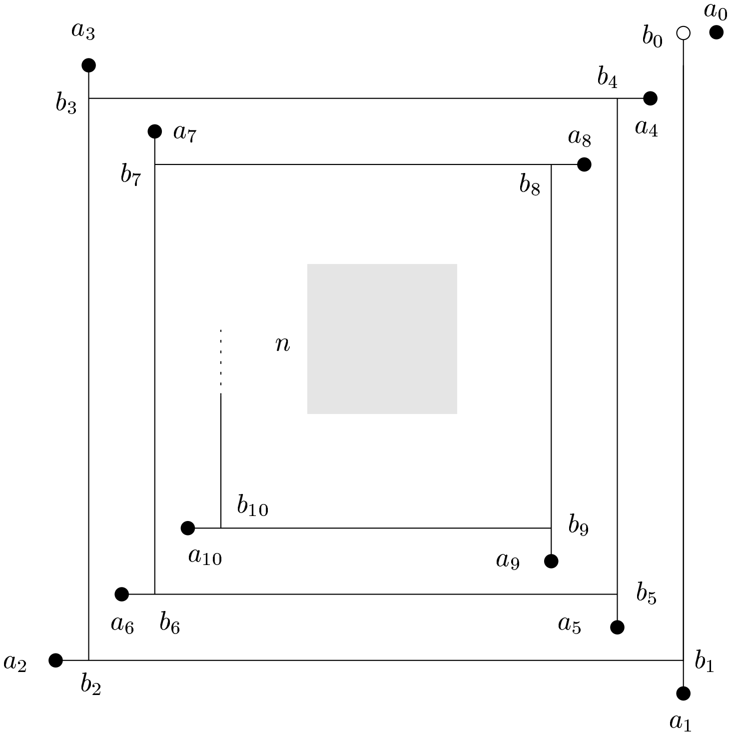


Figure 7: A configuration with non-robust hull

Proof. The construction is shown in Fig. 7. There is an auxiliary gray square with side n in the middle, and the points a_0, a_1,..., a_n are placed in a spiral-like configuration around the square; the scaling is such that the distance of a_0 and b_0 is 1, as well as each of the distances $a_i b_i$. Since $f(a_1) = 0$ and the distance $b_0 a_1$ is at least n, the convexity of f on the line $b_0 a_1$ implies $f(b_1) \leq \frac{1}{n} f(b_0)$, and induction (along the indicated lines) yields $f(b_i) \leq n^{-i} f(b_0)$. Finally, the triviality of A is easy (one can check that there is no T_4 configuration, or apply the algorithm). $\square$

Of course, one can hope that configurations with this bad behavior are exceptional and that we suffice with much smaller precision for computing the D-convex envelope for “usual” examples. Computational experiments, described next, indicate that one has to be careful even with not too large random configurations.

For various values of n, random n-point sets $A \subset \mathbf{R}^3$ of the form (1) were generated. Recall that they are subsets of the grid $G = \{1, 2, \ldots, n\}^3$. For such an A, the function $f_A\colon G \to \{0,1\}$, with $f_A(x) = 0$ for $x \in A$ and $f_A(x) = 1$ otherwise, was considered. As shown in [MP98], the points of $G \cap \mathrm{co}^{sc}(A)$ are exactly the zero set of the separately convex envelope of f_A *on* G (where the separately convex envelope on G is the largest function $G \to \mathbf{R}$ that is below f_A and satisfies the convexity condition for any triple of points of G lying on a line parallel to a coordinate axis). The function f_A was separately convexified on the grid G by a straightforward iterative algorithm: convexify along the lines parallel to the x-axis, then along the y-axis, then along the z-axis, and repeat until the maximum change of the function's value in a single iteration drops below a small threshold (chosen as 10^{-13} for double-precision arithmetic). Up to small rounding errors, this algorithm provides an upper bound on the values of $C_{sc}f_A$ on G (and, hopefully, should provide good approximation to the actual values of the separately convex envelope, but no error bound seems to be available).

A measure of the accuracy required for correct computation of $\mathrm{co}^{sc}(A)$ by this method is the smallest value of $C_{sc}f_A(x)$ for $x \in G \setminus \mathrm{co}^{sc}(A)$ (where $\mathrm{co}^{sc}(A)$ was determined by the exact combinatorial algorithm). In the experiments, this value was typically quite small even for moderate values of n. For example, while it was typically between 10^{-3} and 10^{-4} for $n = 20$, already for $n = 40$ it was usually below 10^{-7} and values as small as 2×10^{-11} appeared in a few cases. For $n = 50$, the values were often below 10^{-13} and cannot be considered reliable anymore with the double-precision computations.

Still, knowing that the value of the D-convex envelope of the distance function of A (or of some other suitable function) is reasonably large at some point x, we can conclude that $x \notin \mathrm{co}^D(A)$. Approximate computation of D-convex envelopes might thus yield at least a reasonable outer approximation of the D-convex hull. But even here, there is a problem with controlling the error of the approximation of the envelope. The most natural method of computing the D-convex envelope (employed above for separate convexification on a grid), namely iterative one-dimensional convexifications along the directions in D, is likely to provide an upper bound on the values of the envelope. But controlling the error, and getting a reliable bound from below, appears challenging. As was remarked above, no error bounds seem to be available even for separate convexification on a grid.

5 Lipschitz constant is preserved by the D-convex envelope

Recall that for a function $f\colon \mathbf{R}^d \to \mathbf{R}$ and a direction vector v, $C_{\{v\}}f$ denotes the convex envelope of f taken in direction v; that is, f is convexified (independently) along each line parallel to v. For simplicity, we write only $C_v f$ for $C_{\{v\}}f$.

Lemma 5.1 *Let $v \in \mathbf{R}^d$ be a nonzero direction vector. Let f be a real 1-Lipschitz function defined on $\mathbf{R}^d$. Then $C_v f$ is 1-Lipschitz as well. Here "1-Lipschitz" may be taken with respect to an arbitrary translation-invariant metric on $\mathbf{R}^d$.*

The following simple proof was communicated to me by Kirchheim; my previous formulation was more complicated.

Proof. For brevity, denote the function $C_v f$ by g. Let $x, y \in \mathbf{R}^d$ be two arbitrary points; it suffices to prove

$$g(x) \leq g(y) + \rho(x, y), \tag{2}$$

where ρ is the considered translation-invariant metric. Let $\varepsilon > 0$ be arbitrary, and let ℓ_y be the line parallel to v containing y. Since the point $(y, g(y) + \varepsilon)$ is above the convex envelope of f restricted to ℓ_y, there are two points $y_1, y_2 \in \ell_y$ such that y lies between y_1 and y_2 and $g(y) + \varepsilon > tf(y_1) + (1-t)f(y_2)$, $t \in [0, 1]$. Then we have

$$\begin{array}{rcll} g(x) & = & g(y + (x - y)) & \\ & \leq & tf(y_1 + (x - y)) + (1 - t)f(y_2 + (x - y)) & \text{(convexity of } g \text{ on } \ell_x) \\ & \leq & tf(y_1) + (1 - t)f(y_2) + \rho(x, y) & \text{(as } f \text{ is 1-Lipschitz)} \\ & < & g(y) + \varepsilon + \rho(x, y). & \end{array}$$

Since $\varepsilon > 0$ was arbitrary, (2) is proved. $\square$

Corollary 5.2 *Let f be a 1-Lipschitz real function defined on $\mathbf{R}^d$. Then $C_D f$ is 1-Lipschitz as well.*

Proof. Set

$$\bar{f} = \inf\{C_{v_1} C_{v_2} \ldots C_{v_n} f : v_1, v_2, \ldots, v_n \in D,\, n = 1, 2, 3, \ldots\}$$

(a pointwise infimum). Clearly $C_D f \leq \bar{f}$, and it is straightforward to verify that $\bar{f}$ is a D-convex function, hence $\bar{f} = C_f$. At the same time, all the functions $C_{v_1} C_{v_2} \ldots C_{v_n} f$ are 1-Lipschitz by Lemma 5.1, and hence $C_D f$ is 1-Lipschitz as well. $\square$

There is an interesting consequence for D-convex envelopes of distance functions to sets.

Corollary 5.3 *Let $A \subseteq \mathbf{R}^d$ be a set and let $B = \mathrm{co}^D(A)$. Then $C_D \delta_A = C_D \delta_B$, where δ_X denotes the distance function of a set X.*

Proof. Since $A \subseteq B$, we have $\delta_A \geq \delta_B$ and hence also $C_D \delta_A \geq C_D \delta_B$. To see the opposite inequality, we note that $C_D \delta_A$ is a D-convex function and hence it is 0 on B. Moreover, it is 1-Lipschitz by Corollary 5.2, and therefore $C_D \delta_A \leq \delta_B$. Taking D-convex envelopes on both sides of the last inequality yields $C_D \delta_A \leq C_D \delta_B$. $\square$

6 A locality result

The following result was claimed in [MP98] as Corollary 2.9:

Proposition 6.1 *Let $A \subseteq \mathbf{R}^d$ be contained in a (functionally) D-convex set C, which is a disjoint union of compact sets $C_1, \ldots, C_k$. Then $\mathrm{co}^D(A) = \bigcup_{i=1}^k \mathrm{co}^D(A \cap C_i)$.*

As was pointed out by Bernd Kirchheim (private communication, 1998), this is probably not an immediate consequence of the previous theorem in [MP98] (which states that if $C_1, C_2 \subseteq \mathbf{R}^d$ are disjoint compact sets with $C_1 \cup C_2$ being D-convex then C_1 and C_2 are D-convex as well). Here we give a full proof. The result was independently proved, together with some other related properties of D-convex hulls, by Kirchheim [Kir99].

Proof. It suffices to prove the following statement: *Let $B, C \subset \mathbf{R}^d$ be disjoint compact sets whose union is D-convex, and let $K \subseteq B$; then $B \cap \mathrm{co}^D(K \cup C) = \mathrm{co}^D(K)$.* Indeed, in the situation of Proposition 6.1, we set $B = C_1$, $C = C_2 \cup \cdots \cup C_k$, $K = A \cap C_1$, and we use the monotonicity of the D-convex hull.

Put $f = C_D \delta_{B \cup C}$. Fix $\beta > 0$ such that $B_{2\beta} \cap C = \emptyset$ (where B_ε denotes the ε-neighborhood of B), and let $S = \overline{B_{2\beta} \setminus B_\beta}$. By Theorem 3.1, f is positive on S, and so it is bounded away from 0 there, by the compactness of S.

Let $f_K = C_D \delta_K$. This f_K is positive on the compact set S (since $\mathrm{co}^D(K) \subseteq B$). Choose $\eta > 0$ so that $\eta f_K \leq f$ on S. Define a function g as follows:

$$g = \begin{cases} \max(\eta f_K, f) & \text{on } B_{2\beta} \\ f & \text{elsewhere.} \end{cases}$$

Since $f = 0$ on $B \cup C$, the zero set of g contains C, and its intersection with B equals the zero set of f_K, i.e. $\mathrm{co}^D(K)$. It remains to check that g is D-convex. On $B_{2\beta}$, g is D-convex as the maximum of two D-convex functions. Outside of B_β, we have $g = f$. If a line ℓ intersects both B_β and the complement of $B_{2\beta}$, it shares a segment of length at least β with S; consequently, g is D-convex everywhere on $\mathbf{R}^d$. $\square$

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References

[AH86] R. Aumann and S. Hart. Bi-convexity and bi-martinagles. *Israel. J. Math.*, 54(2):159–180, 1986.

[BFJK94] K. Bhattacharya, N. B. Firoozye, R.D. James, and R.V. Kohn. Restrictions on microstructure. *Proc. Royal Soc. Edinburgh*, 124A:843–878, 1994.

[CT93] E. Casadio-Tarabusi. An algebraic characterization of quasiconvex functions. *Ricerche Mat.*, 42:11–24, 1993.

[DKMŠ99] G. Dolzmann, B. Kirchheim, S. Müller, and V. Šverák. The two-well problem in three dimensions. Preprint 21/1999, Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany, 1999.

[Kir99] B. Kirchheim. On the geometry of rank-one convex hulls. Manuscript, Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany, 1999.

[KKB99] B. Kirchheim, J. Kristensen, and J. Ball. Regularity of quasiconvex envelopes. Manuscript, 1999.

[Let99] B. Letocha. Directional convexity (in Czech). M. Sc. Thesis, Department of Applied Mathematics, Charles University, Prague, 1999.

[Mor52] C. B. Morrey. Quasi-convexity and the lower semicontinuity of multiple integrals. *Pacific J. Math.*, 2:25–53, 1952.

[MP98] J. Matoušek and P. Plecháč. On functional separately convex hulls. *Discr. Comput. Geom.*, 19:105–130, 1998.

[MŠ99a] S. Müller and V. Šverák. Convex integration for Lipschitz mappings and counterexamples to regularity. Preprint 26/1999, Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany, 1999.

[MŠ99b] S. Müller and V. Šverák. Convex integration with constraints and applications to phase transitions and partial differential equations. Preprint 28/1999, Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany, 1999.

[Mül98] S. Müller. Variational models for microstructure and phase transitions. Lecture Note Nr. 2/1998, Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany, `http://www.msi.mpg.de`, 1998.

[Mül99] S. Müller. Rank-one convexity implies quasiconvexity on diagonal matrices. Preprint 29/1999, Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany, 1999.

[Sch74] V. Scheffer. Regularity and irregularity of solutions to nonlinear second order elliptic systems of partial differential equations and inequalities. Dissertation, Princeton University, 1974.

[Šve92] V. Šverák. New examples of quasiconvex functions. *Proc. of Royal Soc. Edinburgh*, 120A:185–189, 1992.

[Tar93] L. Tartar. On separately convex functions. In *Microstructure and Phase Transition, The IMA volumes in mathematics and its applications, vol. 54 (D. Kinderlehrer et al. eds)*, pages 191–204. Springer, 1993.