Lower bound on the minus-domination number*

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Abstract

For a graph G, a function $f : V(G) \to \{-1, 0, +1\}$ is called a *minus-dominating function* of G if the closed neighborhood of each vertex of G contains strictly more 1's than -1's. The *minus-domination number* $\gamma^-(G)$ of G, as defined by Henning and Slater, is the minimum, over all minus-dominating functions f of G, of $\sum_{v \in V(G)} f(v)$. As observed by Füredi and Mubayi, a well-known probabilistic bound for the size of a transversal of a set system implies that $\gamma^-(G) = O\left(\frac{n}{r} \log r\right)$ for any graph G on n vertices of minimum degree r.

We prove that there exist r-regular *multigraphs* G on n vertices, in which each vertex has at least $\frac{r}{2}$ distinct neighbors, and such that $\gamma^-(G) \geq c\,\frac{n}{r} \log r$ for some constant $c > 0$. (For a multigraph, the closed neighborhood of a vertex is considered as a multiset in the definition of a minus-dominating function.)

Keywords: domination in graphs, minus-domination number, regular graph, probabilistic method, hypergeometric distribution

1 Introduction

Let G be an (undirected) graph on n vertices. For a vertex $v \in V(G)$, the *closed neighborhood* $N[v]$ of v is the set consisting of v and all of its neighbors. A *minus-dominating function* of G is any function $f : V(G) \to \{-1, 0, +1\}$ such that for every vertex $v \in V(G)$, we have $f(N[v]) > 0$ (here and in the sequel, we use the notation $f(S) = \sum_{x \in S} f(x)$ for a subset S of the domain of f). The *minus-domination number* of G is defined as

$$\gamma^-(G) = \min\{f(V(G)) : f \text{ is a minus-dominating function of } G\}.$$

This variant of the usual domination number has recently been studied in several papers (e.g., [**?**], [**?**], [**?**]).

One of the main questions related to this notion is the largest possible value of $\gamma^-(G)$ for r-regular n-vertex graphs (or for n-vertex graphs of minimum degree r) in dependence on n and on r. By an easy double-counting argument, one can see that $\gamma^-(G) \geq \frac{n}{r+1}$ for any r-regular n-vertex graph G. On the other hand, as was observed by Füredi and Mubayi [**?**], $\gamma^-(G) = O(\frac{n}{r} \log r)$ for any n-vertex graph of minimum degree r. Indeed, consider the set system with $V(G)$ as the ground set and with the n closed neighborhoods $N[v]$, $v \in V(G)$, as sets. By a simple and well-known probabilistic

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argument, it is possible to pick a transversal of size $O(\frac{n}{r}\log r)$ for such a set system. The vertices of the transversal are assigned $+1$'s and the other vertices get 0's, which defines a minus-dominating function of G. Füredi and Mubayi [?] conjectured that this bound is asymptotically the best possible.

In this note, we partially confirm their conjecture. Namely, we show the existence of r-regular n-vertex *multigraphs* with minus-domination number at least $\Omega(\frac{n}{r}\log r)$. In the definition of a minus-dominating function, we consider the closed neighborhood of a vertex as a multiset (where each neighbor occurs with an appropriate multiplicity). That is, if v is connected to u by k edges, the value $f(u)$ is counted k times in $f(N[v])$. Of course, if we do not put any further conditions, it is trivial to construct an r-regular multigraph with a large minus-domination number: just partition the vertices into pairs and connect each pair by r edges (then the minus-domination number is $\frac{n}{2}$). To avoid such trivialities, we insist that each vertex has at least $\frac{r}{2}$ distinct neighbors. (Then the $O(\frac{n}{r}\log r)$ upper bound using the transversal argument applies.) We prove

Theorem 1.1 *There are constants C and $c > 0$ such that for all integers $r \geq C$ and for all n's that are multiples of $4r$, there exists a bipartite r-regular n-vertex multigraph G, in which each vertex has at least $\frac{r}{2}$ distinct neighbors and such that $\gamma^-(G) \geq c\frac{n}{r}\log r$.*

To prove the theorem, it suffices to treat the case $n = 4r$. The larger n's can be handled by putting together the appropriate number of disjoint copies of the multigraph for $n = 4r$. For a more convenient notation, we will write $2n$ instead of n, and we will aim at constructing a bipartite r-regular multigraph G with both classes of size $n = 2r$ and with $\gamma^-(G) = \Omega(\log r)$. (The constant 2 has no particular significance here.)

Remarks. The definition of the minus-domination number can be naturally extended to an arbitrary set system $\Sigma = (X, \mathcal{S})$. Namely, $f : X \to \{-1, 0, +1\}$ is a minus-dominating function of Σ if $f(S) > 0$ for each $S \in \mathcal{S}$. If Σ is a system of n sets, each of size at least r, on n points, then the argument with transversal still gives the $O(\frac{n}{r}\log r)$ upper bound for the minus-domination number $\gamma^-(\Sigma)$. Here a more or less standard probabilistic approach, which we now outline, gives the matching $\Omega(\frac{n}{r}\log r)$ lower bound.

First, let X be an n-point set, $n = 2r$, and let $\mathcal{R}$ be a system obtained by picking a random r-element subset of X independently $100n$-times. For a fixed mapping $f : X \to \{-1, 0, 1\}$, let P_f denote the probability of f being a minus-dominating function for $(X, \mathcal{R})$. Suitable estimates show that

$$\sum_f P_f \to 0$$

as $r \to \infty$, where the summation is over all mappings f such that $f(X) \leq c\log r$ with a sufficiently small constant $c > 0$. Consequently, there is a system Σ_0 of at most $100n$ sets of size r on n points with $\gamma^-(\Sigma_0) > c\log r$. The constant 100 can be made smaller, but the calculation doesn't work anymore when the number of sets is the same as the number of points.

For set systems, one can get around this by the following trick. Consider a set system Σ_0 with $100n$ sets on n points as above. To get a system of $101n$ sets of size r on $101n$ points, say, add $100n$ new points to X, partitioned into 200 sets of size r each (and choose the remaining $n - 200$ sets arbitrarily). Clearly, the resulting system still has γ^- at least of the order $\log r$. But this or similar tricks don't seem to work for the problem of minus-domination number for graphs.

The basic idea of the proof of Theorem **??** is the same as in the construction of the set system Σ_0 above, i.e. showing that for a suitable random graph, we have $\sum_f P_f \to 0$, where, again, P_f is the probability that f is a minus-dominating function and the sum extends over all mappings with at most $c \log r$ more $+1$'s than -1's. But to make the calculation work, we will need the graph to be r-regular. There are various models of random r-regular graphs, but all of those I have considered either produce multiple edges with high probability, or they appear hopelessly complicated for calculations, or they are not "random enough" for the purposes of the above argument. Thus, the argument currently yields multigraphs only.

2 Preliminaries on the hypergeometric distribution

Here we will recall some estimates for hypergeometric distributions, following the treatment in [**?**].

Let k, m, and N be positive integers with $\max(k,m) \le N$. We have N urns, labeled 1 through N, and we put m balls into m different urns at random (draws without replacement). Some k of the urns are "distinguished," and we let X denote the number of balls in the distinguished urns.

The random variable X has expectation $\mathbf{E}X = \frac{km}{N}$ and variance

$$\operatorname{Var} X = \frac{km(N-k)(N-m)}{N^2(N-1)} \le \frac{km}{N} = \mathbf{E}X. \tag{1}$$

By results of [**?**], X has the same distribution as the sum of certain k independent indicator random variables (i.e. attaining only values 0 and 1). Consequently, various tail estimates like Bernstein's inequality, involving the expectation and the variance, can be applied for X. Specifically, we will need

$$\Pr[\,|X - \mathbf{E}X| \ge t\,] \le 2\exp\left(-\frac{t^2}{2(\operatorname{Var} X + t/3)}\right). \tag{2}$$

We will also need lower bounds for the tail estimates.

Lemma 2.1 *Let X be a sum of independent 0/1 random variables and let $\sigma = \sqrt{\operatorname{Var} X} \ge 200$. Then for any $t \in [0, \frac{\sigma^2}{100}]$, we have*

$$\Pr[X \ge \mathbf{E}X + t] \ge ce^{-t^2/3\sigma^2}$$

for a suitable constant $c > 0$.

Proof. We use a (much more precise) estimate due to Feller [**?**]. Let us write $x = \frac{t}{\sigma}$. Feller proves (eq. (2.10)) that for X as in the lemma (actually, under much more general conditions) and for $0 \le t < \sigma^2/12$,

$$\Pr[X \ge \mathbf{E}X + t] = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{x} \cdot e^{-\frac{x^2}{2}(1+Q(x))} \left[1 - \frac{\xi}{x^2} + \sqrt{2\pi}\,\theta\,\frac{x}{\sigma}\right], \tag{3}$$

where θ is a quantity (dependent on x) with $|\theta| < 9$, $\xi = \xi(x) \in (0,1)$, and $Q(x) = \sum_{\nu=1}^{\infty} q_\nu x^\nu$, with some $q_\nu < \frac{1}{7}\left(\frac{12}{\sigma}\right)^\nu$. For $x \le \frac{\sigma}{100}$, we have $|Q(x)| \le \frac{1}{7}\sum_{\nu=1}^{\infty}\left(\frac{12x}{\sigma}\right)^\nu \le \frac{1}{7}\sum_{\nu=1}^{\infty}\left(\frac{12}{100}\right)^\nu \le 0.02$.

First consider the case $2 \leq x \leq \frac{\sigma}{100}$. Then $1 - \xi/x^2 + \sqrt{2\pi}\,\theta\,\frac{x}{\sigma} \geq \frac{1}{2}$, and so $\Pr[X \geq \mathbf{E}X + t] \geq \frac{1}{6x}\, e^{-0.49x^2}$. This implies the bound in the lemma for $x \geq 2$ (with a sufficiently small $c > 0$), and the case $x \leq 2$ follows from the result for $x = 2$ by monotonicity (perhaps taking c still smaller). $\square$

3 Proof of Theorem ??

As was remarked below Theorem **??**, we set $n = 2r$, and we construct an r-regular bipartite multigraph with both classes of size n. We use the so-called *configuration model* (see e.g. Bollobás [**?**]) to produce such a multigraph at random. That is, we take a set V of $n = 2r$ vertices $v_1, \ldots, v_n$ and a set U of n vertices $u_1, \ldots, u_n$. We imagine that each v_i has r "paws" numbered 1 through r, and similarly each u_j has r numbered paws. We choose a random perfect matching between the set of the the nr paws of the v_i's and the set of the nr paws of the u_j's, and for each pair of matched paws, we connect the corresponding vertices by an edge. In this way, each v_i and each u_j has degree exactly r, but with high probability, multiple edges will arise.

Note that the probability that some vertex of the above random multigraph has at most $\frac{r}{2}$ different neighbors is bounded from above by

$$4r\binom{2r}{r/2}\frac{\frac{r^2}{2}(\frac{r^2}{2}-1)\cdots(\frac{r^2}{2}-r+1)}{2r^2(2r^2-1)\cdots(2r^2-r+1)} \leq 4r(4e)^{r/2}4^{-r} \leq 4r(0.7)^{r/2},$$

and so, almost surely, each vertex has at least $\frac{r}{2}$ distinct neighbors for r sufficiently large.

Next, we want to "kill" all the potential minus-dominating functions. Let us put $\Delta = c\log r$ for a small positive constant c. For a parameter a, $\Delta \leq 2a + \Delta \leq n$, by an *$a$-minuses mapping* on U we mean a mapping $f : U \to \{-1, 0, 1\}$ attaining exactly a values -1, $a + \Delta$ values $+1$, and the remaining $n - 2a - \Delta$ values 0. Similarly we define a b-minuses mapping on V. To prove Theorem **??**, it suffices to show that

$$\sum_{f,g} \Pr\left[f \cup g \text{ is a minus-dominating function of } G\right] \to 0$$

for $r \to \infty$, where the sum is over all a-minuses mappings f on U and b-minuses mappings g on V, $\Delta \leq 2a + \Delta \leq n$, $\Delta \leq 2b + \Delta \leq n$. Indeed, if $h : U \cup V \to \{-1, 0, 1\}$ is any minus-dominating function of G, we first note that by the r-regularity of G, we have $h(U) \geq 0$ and $h(V) \geq 0$. If we assume $h(U \cup V) \leq \Delta$, then $h(U) \leq \Delta$ and $h(V) \leq \Delta$ too, and we may actually suppose that $h(U) = h(V) = \Delta$ (by changing the appropriate number of 0's or -1's to $+1$'s). Hence, we may assume that h restricted to U is an a-minuses mapping and h restricted to V is a b-minuses mapping for suitable a and b.

First, we estimate the number of mappings we have to deal with.

Lemma 3.1 *The number of a-minuses mappings on U is bounded by $n^{3\Delta}$ for $a \leq \Delta$, and by*

$$\left(\frac{3n}{a}\right)^{3a}$$

for $a > \Delta$.

Proof. The number of a-minuses mappings is $\binom{n}{a}\binom{n-a}{a+\Delta}$. The bound for $a \le \Delta$ is immediate. For $a > \Delta$, we use the estimate $\binom{n}{k} \le \left(\frac{en}{k}\right)^k$, obtaining

$$\binom{n}{a}\binom{n-a}{a+\Delta} \le \left(\frac{en}{a}\right)^a \left(\frac{en}{a}\right)^{2a} < \left(\frac{3n}{a}\right)^{3a}.$$

□

Next, we assume that an a-minuses mapping f on U and a b-minuses mapping g on V are fixed, and we want to estimate the probability that $f \cup g$ is a minus-dominating function for the random multigraph G. Put $U^+ = f^{-1}(+1)$, $U^- = f^{-1}(-1)$, and similarly for V^+, V^-. Thus, we have $|U^-| = a$, $|U^+| = a + \Delta$, $|V^-| = b$, $|V^+| = b + \Delta$.

We distinguish two cases: first, when both $a \le K\Delta$ and $b \le K\Delta$, and second, when at least one of a, b exceeds $K\Delta$. Here K is a suitable large constant. The constant c in $\Delta = c \log r$ will be determined in dependence on K.

The case $a, b \le K\Delta$. We put $V_1 = V \setminus V^+$. We observe that if $f \cup g$ is a minus-dominating function of G, then any vertex $v \in V_1$ must have a neighbor in U^+. Therefore, U^+ is a transversal for the neighborhoods of all vertices in V_1. We have $|U^+| \le (K+1)\Delta$ and $|V_1| \ge n - (K+1)\Delta \ge \frac{n}{2}$. For technical convenience, we pick a subset $V_0 \subseteq V_1$ of exactly $\frac{n}{2}$ elements. The following lemma is used to deal with the first case:

Lemma 3.2 *Let $U^+ \subset U$ be a set of $(K+1)\Delta$ vertices, and let $V_0 \subset V$ be a set of $\frac{n}{2} = r$ vertices. Then the probability that each vertex of V_0 has a neighbor in U^+ is at most $e^{-\sqrt{r}}$ (provided that r is sufficiently large).*

This easy lemma is proved in Section **??** below. To handle the case $a, b \le K\Delta$, we note that there are at most $(K\Delta)^2 n^{O(\Delta)} = e^{O(\log^2 r)}$ choices for $f \cup g$ by Lemma **??**, and therefore the probability that any $f \cup g$ with $a, b \le K\Delta$ is minus-dominating tends to 0.

The case $a > K\Delta$. It remains to deal with the case when $a > K\Delta$ or $b > K\Delta$. By symmetry, it suffices to suppose $a > K\Delta$ (while losing a factor of at most 2 in the probability estimate). Unlike to the previous case, here we will only consider the values of the mapping f and ignore g completely. Call a vertex $v \in V$ *strictly negative* if it has strictly more edges going to U^- than to U^+. We want to show that the probability of no vertex being strictly negative is overwhelmingly small. This is formulated in the following lemma (the bounds are far from the truth in some ranges of the parameters but sufficient for our purposes). For a simpler notation, we will use the same constant K in distinguishing several more cases, although there is no intrinsic reason to use the same constant.

Lemma 3.3 *Let $|U^-| = a > K\Delta$ and $|U^+| = a + K\Delta$. Then the probability that there is no strictly negative vertex in V is bounded by*

$$\begin{array}{ll} e^{-\sqrt{r}} & \text{for } \Delta K < a \le K\Delta^2, \\ e^{-\delta r} & \text{for } K\Delta^2 < a \le \frac{r}{K}, \\ \varepsilon^r & \text{for } a > \frac{r}{K}, \end{array}$$

where $\delta > 0$ is a specific positive constant (independent of K), and $\varepsilon > 0$ can be chosen as small as desired, provided that $r > r_0(\varepsilon)$ for a suitable $r_0(\varepsilon)$.

Given these estimates and Lemma **??**, the proof of Theorem **??** is now finished by a very simple calculation. □

4 Proofs of the lemmas

Proof of Lemma ??. Let P denote the set of the paws of the vertices of U (each vertex has r numbered paws), let P^+ be the paws of the vertices of U^+, and Q_0 the paws of V_0. The matching of the paws can be viewed as choosing a random injective mapping $\eta : Q_0 \to P$, and we want to estimate the probability of each $v \in V_0$ having at least one paw mapped to P^+. Let $V_0 = \{v_1, v_2, \ldots, v_r\}$, and let A_i denote the event "at least one paw of v_i is mapped to P^+." We want to estimate

$$\Pr\left[A_1 \cap A_2 \cap \ldots \cap A_r\right] = \prod_{i=1}^{r} \Pr[A_i \,|\, A_1 \cap \ldots \cap A_{i-1}].$$

For each of the conditional probabilities on the r.h.s., we upper-bound the complementary probability

$$\Pr\Big[\overline{A_i} \,\Big|\, A_1 \cap \ldots \cap A_{i-1}\Big]. \tag{4}$$

We imagine that the random mapping $\eta : Q_0 \to P$ is chosen in r stages; at the ith stage, the r paws of v_i are mapped to P_i, the set of the paws in P not used in the previous $i-1$ stages (i.e. by any of the vertices $v_1, v_2, \ldots, v_{i-1}$). Further, let $P_i^+ = P^+ \cap P_i$. We have $|P_i| \geq nr - (i-1)r \geq r^2$ and $|P_i^+| \leq (K+1)\Delta r$. Hence, the conditional probability (**??**) that none of the r paws of v_i gets mapped to P_i^+ is at least

$$\begin{aligned}
\frac{\binom{r^2-(K+1)\Delta r}{r}}{\binom{r^2}{r}} &= \frac{r^2-(K+1)\Delta r}{r^2} \cdot \frac{r^2-(K+1)\Delta r-1}{r^2-1} \cdots \frac{r^2-(K+1)\Delta r-r+1}{r^2-r+1} \\
&\geq \left(\frac{r^2-(K+1)\Delta r - r}{r^2-r}\right)^r \geq \left(1 - \frac{2K\Delta}{r}\right)^r \geq e^{-3K\Delta} \\
&= e^{-3Kc\log r} \geq r^{-1/2}
\end{aligned}$$

for $c \leq \frac{1}{6K}$. Consequently, we get

$$\Pr\left[A_1 \cap A_2 \ldots \cap A_r\right] \leq \left(1 - r^{-1/2}\right)^r \leq e^{-\sqrt{r}}.$$

Lemma **??** is proved. □

Proof of Lemma ??. For technical reasons, we will only consider half of the vertices in V, i.e. the set $V_0 = \{v_1, v_2, \ldots, v_r\}$. Let A_i denote the event "v_i is *not* a strictly negative vertex." Thus, we want to upper-bound the probability of $A_1 \cap A_2 \cap \ldots \cap A_r$.

In the proof, we will need to distinguish two cases, namely $a \leq \frac{r}{K}$ and $a > \frac{r}{K}$. We begin with the first case (of not too large a). The second case, when a is close to r, will be treated similarly, but we will need an extra trick to get a sufficiently good estimate. So, for the time being, we assume $a \leq \frac{r}{K}$.

As in the previous lemma, we let Q_0 be the paws of V_0, and let P, P^+, P^- be the paws of U, U^+, U^-, respectively. As before, we consider a random injective map $\eta : Q_0 \to P$, and we imagine that it is chosen in r stages, the r paws of v_i being mapped at the ith stage. We also use the notation P_i, P_i^+, P_i^- for the paws of P, P^+, P^- not used at stages 1 through $i-1$, respectively.

Essentially, we want to estimate the probability of A_i conditioned on $A_1 \cap \ldots \cap A_{i-1}$. Note that under the condition $A_1 \cap \ldots \cap A_{i-1}$, we have $|P_i^+| - |P_i^-| \leq |P^+| - |P^-| \leq r\Delta$. But the estimates would go wrong if the paws of P^- were exhausted too quickly during the first stages.

Thus, we introduce the auxiliary event B_i, standing for "$|P_i^-| \geq \frac{1}{4}ar$ and $|P_i^+| \geq \frac{1}{4}ar$." We have $\Pr[\,\overline{B_i}] \leq \Pr[\,\overline{B_r}]$; let us denote the latter probability by q. The random variable $|P^- \setminus P_r^-|$ has hypergeometric distribution with $N = nr = 2r^2$ urns, $m = r^2$ balls, and $k = ar$ distinguished urns (see Section **??**). Its expectation is $km/N = \frac{1}{2}ar$, the variance is $O(ar)$, and consequently, the probability $\Pr[|P_r^-| < \frac{1}{4}ar] = \Pr[|P^- \setminus P_r^-| > \frac{3}{4}ar]$ is bounded by $e^{-\Omega(ar)} = e^{-\Omega(r \log r)}$ using (**??**). Similar considerations apply to the size of P_r^+. Thus, $\Pr[\,\overline{B_i}\,] \leq q \leq e^{-\Omega(r \log r)}$.

In the sequel, we will be able to upper-bound the probabilities $p_i = \Pr[A_i \,|\, A_1 \cap \ldots \cap A_{i-1} \cap B_i]$ for all i. We need to check that the additional conditioning on B_i doesn't hurt. By induction, we show that

$$\Pr[A_1 \cap \ldots \cap A_i] \leq iq + \prod_{j=1}^{i} p_j.$$

We have

$$\begin{aligned}
\Pr[A_1 \cap \ldots \cap A_i] &\leq \Pr[A_1 \cap \ldots \cap A_i \cap B_i] + \Pr[\,\overline{B_i}] \\
&\leq \Pr[A_i \,|\, A_1 \cap \ldots \cap A_{i-1} \cap B_i] \Pr[A_1 \cap \ldots \cap A_{i-1} \cap B_i] + q \\
&\leq p_i \Big((i-1)q + \prod_{j=1}^{i-1} p_j \Big) + q \\
&\leq iq + \prod_{j=1}^{i} p_j .
\end{aligned}$$

Hence,

$$\Pr[A_1 \cap \ldots \cap A_i] \leq e^{-\Omega(r \log r)} + \prod_{j=1}^{r} \Pr[A_j \,|\, A_1 \cap \ldots \cap A_{j-1} \cap B_j]. \tag{5}$$

Next, we want to upper-bound the conditional probability $\Pr[A_i \,|\, A_1 \cap \ldots \cap A_{i-1} \cap B_i]$. Since this probability is close to 1, we will actually lower-bound its complement. So we have a set P_i with two disjoint subsets P_i^+ and P_i^-, and we know that

$$\begin{array}{rcccl}
r^2 & \leq & |P_i| & \leq & 2r^2 \\
\frac{1}{4}ar & \leq & |P_i^-| & \leq & ar \\
\frac{1}{4}ar & \leq & |P_i^+| & \leq & ar + \Delta r.
\end{array}$$

We consider a random injective mapping of the r paws of v_i to P_i, and we want to lower-bound the probability that more paws are mapped to P_i^- than to P_i^+.

First, we want to restrict the problem to $P_i^+ \cup P_i^-$. Let X denote the number of paws mapped into $P_i^+ \cup P_i^-$. This corresponds to the urn scheme in Section **??** with $N = |P_i|$ urns, $k = |P_i^+ \cup P_i^-|$ distinguished urns, and $m = r$ balls. We get $\mathbf{E}X = km/N \in [\frac{a}{4}, 3a]$ and $\operatorname{Var} X \leq \mathbf{E}X \leq 3a$. By the tail estimate (**??**) in Section **??**, we see that with probability at least $e^{-\Omega(a)} \geq \frac{1}{r}$, X is within $\frac{a}{8}$ of its expectation. Consequently, if C_{ij} denotes the event of exactly j paws of v_i going to $P_i^+ \cup P_i^-$, we have

$$\Pr[A_i \,|\, A_1 \cap \ldots \cap A_{i-1} \cap B_i] \leq \frac{1}{r} + \max_{|j - \mathbf{E}X| \leq a/8} \Pr[A_i \,|\, A_1 \cap \ldots \cap A_{i-1} \cap B_i \cap C_{ij}]. \tag{6}$$

With the number of paws going to $P_i^+ \cup P_i^-$ fixed to some j, $\frac{a}{8} \leq j \leq 4a$, we can restrict our attention to the random variable Y, which is the number of paws going to

P_i^- for a random injective mapping of j paws into $P_i^+ \cup P_i^-$. This again corresponds to the urn scheme, this time with $N = |P_i^+ \cup P_i^-| \le 2|P_i^-| + r\Delta$ urns, $k = |P_i^-| \ge \frac{1}{4}ar$ distinguished urns, and $m = j$ balls. This time we have

$$\mathbf{E}Y = \frac{km}{N} \ge \frac{km}{2k + r\Delta} > \frac{m}{2} - \frac{\Delta r}{2k} \ge \frac{m}{2} - \frac{2\Delta m}{a} \ge \frac{m}{2} - 8\Delta.$$

We are interested in lower-bounding the probability $\Pr[Y > \frac{m}{2}]$. This means a positive deviation from the expectation by no more than 8Δ. Since $m = o(N)$, $k = O(ar)$, and $N \ge k + \frac{1}{4}ar$, from the expression (**??**) for the variance of a hypergeometric random variable we get

$$\operatorname{Var} Y \ge \frac{km}{N} \cdot \frac{N-m}{N} \cdot \frac{N-k}{N} \ge \frac{km}{N} \cdot \Omega(1) = \Omega(m) = \Omega(a),$$

where the constant hidden in the Ω notation is absolute (independent of K). We now use Lemma **??** with $t = 8\Delta$ and $\sigma = \Omega(\sqrt{a})$. Since $a > K\Delta$, the assumption $t \le \sigma^2/100$ holds. For $a > K\Delta^2$, we have $t/\sigma < 1$ and hence $\Pr[Y \ge \mathbf{E}Y + t]$ is at least a small constant. For $K\Delta < a \le K\Delta^2$, we have $t/\sigma \le \sqrt{\Delta}$, and so $\Pr[Y \ge \mathbf{E}Y + t] = \Omega(e^{-t^2/3\sigma^2}) = \Omega(e^{-\Delta}) \ge r^{-1/2}$, say (for r large). In both cases, the additional contribution of $\frac{1}{r}$ in (**??**) is negligible. Finally, plugging the resulting estimates for $\Pr[A_i \,|\, A_1 \cap \ldots \cap A_{i-1} \cap B_i]$ into (**??**), we get the assertions of Lemma **??** for the case $a \le \frac{r}{K}$.

It remains to deal with the case $a > \frac{r}{K}$. Here we would need to bound the quantity p_i in the previous part of the proof, i.e. the conditional probability of v_i not being strictly negative, by a very small constant (at least that is what we need for $a \approx r$). But such a bound doesn't hold in general, since for $a \gg \Delta^2$, the probability of v_i being strictly negative is close to $\frac{1}{2}$.

We circumvent this as follows. If Y denotes, as above, the number of paws of v_i going to P_i^-, we observe that Y has standard deviation much larger than Δ, and hence the probability of Y deviating from its expectation by considerably more than Δ is fairly close to 1. A positive deviation, which is what we need, has probability close to $\frac{1}{2}$ only. But the deviation cannot be large and negative all the time since we only have Δr more paws in P^+ than in P^-.

To make this precise, call an index $i \in \{1, 2, \ldots, r\}$ *restricted* if the number of paws of v_i going to P_i^+ is by at most $K\Delta$ larger than the number of those going to P_i^-. If no v_i is strictly negative, then there are at most $\frac{r}{K}$ indices that are not restricted.

Let $I \subset \{1, 2, \ldots, r\}$ be a fixed set of $\frac{r}{K}$ indices. For $i \notin I$, let A_i^I be the event "A_i occurs and i is restricted," and for $i \in I$, let $A_i^I = A_i$. In other words, for $i \notin I$, A_i^I means $\frac{m}{2} - K\Delta \le Y \le \frac{m}{2}$, where Y is the number of paws of v_i going to P_i^- and m is the number of paws of v_i going to $P_i^+ \cup P_i^-$.

We now do the considerations of the previous part of the proof with the A_i^I's instead of the A_i's. To get a sufficiently good resulting bound, it is enough to show that for $i \notin I$,

$$\Pr[A_i^I \,|\, A_1^I \cap \ldots \cap A_{i-1}^I \cap B_i] = o(1). \tag{7}$$

As before, we may restrict ourselves to the cases when m, the number of v_i's paws going to $P_i^+ \cup P_i^-$, is fixed and it is close to its expectation (and consequently $m = \Theta(a)$). Then, as above, $\operatorname{Var} Y = \Omega(a)$. To prove (**??**), it suffices to show that the probability of Y falling into any given interval of length $K\Delta = o(\sigma)$ is $o(1)$, where $\sigma = \sqrt{\operatorname{Var} Y} = \Omega(\sqrt{a})$. By the Central Limit Theorem, the appropriately normalized distribution function of Y converges to the distribution function of the normal distribution $N(0,1)$ for $r \to \infty$.

For the normal distribution $N(0,1)$, any interval of length $o(1)$ only contains $o(1)$ of the total probability, and hence an interval of length $o(\sigma)$ contains $o(1)$ of the total probability of Y. (Quantitative bounds can be obtained from the results of Feller [**?**] mentioned in Section **??**.)

Thus, we have proved that (**??**) holds for all $i \notin I$, and from this we get $\Pr[A_1^I \cap \ldots \cap A_r^I] \leq o(1)^r$. We know that if $A_1 \cap \ldots \cap A_r$ occurs, then $A_1^I \cap \ldots \cap A_r^I$ occurs too for some choice of the set I of at most $\frac{r}{K}$ indices. The number of choices of I is $\binom{r}{r/K} \leq O(K)^r$, and so we finally get that $\Pr[A_1 \cap \ldots \cap A_r] \leq o(1)^r$. This concludes the proof of Lemma **??** for the case $a > \frac{r}{K}$. $\square$

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