On the chromatic number of Kneser hypergraphs*

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Abstract

We give a simple and elementary proof of Kříž's lower bound on the chromatic number of the Kneser r-hypergraph of a set system $\mathcal{S}$.

1 Introduction

Let $\mathcal{S}$ be a system of subsets of a finite set X. The *Kneser r-hypergraph* $\mathrm{KG}_r(\mathcal{S})$ has $\mathcal{S}$ as the vertex set, and an r-tuple $(S_1, S_2, \ldots, S_r)$ of sets in $\mathcal{S}$ forms an edge if $S_i \cap S_j = \emptyset$ for all $i \neq j$. In particular, $\mathrm{KG}(\mathcal{S}) = \mathrm{KG}_2(\mathcal{S})$ is the *Kneser graph* of $\mathcal{S}$. Kneser [8] conjectured in 1955 that $\chi(\mathrm{KG}(\binom{[n]}{k})) \geq n-2k+2$, $n \geq 2k$, where $\binom{[n]}{k}$ denotes the system of all k-element subsets of the set $[n] = \{1, 2, \ldots, n\}$, and χ denotes the chromatic number. This was proved in 1978 by Lovász [12], as one of the earliest and most spectacular applications of topological methods in combinatorics. Several other proofs have been published since then, all of them topological; among them, at least those of Bárány [2], Dol'nikov [5] (also see [6], [7]), and Sarkaria [13] can be regarded as substantially different from each other and from Lovász' original proof. Erdős' generalization of Kenesr's conjecture to hypergraphs, dealing with the chromatic number of $\mathrm{KG}_r(\binom{[n]}{k})$, was established by Alon, Frankl, and Lovász [1].

Kříž [11], [10] proved a remarkable lower bound for the chromatic number of $\mathrm{KG}_r(\mathcal{S})$ for an arbitrary set system $\mathcal{S}$, which easily implies the correct bound in the case when $\mathcal{S} = \binom{[n]}{k}$ considered by Alon et al. (for $r = 2$, the result was obatined earlier by Dol'nikov [5]).

To state this result, we first recall that a mapping $c: V \to [m]$ is a *(proper) coloring* of a hypergraph $\mathcal{H} = (V, E)$ if none of the edges $e \in E$ is monochromatic under c. The *chromatic number* $\chi(\mathcal{H})$ of $\mathcal{H}$ is the smallest m such that a proper coloring $c: V \to [m]$ exists. We define the *r-colorability defect* of $\mathcal{H} = (V, E)$ as the smallest number of vertices that must be removed so that the edges living comletely on the remaining points form an r-colorable hypergraph, i.e.

$$\mathrm{cd}_r(\mathcal{H}) = \min\Big\{|Y| : \chi((V \setminus Y, \{e \in E : e \cap Y = \emptyset\})) \leq r\Big\}.$$

Kříž's result can be stated as follows.

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Theorem 1.1 (Dol'nikov [5] for $r = 2$; Kříž [11],[10]) *For any finite set system $(X, \mathcal{S})$ and any $r \geq 2$, we have*

$$\chi(\mathrm{KG}_r(\mathcal{S})) \geq \frac{1}{r-1} \cdot \mathrm{cd}_r((X, \mathcal{S})).$$

The proof in [11] does not work in the generality stated there (as was pointed out by Živaljević) but Theorem 1.1 remains valid [10]. We remark that $\mathrm{KG}_r(\mathcal{S})$ is denoted by $[\mathcal{S}, r]$ in [11], and $\mathrm{cd}_r(\mathcal{S})$ is denoted by $w(\mathcal{S}, r)$ there and called the *r-width*.

In this paper, we present another proof of Theorem 1.1. The basic approach is similar to that of Kříž, but our proof is somewhat simpler and, hopefully, more accessible to non-specialists in topology.

We only assume reader's familiarity with a few basic topological notions (such as simplicial complex and its geometric realization); more special topological notions are reviewed in Section 2, in very concrete form just suitable for our purposes. We refer to Börner [3] or Živaljević [14], [15] for wider backgroung and for nice nice recent overviews of topological methods in combinatorics.

2 Preliminaries

Simplicial complexes. For our purposes, a (finite) simplicial complex K is a hereditary family (including the empty set) of subsets of a finite set; the sets in K are called *simplices*. The *dimension* of a simplex is the number of its vertices minus 1. The vertex set of K is denoted by $V(\mathsf{K})$, and the polyhedron of a geometric realization of K is denoted by $\|\mathsf{K}\|$.

Let $(V, \leq)$ be a partially ordered set. The *order complex* of $(V, \leq)$ is the simplicial complex with vertex set V and with all chains under $\leq$ (i.e. subsetes of V linearly ordered by $\leq$) as simplices. The *first barycentric subdivision* of a simplicial complex K, denoted by $\mathrm{sd}(\mathsf{K})$, is the order complex of the set of all nonempty simplices of K ordered by inclusion. The polyhedra of K and of $\mathrm{sd}(\mathsf{K})$ are canonically homeomorphic.

Let K, L be simplicial complexes. A *simplicial map* $f : \mathsf{K} \to \mathsf{L}$ is a map $V(\mathsf{K}) \to V(\mathsf{L})$ such that the image of any simplex of K is contained in a simplex of L. A simplicial map induces a map $\|\mathsf{K}\| \to \|\mathsf{L}\|$ of topological spaces.

The *join* of simplicial complexes K and L with $V(\mathsf{K}) \cap V(\mathsf{L}) = \emptyset$ is the simplicial complex with vertex set $V(\mathsf{K}) \cup V(\mathsf{L})$ and with simplices $F \cup G$ for all $F \in \mathsf{K}$ and $G \in \mathsf{L}$. If $V(\mathsf{K})$ and $V(\mathsf{L})$ are not disjoint then $\mathsf{K} * \mathsf{L}$ is the join of K with an isomorphic copy of L whose vertex set is disjoint from $V(\mathsf{K})$. If $\mathsf{K}_1, \mathsf{K}_2, \mathsf{L}_1, \mathsf{L}_2$ are simplicial complexes, $V(K_1) \cap V(L_1) = \emptyset$, and $f \colon \mathsf{K}_1 \to \mathsf{K}_2$ and $g \colon \mathsf{L}_1 \to \mathsf{L}_2$ are simplicial maps, then $f * g \colon \mathsf{K}_1 * \mathsf{L}_1 \to \mathsf{K}_2 * \mathsf{L}_2$ is the simplicial map given by $(f * g)(v) = f(v)$ for $v \in V(K_1)$ and $(f * g)(v) = g(v)$ for $v \in V(L_1)$.

Connectivity. Let X, Y be topological spaces, $k \geq 0$ stands for an integer. All mappings between topological spaces are implicitly assumed to be continuous. X is *k-connected* if for any $j = 0, 1, \ldots, k$ any mapping f of the j-dimensional sphere S^j into X can be extended to a mapping of the $(j+1)$-dimensional ball into X.

$\mathbf{Z}_p$-spaces. A $\mathbf{Z}_p$*-space* is a pair (X, ν), where $\nu : X \to X$ is a homeomorphism $X \to X$ with $\nu^p = \mathrm{id}_X$; ν is called a $\mathbf{Z}_p$*-action* on X. The $\mathbf{Z}_p$-action ν is called *free* if for each $x \in X$ the points $x, \nu(x), \nu^2(x), \ldots, \nu^{p-1}(x)$ are pairwise distinct. For prime p, it suffices to require $\nu(x) \neq x$ for all x. A simplicial $\mathbf{Z}_p$-complex is a simplicial complex K with a $\mathbf{Z}_p$-action on $\|\mathsf{K}\|$ given by a simplicial map $\mathsf{K} \to \mathsf{K}$.

For $\mathbf{Z}_p$-spaces (X, ν), (Y, ω), a $\mathbf{Z}_p$-mapping $f \colon (X, \nu) \to (Y, \omega)$ is a mapping of X into Y which commutes with the $\mathbf{Z}_p$-actions, i.e. $f \circ \nu = \omega \circ f$.

The $\mathbf{Z}_p$-index. For integers k and p, we define the simplicial complex $\mathsf{E}_{k,p}$ whose maximal simplices are the edges of the complete $(k{+}1)$-uniform $(k{+}1)$-partite hypergraph with classes of size p. More formally, the vertex set is $[k{+}1] \times [p]$ and the simplices have the form $\{(j_1, i_1), (j_2, i_2), \ldots, (j_q, i_q)\}$, $1 \leq j_1 < j_2 < \cdots < j_q \leq k{+}1$ and $i_t \in [p]$, $t = 1, 2, \ldots, q$. The mapping $\omega \colon V(\mathsf{E}_{k,p}) \to V(\mathsf{E}_{k,p})$ given by $(j, i) \mapsto (j, i{+}1)$, where $p{+}1$ means 1, is a free simplicial $\mathbf{Z}_p$-action on $\mathsf{E}_{k,p}$. In particular, $\mathsf{E}_{k,2}$ is the k-dimensional sphere represented as the unit sphere of the L_1-norm, and the $\mathbf{Z}_2$-action is the antipodality $x \mapsto -x$. The important property of $\mathsf{E}_{k,p}$ is that its polyhedron is a k-dimensional, $(k{-}1)$-connected free $\mathbf{Z}_p$-space;[1] any (paracompact) k-dimensional $(k{-}1)$-connected free $\mathbf{Z}_p$-space would do equivalently in the definition below.

For a free $\mathbf{Z}_p$-space (X, ν), the $\mathbf{Z}_p$-index is defined by

$$\mathrm{ind}_{\mathbf{Z}_p}(X) = \min\{k : \text{ there is a } \mathbf{Z}_p\text{-map } (X, \nu) \to (\|\mathsf{E}_{k,p}\|, \omega)\}$$

(the action ν is not shown in the notation $\mathrm{ind}_{\mathbf{Z}_p}$ but is understood from context). This kind of index, under the name *genus*, was introduced by Krasnosel'skiĭ [9] (for $\mathbf{Z}_2$-spaces); our presentation follows [14].

The key fact about $\mathbf{Z}_p$-index is $\mathrm{ind}_{\mathbf{Z}_p}(\|\mathsf{E}\|_{k,p}) = k$, i.e. there is no $\mathbf{Z}_p$-map $\|\mathsf{E}_{k,p}\| \to \|\mathsf{E}_{k-1,p}\|$. For $p = 2$, this is one of the versions of the well-known Borsuk–Ulam theorem, and for larger p, it is a particular case of a theorem of Dold [4]; see e.g. [15] for a sketch of proof using only basic homology theory.

Clearly, if there is a $\mathbf{Z}_p$-map $(X, \nu) \to (Y, \omega)$ then $\mathrm{ind}_{\mathbf{Z}_p}(X) \leq \mathrm{ind}_{\mathbf{Z}_p}(Y)$. For a free simplicial $\mathbf{Z}_p$-complex, we have $\mathrm{ind}_{\mathbf{Z}_p}(\|K\|) \leq \dim(\mathsf{K})$ (this can be shown easily using the $(k{-}1)$-connectedness of $\mathsf{E}_{k,p}$; see, for example, [15]). For free simplicial $\mathbf{Z}_p$-complexes K and L, we have

$$\mathrm{ind}_{\mathbf{Z}_p}(\mathsf{K} * \mathsf{L}) \leq \mathrm{ind}_{\mathbf{Z}_p}(\mathsf{K}) + \mathrm{ind}_{\mathbf{Z}_p}(\mathsf{L}) + 1, \tag{1}$$

where the $\mathbf{Z}_p$-action on $\mathsf{K} * \mathsf{L}$ is the join of the $\mathbf{Z}_p$-actions on K and on L. This is easily derived from the isomorphism of $\mathsf{E}_{k,p} * \mathsf{E}_{\ell,p}$ with $\mathsf{E}_{k+\ell+1,p}$.

3 Proof of Theorem 1.1

First, let $r = p$ be a prime number. Let $X = [n]$, and let $\mathcal{S}$ be a set system on X with $\mathrm{cd}_p((X, \mathcal{S})) > \ell$.

[1] The $(k-1)$-connectedness can be derived in several ways, for example by representing $\mathsf{E}_{k,p}$ as the $(k{+}1)$-fold join $[p]^{*(k+1)}$, where $[p]$ is the p-point discrete space, and use the fact that the join of a j-connected simplicial complex and of an ℓ-connected simplicial complex is $(j + \ell + 2)$-connected (see e.g. [3]).

We define a partial ordering $\leq$ on the set of all ordered p-tuples $(A_1, A_2, \ldots, A_p)$ of subsets of X by letting $(A_1, \ldots, A_p) \leq (A'_1, \ldots, A'_p)$ iff $A_i \subseteq A'_i$ for all $i = 1, 2, \ldots, p$.

Consider the set of all ordered p-tuples $(A_1, A_2, \ldots, A_p)$ such that the A_i are pairwise disjoint subsets of X whose union covers all but at most ℓ points of X, and let $\mathsf{K} = \mathsf{K}(X, p, \ell)$ be the order complex of this set with the ordering $\leq$ defined above. A simplicial free $\mathbf{Z}_p$-action ν is defined on K by the cyclic shift:

$$\nu: (A_1, \ldots, A_p) \mapsto (A_2, A_3, \ldots, A_p, A_1).$$

Suppose that $c: \mathcal{S} \to [m]$ is a proper m-coloring of the Kneser p-hypergraph $\mathrm{KG}_p(\mathcal{S})$. This time we consider the set of all ordered p-tuples $(C_1, \ldots, C_p)$ of subsets of $[m]$ with $\bigcup_{i=1}^p C_i \neq \emptyset$ and $\bigcap_{i=1}^p C_i = \emptyset$. Let L be the order complex this set with the componentwise inclusion ordering $\leq$ as above. The simplicial $\mathbf{Z}_p$-action on L, again given by the cyclic shift of coordinates (i.e. $(C_1, \ldots, C_p) \mapsto (C_2, \ldots, C_p, C_1)$), is free—here we use that p is a prime.

Using the m-coloring c, we are going to define a simplicial $\mathbf{Z}_p$-map $f: \mathsf{K} \to \mathsf{L}$. For a subset $A \subseteq X$, let

$$g(A) = \{c(S) : S \subseteq A,\, S \in \mathcal{S}\}$$

and for a vertex $(A_1, A_2, \ldots, A_p)$ of K, put

$$f((A_1, A_2, \ldots, A_p)) = (g(A_1), g(A_2), \ldots, g(A_p)).$$

If c is a proper coloring, then no p pairwise disjoint sets of $\mathcal{S}$ can have the same color, and it follows that $\bigcap_{i=1}^p g(A_i) = \emptyset$. Since we assume $\mathrm{cd}_p((X, \mathcal{S})) > \ell$, for any ordered p-tuple $(A_1, \ldots, A_p) \in V(\mathsf{K})$, there are $i \in [p]$ and $S \in \mathcal{S}$ with $S \subseteq A_i$. Therefore, $\bigcup_{i=1}^p g(A_i) \neq \emptyset$, so $f((A_1, \ldots, A_p)) \in V(\mathsf{L})$ and and it is easy to see that f is a simplicial $\mathbf{Z}_p$-map $\mathsf{K} \to \mathsf{L}$.

It remains to bound the indices $\mathrm{ind}_{\mathbf{Z}_p}(\mathsf{K})$ and $\mathrm{ind}_{\mathbf{Z}_p}(\mathsf{L})$. As for the latter, we have $\mathrm{ind}_{\mathbf{Z}_p}(\mathsf{L}) \leq \dim(\mathsf{L}) = m(p{-}1)$. Indeed, supposing that $(C_1, \ldots, C_p)$ is the largest element in a chain of vertices of L, each $j \in [m]$ is in at most $p{-}1$ of the C_i, and each time we pass to a smaller element of the chain, some $j \in [m]$ is omitted from at least one of the sets; thus, the chain has at most $m(p{-}1){+}1$ elements.

The $\mathbf{Z}_p$-index of K can be bounded from below in several ways (homology computation, inductive argument showing an appropriate connectivity, shelling argument); we use a simple approach inspired by Sarkaria's papers.

First we consider the larger complex $\mathsf{K}_0 = \mathsf{K}(X, p, n{-}1)$, with all p-tuples of pairwise disjoint subsets of X, not all of them empty, as vertices. It is well-known that $\mathrm{ind}_{\mathbf{Z}_p}(\mathsf{K}_0) = n{-}1$ (for those familiar with deleted joins, we remark that we remark that K_0 is the first barycentric subdivision of the p-fold 2-wise deleted join of the $(n{-}1)$-simplex—see e.g. [13]). In fact, K_0 is $\mathbf{Z}_p$-isomorphic to $\mathrm{sd}(\mathsf{E}_{n-1,p})$: the isomorphism $\varphi: V(\mathrm{sd}(\mathsf{E}_{n-1,p})) \to V(\mathsf{K}_0)$ is given by $\{(j_1, i_1), (j_2, i_2), \ldots, (j_q, i_q)\} \mapsto (A_1, A_2, \ldots, A_p)$, where $A_i = \{j_t : i_t = i, t = 1, 2, \ldots, q\}$.

Let K_1 be the subcomplex of K_0 with $V(\mathsf{K}_1) = V(\mathsf{K}_0) \setminus V(\mathsf{K})$ and with simplices inherited from K_0. The vertices of K_1 are p-tuples $(A_1, \ldots, A_p)$ of disjoint sets with $|\bigcup_{i=1}^{p} A_i| \le n-\ell-1$, and $\mathrm{ind}_{\mathbf{Z}_p}(\mathsf{K}_1) \le \dim(\mathsf{K}_1) = n-\ell-2$. We have $\mathsf{K}_0 \subseteq \mathsf{K} * \mathsf{K}_1$, and so by (1)

$$\mathrm{ind}_{\mathbf{Z}_p}(\mathsf{K}) \ge \mathrm{ind}_{\mathbf{Z}_p}(\mathsf{K}_0) - \mathrm{ind}_{\mathbf{Z}_p}(\mathsf{K}_1) - 1 = n - 1 - (n - \ell - 2) - 1 = \ell.$$

Since we have constructed the $\mathbf{Z}_p$-map $f: \mathsf{K} \to \mathsf{L}$, we have $\ell \le \mathrm{ind}_{\mathbf{Z}_p}(\mathsf{K}) \le \mathrm{ind}_{\mathbf{Z}_p}(\mathsf{L}) \le m(p-1)$. This proves Theorem 1.1 for all prime r.

The non-prime case is handled by a simple combinatorial argument, which is given in [10] and which we omit. $\square$

Remark. As we have seen, the simplicial complex K_0 is the subdivision of $E_{n-1,p}$; in particular, for $p = 2$, it is an $(n-1)$-sphere. The subcomplex K_1 is the subdivision of the $(n-\ell-2)$-skeleton of $E_{n-1,p}$. For $p = 2$, the simplicial complex K also has a nice interpretation (noted by G. Ziegler): it can be regarded as the subdivision of the ℓ-skeleton of the cube $[0,1]^n$ i(nterpreted as a cell complex, with faces being the usual faces of the cube, i.e. cubes of various dimensions). Indeed, a vertex (A, B) of K can be encoded by a sequence $v \in \{0, 1, *\}^X$, where $v_x = 0$ if $x \in A$, $v_x = 1$ if $x \in B$, and $v_x = *$ otherwise. Each such v specifies a face of the n-cube.

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