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The Coloring Poset and its On - Line Universality

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Abstract

The coloring poset $\mathcal{C}$ is the class of all (finite) graphs together with the ordering given by the existence of homomorphism. Based on the study of extension properties of $\mathcal{C}$ we show that any poset can be *on-line* represented in $\mathcal{C}$. Particularly, this implies that the coloring poset $\mathcal{C}$ (of all finite graphs) is universal for all countable posets, a result obtained by category theory techniques by Hedrlín and Kučera.

1 Introduction

This is a paper on graph homomorphisms: given graphs $G = (V, E)$ and $G' = (V', E')$ a mapping $f : V \to V'$ is said to be a *homomorphism* if $\{f(x), f(y)\} \in E'$ whenever $\{x, y\} \in E$.

All graphs considered in this paper are considered undirected and finite. (But some other objects, like posets, may be countable.) While the finitness is in a way necessary (see remarks at the end of this paper) the undirected graph assumption should be seen as an indication of the strength of our method (besides the fact that undirected graphs present perhaps the most frequently studied combinatorial model).

From a combinatorial point of view the homomorphisms are related to various coloring problems extending the obvious (yet basic) observation that a graph G is homomorphic to the complete graph K_n if and only if its chromatic number $\chi(G)$ is at most k. Motivated by this a homomorphism $G \to H$ is also called a H-coloring. This notion not only covers some frequently studied graph invariants (like circular

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chromatic number, multicoloring, oriented coloring, T-coloring etc., see [24, 28, 22]) but also led to an interesting research on the boundary of theoretical computer science and combinatorics, see e.g. [9, 3, 2, 1].

This also motivated some peculiar terminology when for example the class of all H-colorable graphs is called a *colour class* often denoted by $\mathcal{C}_H$ or simply by $\to H$. Each colour class is determined by the target graph H and thus $\mathcal{C}_H$ is a subclass of $\mathcal{C}'_H$ if and only if $H \to H'$. It follows that the poset generated by the colour classes is closely related to the quasiordering of all graphs ordered by the inclusion. This leads to the following notion: We say that a graph G is a *core* graph if every homomorphism $G \to G$ is an automorphism. It is easy to see that every finite graph G contains (up to an isomorphism) uniquelly determined subgraph G' such that $G \to G'$ and G' is a core. Such a graph G' is called *core of* G. (Proof is easy: for a graph $G = (V, E)$ simply take the homomorphic image $f(G) = (f(V), \{\{f(x), f(y)\}; \{x, y\} \in E\})$ of a homomorphism $f : G \to G$ with the minimal number of vertices and observe that any two such homomorphic images are isomorphic; see [10]).

For cores the quasi order defined by the existence of homomorphism becames a *Coloring Poset* which will be denoted by $\mathcal{C}$, with the ordering denoted by $\leq$. Explicitely, for graphs G and H we have $G \leq H$ if and only if $G \to H$. We shall also refer to $\leq$ as *homomorphism order.*

The study of Coloring Poset $\mathcal{C}$ brings somehow a more general perspective and led to notions of *density*, *homomorphism duality* and others, see [13] for a recent servey. In fact $\mathcal{C}$ has been studied intensively even earlier from a different point of view as a special category: every poset is a *category* where for every pair of objects we have at most one morphism between them; such categories are called *thin* (see [21] and [13] for introduction to combinatorial aspects of category theory). Particularly, the following fundamental result has been proved by Z. Hedrlín and L. Kučera in the framework of the important project of embedding of categories (see [21] for a detailed account of this):

Theorem 1.1 *Every countable poset is an induced subposet of* $\mathcal{C}$.

More formally, for every countable poset (X, R) *we can find an injective mapping which assigns to every* $x \in X$ *a finite graph* G_x *such that* $(x, y) \in R$ *if and only if* $G_x \leq G_y$.

In the other words $\mathcal{C}$ is countable *universal* poset (for all countable posets).

Thus the poset $\mathcal{C}$ plays a similar role as the *Rado graph* $\mathcal{R}$ which is the countable universal graph. This property (of Rado graph) is easy to prove as the Rado graph has the *graph extension property*: for every finite graph G and any $x \in V(G)$ holds: any subgraph of $\mathcal{R}$ isomorphic to $G - x$ may be extended to a subgraph isomorphic to G. By the same token we can construct universal countable poset $\mathcal{U}$: we start with the singleton poset and having constructed a finite portition $\mathcal{U}_n$ of $\mathcal{U}$ we let $\mathcal{U}_{n+1}$ be the poset which is formed by all possible singleton extensions of of $\mathcal{U}_n$. $\mathcal{U}$ will be formed as the union of all $\mathcal{U}_n$. (This is a bit sletchy but this is fairly standard, see e.g. [4]. The class of all posets has (unbounded) extension property.)

However the situation is far from beeing so simple for poset $\mathcal{C}$. The poset $\mathcal{C}$ has substantially more algebraical structure than $\mathcal{U}$ (e.g. it is a lattice with respect to products and sums; it has also powers; all this will be explained in a greater detail below) and as a result of this the extension property does not hold for $\mathcal{C}$ in a very strong sense.

The proof of Theorem 1.1 is complicated. In fact it took several papers (see e.g. [6]) to achieve it (and the validity of Theorem 1.1 was a long-standing open problem, see e.g. [7, 21]). The solution was achieved only in the broader context of embedding of categories where one introduces several (many) intermediate steps. The whole proof takes cca 25 pages in the monography [21] and it is one of the main results of this book (and also [21] is the only available proof in print).

Here we give an independent and direct (and we believe a simpler) proof of Theorem 1.1 which grew out of the study of extension properties of $\mathcal{C}$.

Perhaps more importantly this proof does not use many ad hoc ideas and instead (as we are going to demonstrate, see e.g. Section 2) it is related to some of the fundamental properties of $\mathcal{C}$ - *density, product conjecture and extendability* of the class $\mathcal{C}$. It follows that the class of graphs which represent $\mathcal{C}$ can be built from general ("Erdős-type graphs") which can be generated at random.

The paper is organized as follows Section 2 we review the extension properties of $\mathcal{C}$ and relate it to *density* problems and the *product* (Hedetniemi - Lovász) *conjecture.* In Section 3 we define on line representability of posets (by $\mathcal{C}$ and show that this is equivalent to our problem. In Section 2 and 3 we also introduce ingredients which will be needed for our main result (Homomorphism Cancelation Lemma in Section 2 and Hereditary Suspension Graph in Section 3).

In Section 4 we prove the main result (stated there as Theorem 4.1) which has Theorem 1.1 as its corollary. Section 5 contains the statement of several remarks and extensions.

2 Extension and Cancelation Properties of $\mathcal{C}$

A poset $\mathcal{P}$ is said to be *k-extendable* (or to have *k-extension property*) if any its subposet P with k elements and for any poset P' with $k+1$ elements which contain P as an induced subposet (i.e. P' is a single poin extension of P) one can find a copy of P' in $\mathcal{P}$ which contains P(i.e. we can get P' from P by addition of one element of $\mathcal{P}$). $\mathcal{P}$ is said to be *extendable* (or to have *k-extension property*) if it is k-extendable for every positive k.

The k-extension properties of coloring poset $\mathcal{C}$ are interesting already for small values of k and they were studied under different names extensively. Perhaps the k-extendability is a convenient classification for this type of results. However here we give only examples which are related to our main theme.

Extending any poset given by graphs G_1, G_2 to a poset with graphs G_1, G_2, H where H is homomorphic to both G_1 and G_2 is easy: let H be categorical product $G_1 \times G_2$. Similarly, extending any poset given by graphs G_1, G_2 to a poset with graphs G_1, G_2, H where both G_1 and G_2 are homomorphic to H is easy: let H be the disjoint union $G_1 + G_2$.

However 2-extendability also includes the case of an extension of poset $G_2 \to G_1 \not\to G_1$ to a poset with graphs G_1, G_2, H where $H \to G_1$ while there is no homomorphism between G_2 and H.

H always exists and in fact it can be chosen with arbitrarily large girth. This result is called *Sparse Incomparability Lemma* and it was proved in [16] (and independently in [29], see [13]).

Let us consider 2-extendability for the case of an extension of poset $G_2 \to G_1 \not\to G_1$ to a poset with graphs G_1, G_2, H where $G_2 \to H \to G_1$. This is called *density problem* and the answer can be negative: consider $G_1 = K_1$ and $G_2 = K_2$. In such a case we say that the pair (G_1, G_2) forms a gap. Welzl [27] proved that for undirected graphs there are no other gaps. However gaps are abundant for oriented graphs (and relational structures) and they have been completely characterized only recently C. Tardif and the author [18], see also [17].

3-extendability includes the following situation: Given graphs G_1, G_2, G_3 such that $G_i \not\to G_3$, $i = 1, 2$, does there exists a graph H such that $H \to G_i$, $i = 1, 2$, while $H \not\to G_3$. In the other words we want $H \to G_1$, $H \to G_2$, and $H \not\to G_3$. However $H \to G_1$, $H \to G_2$ implies $H \to G_1 \times G_2$ and thus this is the same as to ask whether $G_1 \times G_2 \not\to G_3$ providing that $G_i \not\to G_3$, $i = 1, 2$. Such graphs G_3 are called *productive* [15] (as the class of all graphs G which are not homomorphic to G_3 is productive; sometimes the name *multiplicative* graphs is used). The explicite characterization of productive graphs is a very difficult problem already in the “simplest” instances $G_3 = K_k$. This particular case is known as the *Hedetniemi - Lovász problem*, see e.g. [23] for a survey. There are infinitely many non-productive graphs: these include of course graphs $G_1 \times G_2$ where G_1 and G_2 are not homomorphic to each other, but also particular instances of Kneser graphs (e.g $K\binom{3k}{k}$; see [25]), and infinitely many further examples. For example we can take graph $K_k \times G$ where G is any graph vertex critical graph with $\chi(G) > k$. In the graph $K_k \times G$ collapse the set $V(K_k) \times \{x\}$ into a single vertex, say x, for all x in an independent set A of the graph G. Call the resulting graph H. H is not productive. This seem to give the smallest non-productive (undirected) graph (with 23 vertices).

Among 3-extension properties the productivity is the only essential problem.

The categorical (or direct) product is denoted by $\times$, the product of more factors is denoted by $\prod_{i=1}^{n} G_i$ or $\prod_{i \in I} G_i$. The disjoint union of graphs G and H is denoted by $G + H$ and for more factors we use $\sum_{i=1}^{n} G_i$ or $\sum_{i \in I} G_i$.

Let us consider the following instance of $(m+n)$-extension property:

Given graphs $G_1, G_2, \ldots, G_m, H_1, H_2, \ldots, H_n$, $G_i \to H_j$ while $H_j \not\to G_i$ for every $1 \leq i \leq m, 1 \leq m \leq n$, we want to find a graph F such that $G_i \to F \to H_j$ simultaneously fro all $1 \leq i \leq m, 1 \leq m \leq n$.

Clearly for $m = n = 1$ this is the density problem and thus we can call the above problem (m,n)*-density problem*. The (m,n)-density problem is not possible to reduce to $(1,1)$-density problem by simply considering graphs $G = \sum_{i=1}^{n} G_i$ and $H = \prod_{i=1}^{n} H_i$ (because of the non-productivity properties of graphs H_i).

However we can modify one of the proofs of density (due to Nešetřil and Perles) ([14] and, implicitely, [16]) to yield this. We state the (m,n)-density result explicitely as it inspired the proof of the main result (Theorem 4.1) of this paper. (Another motivation came from unpatience that such a fundamental result as Theorem 1.1 did not have an accessible proof which we badly needed for a forthcomming book with P. Hell and X. Zhu).

Proposition 1 *((m,n)-density)*
Given graphs $G_1, G_2, \ldots, G_m, H_1, H_2, \ldots, H_n$, $G_i \to H_j$ while $H_j \not\to G_i$ for every $1 \leq i \leq m, 1 \leq m \leq n$. Then there exists a graph F such that $G_i \to F \to H_j$ and $H_j \not\to F \not\to G_i$ simultaneously fro all $1 \leq i \leq m, 1 \leq m \leq n$.

Proof. Put $a = \sum_{i=1}^{m} |G_i|$ and $b = \prod_{j=1}^{n} |V(H_j)|$ and let H be any graph with $\chi(H) > a^b$ and without odd cycles of length $\leq b$.

Define F by the following formula:

$$F = (\sum_{i=1}^{m} G_i) + ((\prod_{j=1}^{n} H_j) \times H) \tag{1}$$

Then the existence of homomorphism $G_i \to F \to H_j$ is clear while $H_j \not\to F$ follows by the girth assumption. $F \not\to G_i$ may be obtained as follows: To the contrary assume that $h : ((\prod_{j=1}^{n} H_j) \times H) \to \sum_{i=1}^{m} G_i$ and define the mappings $h_y : V(\prod_{j=1}^{n} H_j)) \to V(\sum_{i=1}^{m} G_i)$ by $h_y(x) = h(x,y)$. The mapping h_y need not be a homomorphism but if we interpret the mapping h_y as a color of the vertex y we get that there are vertices y, y' which form an edge in H such that $h_y = h_{y'}$. However then the mapping $g = h_y = h_{y'}$ is necessary a homomorphism $g : \prod_{j=1}^{n} H_j \to \sum_{i=1}^{m} G_i$ (as $\{g(x), g(x')\} = \{h_y(x), h_{y'}(x')\} \in E(F)$ for any edge $\{x, y\} \in E(\prod_{j=1}^{n} H_j)))$. ♡

The following result seems to be a usefull generalization of the previous Proposition 1 and of particular cases which appeared in e.g [5, 16, 17, 20]. A graph G is said to be *pointed* for a graph F if for any tho homomorphisms $f, f' : F \to G$ and for any vertex $x \in V(G)$ holds:

If $f(y) = f'(y)$ for every $y \neq x$ then also $f(x) = f'(x)$.

We have:

Theorem 2.1 *(Homomorphism Cancelation Property)*

Let F, G be graphs, let H be any graph with $\chi(H) > |V(H)|^{|V(G)|}$. Then

i. If there exists a homomorphism $h : G \times H \to F$ the there is a homomorphism $g : G \to F$.

ii. If G is connected and pointed for F then the homomorphism g is uniquelly determined by h.

Proof. The proof of *i.* is similar to the proof of Proposition 1.

To prove *ii.* it suffices to prove the following: if h_y and h_z are homomorphisms and $\{y, z\} \in E(H)$ then $h_= h_z$.

Thus assume to the contrary that there are homomorphisms $h_y, h_z : G \to F$ such that $h_y(x_0) \neq h_z(x_0)$ for a vertex $x_0 \in V(G)$. Define mapping $h : V(G) \to V(F)$ as $h(x) = h_y(x)$ for $x \neq x_0$ and $h(x_0) = h_z(x_0)$. Then h is a homomorphism $G \to H$. (It suffices to check edges of G incident with x_0: If $\{x, x_0\} \in E(G)$ then $\{(x, y), (x_0, z)\} \in E(G \times H)$ and thus $\{h_y(x), h_z(x_0)\} = \{h(x), h(x_0)\} \in E(F)$.) However this is a contrary with the fact that F is G-pointed. ♡

Note that if all degrees of G are > 2 and F has no C_4 then F is pointed for G, so the conditions of Theorem 2.1 are easy to satisfy.

We shall use Homomorphism Cancelation Property (HCP for short) 2.1 many times in this paper.

Also the following notion and property will be needed in the proof of Theorem 4.1.

A (finite or infinite)set $\{S_i : i = 1, 2, \ldots, n, \ldots\}$ of graphs is said to be *inverse set* if for any two distinct indices i, j holds: For $i < j$ there is no homomorphisms $S_i \to S_j$ while for $j > i$ here is a homomorphisms $S_j \to S_i$.

Proposition 2 *For every k and every increasing sequence $(l_i; i = 1, 2, \ldots)$ of integers ≥ 3 there exists an inverse set $\{S_i : i = 1, 2, \ldots\}$ of connected graphs and vertices $s(i) \in V(S_i)$ such that each S_i satisfies:*

i. $\chi(S_i) = k$;

ii. The odd girth of S_i is l_i;

iii. For every $j > i$ there exists a homomorphism $f : S_j \to S_i$ such that $f(s(j)) = s(i)$.

Proof.

We construct S_i on-line as follows.

Let $S_1, S_2, \ldots, S_{n-1}$ with vertices $s(1), s(2), \ldots, s(n-1)$ be already constructed inverse set. Put $a = \sum_{i=1}^{n-1} |G_i|$ and let H be any graph with $\chi(H) > a^a$ and without odd cycles of length $\leq a$. Define S_n by the formula:

$$S_n = S_{n-1} \times H, \tag{2}$$

Let $s(n)$ be any vertex in the set $\{s(n-1)\} \times V(H)$.

Then for $i < n$ there is no homomorphism $S_i \to S_n$ (by the odd girth assumption) and for $n > j$ the projection to S_{n-1} together with induction assumption gives a homomorphism $S_n \to S_j$. ♡

3 On - Line Representations

Whole America can be On - Line. Our goal is a bit more modest and it is captured by the following definiton:

By an *on-line representation* of a poset $\mathcal{P}$ we mean that one can construct a representation of $\mathcal{P}$ under the circumstances that the elements of $\mathcal{P}$ are revealed one by one (without a priori knowledge about the whole poset $\mathcal{P}$). (The on-line representability by paths was considered in [19].

The on-line representation of can be thought as a game of two players A and B (with usual names Bob and Alice). Bob (as usual - the destroyer) selects $\mathcal{P}$ and reveals the elements of $\mathcal{P}$ one by one to Alice. Whenever an element x of $\mathcal{P}$ is revealed, the relation among x and previously revealed elements is also revealed. Alice is required to contstruct a finite graph G_x. Alice wins if the graphs G_x which she constructed during the game represent poset $\mathcal{P}$.

Clearly this make sense both for finite and infinite poset $\mathcal{P}$. However we are assuming that infinite poset $\mathcal{P}$ is revealed in such a way that at each step the number of previously revealed points is finite. Thus we allow at each step only finite time - however time may run forever; clearly this amounts to saying that the points of $\mathcal{P}$ are indexed by natural numbers. It is easy to see that the Density Theorem [27, 17] implies that Alice can win for every linearly ordered set (she just has to avoid selecting the single jump). In fact the extension property (more precisely, k-extension property for every $k > 0$) guarantees that Alice can win. In fact she can even let Bob to choose the first graph! In fact she can win even if Bob is choosing all graphs G_x providing the choice is consistent with $\mathcal{P}$; there cannot be any trap. However, unlike graphs, $\mathcal{C}$ does not have extension property, and thus this formulation of the on-line representability (one which gives Alice more chances) is justified.

The advantage of on-line representability is that it reduces the infinite problem to finite case by the following theorem (compare [19]):

Proposition 3 *The following three statements are equivalent:*
i. Every countable poset is representable by $\mathcal{C}$*;*
ii. Every countable poset is on-line representable by $\mathcal{C}$*;*
iii. Every finite poset is on-line representable by $\mathcal{C}$*.*

Proof. Obviously *ii.* implies *i.*. Also *iii.* implies *ii.*, as at each step the revealed part of $\mathcal{C}$ is finite. Also *i.* implies *iii.* by an easy argument: Let $\mathcal{U}$ be universal countable poset (for countable posets). Let us consider a representation $x \to G_x$ of $\mathcal{U}$ and we let Alice to play at each step according to the representation of $\mathcal{U}$. As $\mathcal{U}$ has k-extension property for every k this procedure cannot end in a deadlock. ♡

Advancing the proof of Theorem 4.1 we introduce the following notation:

We shall deal with disconnected graphs with many components. These graphs will be built recursively ("on-line") and we shall have to preserve their hierarchical structure. We use the following model:

A *hierarchical structure* of a multiset $(G_i; i \in I)$ of graphs is the graph $\sum_{i\in I} G_i$ together with a system of subsets $(I_x; x \in \mathcal{S})$ which satisfy
i. Every $i \in I$ and I itself belongs to an $I(x)$ for $x \in \mathcal{S}$;
ii. For every $x, x' \in \mathcal{S}$ the sets $I(x)$ and $I(x')$ are either disjoint or in the inclusion (which means that the sets $(I(x); x \in \mathcal{S})$ may be visualized by a tree).

Such a hierarchical structure will be denoted by $((G_i; i \in I), \mathcal{S})$

For a vertex x of $\mathcal{S}$ we define *height* $h(x)$ inductively as follows: $h(x) = i+1$ if all sets $I(y) \subset I(x)$, $y \in \mathcal{S}$, satisfy $h(y) \leq i$ while there is a predecessor of y of height i.

If $((G_i; i \in I), \mathcal{S})$ and $((G'_i; i \in I'), \mathcal{S}')$ are hierarchical structures, then a homomorphism from $((G_i; i \in I), \mathcal{S})$ to $((G'_i; i \in I'), \mathcal{S}')$ is a pair (f, f_H) where f is a homomorphism

$$\sum_{i\in I} G_i \to \sum_{i\in I'} G'_i$$

and f_H is a mapping $f_H : \mathcal{S} \to \mathcal{S}'$ which satisfies the following condition:
i. h does not increase the heights of vertices (i.e. $h(f_H(x)) \leq h(x)$ for every $x \in \mathcal{S}$);
ii. For every point x of $\mathcal{S}$ f maps $\sum_{i\in I(x)} G_i$ to the graph $\sum_{i\in I(f_H(x))} G'_i$.

(In the other words, the homomorphism f preserves the hierarchical structure of the multiset $(G_i; i \in I)$.)

It is easy to code a hierarchical structure by *suspension graphs* and we do this inductively as follows:

First, let for every positive d be give a graph S_d with a specified vertex $s(d)$. (We already know by Proposition 2 how to construct inverse set on-line.)

If $\mathcal{S}$ is of height 1 then we let $(\sum(G_i : i \in I))_{HS}$ denote the graph which arises from $(\sum_{i\in I} G_i) + S_1$ adding disjoint paths of length 2 joining every vertex $x \in \cup_{i\in I} V(G_i)$ with the fixed vertex $s(1)$ of S_1.

If $\mathcal{S}$ is of height $d > 1$ then denote by C_- the set of predecessors of I (this is a deliberately clumsy notation; it is in accordance with the notation which we have to introduce below). We let

$$(\sum(G_i; i \in I))_{HS}$$

to denote the graph which arises from $(\sum_{x\in C_-}(\sum(G_i : i \in I_x))_{HS}) + S_r$ by adding disjoint paths of length $2d$ joining every vertex $x \in \cup_{i\in I} V(G_i)$ with the fixed vertex $s(d)$ of S_d.

(We may call $(\sum(G_i : i \in I))_{HS}$ the *hierarchical suspension graph*, the dependence on the graphs S_d is supressed as these graphs will be clear from the context.)

The following technical statement establishes the fact that homomorphisms between hierarchical structures are coded by homomorphisms between corresponding hierarchical suspension graphs:

Proposition 4 *Let $((G_i; i \in I), \mathcal{S})$ and $((G'_i; i \in I'), \mathcal{S}')$ be hierarchical structures, let every graph in $G_i, i \in I_x$ (or every graph in $G_i, i \in I_x$) with $h(x) = d$ has odd girth $\leq 2d + 1$. Consider the hierarchical suspension graphs $(\sum(G_i : i \in I))_{HS}$ and $(\sum(G'_i : i \in I'))_{HS}$. Suppose that there is no homomorphism between any G_i and S_d (for $i \in I$ and $d > 0$) and any G'_i and S_d (for $i \in I'$ and $d > 0$) , and suppose that graphs S_d ,$d = 1, 2, \ldots$ form an inverse set. Then for every homomorphism*

$$f : (\sum(G_i : i \in I))_{HS} \to (\sum(G'_i : i \in I'))_{HS}$$

there exists a mapping $f_H : \mathcal{S} \to \mathcal{S}'$ such that (f, f_H) is a homomorphism of hierarchical structures $((G_i; i \in I), \mathcal{S}) \to ((G'_i; i \in I'), \mathcal{S}')$.

Proof. The proof is easier than the statement. It follows from the assumption that the homomorphism f mapps every set $\cup_{i\in I(x)} V(G_i)$ to the set $\cup_{i\in I'(y)} V(G'_i)$ for some $y \in \mathcal{S}'$. As vertices of any copy of S_d are mapped to a copy of S'_d for $d' \leq d$. This imples that any suspension graph $(\sum(G_i : i \in I(x)))_{HS}$ is mapped to a suspension graph $(\sum(G'_i : i \in I'(y)))_{HS}$ where x and y are the vertices of T and T' respectively. This defines a mapping $f_H : \mathcal{S} \to \mathcal{S}'$ and it is easy to chcek that (f, f_H) is a homomorphism of hierarchical structures $((G_i; i \in I), \mathcal{S}) \to ((G'_i; i \in I'), \mathcal{S}')$. ♡

4 Posets On-Line

4.0.1 Construction

Let $\mathcal{P}$ be a given poset with points $\{1, 2, 3, \ldots\}$ and relation R. We shall define construct graphs $G_1, G_2, \ldots, G_n, \ldots$ by induction on line. This means that in the construction of G_n we are using only the properties of the poset $\mathcal{P}$ restricted to the set $\{1, 2, 3, \ldots, n\}$, this poset will be denoted $\mathcal{P}_n$

Put $G_1' = C_5 \oplus K_{k-3}$ and $G_1 = (K_k)_T$ ($\oplus$ denotes the complete join of two graphs). Define graphs $G_1', G_2', \ldots$ and $G_1, G_2, \ldots$ by induction. In the inductive step let graphs $G_1, G_2, \ldots G_{n-1}$ and $G_1', G_2', \ldots G_{n-1}'$ be given. Consider the point $n \in \mathcal{P}$ and denote by $C_-(n)$ the set of all points i of $\mathcal{P}_n$ which are covered by n (i.e. $C_(n)$ is the set of all i such that $(i, n) \in R$ and there is no j with $(i, j) \in R, (j, n) \in R$). Similarly, denote by $C_+(n)$ the set of all points of $\mathcal{P}_n$ which cover n.

If both sets $C_-(n)$ and $C_+(n)$ are empty (which corresponds to the case that the point n is isolated in $\mathcal{P}_n$) then let G_n' be any graph G which has girth $= 3$ and with $\chi(G) = k$ and for which there are no homomorphisms $G_i \to G$ and $G \to G_i$ for any $i < n$. This is easy to achieve, we may take any graph H with both its chromatic number and odd girth $> \max|V(G_i)|$. We let $G = G_n'$ to be the graph $H \star (I, a, b)$ which arises form H by replacing every edge of H by a copy of the graph $P_3 \oplus K_{k-3}$.

Thus let $|C_-(n)| + |C_+(n)| > 0$

Denote by $a = |V(G_1)| \times |V(G_2)| \times \ldots \times |V(G_{n-1})|$ and put $b = a^a$. Let H_n be any graph H with the following properties:

1. The odd girth of H is $2n + 1$ and every edge of H belongs to an odd cycle of length $2n + 1$;

ii. The chromatic number $\chi(H) > b$.

Similarly, let S_n be any graph with the following properties:

1. The odd girth of S_n is $2n + 3$;

ii. The chromatic number $\chi(S_n) = k + 1$.

We do not try to optimize at this point. However note that one can construct easily examples of graphs H_n and S_n (for example by the iteration of oriented line graphs - so called *shift* graphs).

Define graphs G_n' by the formula:

$$G_n' = \left(\left(\sum_{i \in C_-(n)} G_i' \right) + \left(\prod_{j \in C_+(n)} G_j' \right) \times H \right) \tag{3}$$

In order to construct G_n we visualize the formula defining G_n' componentwise (as indeed G_n' has many components) as multi-set $(K_i; i \in I)$ (i.e. K_i are all the components of G'_n). This set has two parts: $I = I' + I"$, where $I' = \sum_{i \in C_-(n)} I'(i)$. Denote by $(K_i; i \in I'(i), \mathcal{S}')$ the hierarchical structure inherited from the graph $G_i, i \in C_-(n)$.

The hierarchical structure of G_n will be defined as disjoint sum of hierarchical structures $((K_i; i \in I'(i)), \mathcal{S}')$ and further by the set I (at height n). We let $s(n) \in V(S_n)$ be joined by disjoint paths of length $2n$ to every vertex of $(\prod_{j \in C_+(n)} G'_j) \times H$.

This finishes the construction of the graph G_n.

Symbolically, this can be written as

$$G_n = \left(\sum_{i \in C_-(n)} G_i + (\prod_{j \in C_+(n)} G'_j) \times H) \right)_{HS} \tag{4}$$

This single formula expresses the on-line definition of G_n and ends this construction.

We shall prove the following

Theorem 4.1 *Let graphs $G_1, G_2, \ldots G_{n-1}$ represent a poset $\mathcal{P}_{n-1} = (\{1, 2, \ldots, n-1\}, R')$. Let $\mathcal{P}_n = (\{1, 2, \ldots, n\}, R)$ be any poset which extends P' (i.e. for which $P_n - \{n\} = P_{n-1}$). Then the graph G_n together with the graphs $G_1, G_2, \ldots G_n$ represent the poset $\mathcal{P}_n$.*

Proof.

Before disscussing the representability of $\mathcal{P}$ let us first remark that it follows from the properties of the graph S_i that $S_i \not\to G'_j$ for every $1 \le i, j < n$ (by *ii.*). We have also $G'_j \not\to S_i$ for every $1 \le i, j < n$; this follows for $j \le i$ by the girth assumption and for $j > i$ by Homomorphisms Cancelation Property of graphs H_i: If $G'_j \not\to S_i$ for $j > i$ then take that component of G'_j which has form $G'_l \times H_{l+1} \times H_{l+2} \times \ldots \times H_j$ and then by HCP applied consecutively for $j, j-1, \ldots, l$ we obtain that there exists a homomorphism $G'_l \to S_i$ which a contradiction with the girths assumption on G'_l and S_i (the graph G'_l was chosen at the moment when both sets $C_-(l)$ and $C_+(l)$ were empty and thus G'_l has girth 3.

This means that any homomorphism $f : G_i \to G_j$, $1 \le i, j \le n$ maps vertices of the graph G'_i to vertices of the graph G'_j and all suspension graphs isomorphic to S_l for $l \le i$ in the hierarchical structure of the graph G_i to those suspension graphs in the hierarchical structure of G_j which are isomorphic to the graphs S_l for $l \le i$ (this follows from the fact that suspension graphs form an independent family). As the homomorphism f when restricted to the suspension graphs maps a graph isomorphic to S_l to a graph isomorphic again to S_l, we see that the homomorphism f preserves the hierarchical structure of (components of) G'_i.

After these preparations we prove that the graphs $G_1, G_2, \ldots G_{n-1}, G_n$ represent $\mathcal{P}$.

Obviously $G_i \to G_n$ for every $i \in C_-(n)$ and thus also $G_i \to G_n$ for every i with $(i,n) \in R$.

Similarly $G_n \to G_i$ for every $i \in C_+(n)$ and thus also $G_n \to G_i$ for every i with $(n,i) \in R$.

Now suppose $G_{\check{i}} \to G_n$. As H_n has odd girth $> 2n+1$ and as every edge of $G_{\check{i}}$ is in an odd cycle of length $< 2n-1$, we know that no edge of $G'_{\check{i}}$ maps to

$$(\prod_{i \in C_+(n)} G'_i) \times H$$

and thus

$$G'_{\check{i}} \to \sum_{i \in C_-(n)} G_i$$

But then by the hierarchical suspension property (Proposition 4) we have that $(\check{i}, i) \in R$ for some $i \in C_-(n)$ and thus $(i,n) \in R$.

Finally suppose that $G_n \to G_{\check{i}}$. Put $C_+(n) = \{i_1^+ < i_2^+ < \ldots < i_s^+\}$ and let $i_j < \check{i} < i_{j+1}$ (we allow $j = s$ to cover the case $i_s < \check{i}$ and $j = 0$ to cover the case $\check{i} < i_1$).

By HCP 2.1 (as $|V(G_{\check{i}})| < a$ and $\chi(H) > b$) we get that

$$(\prod_{i \in C_+(n)} G'_i) \times H \to G'_{\check{i}}$$

implies

$$\prod_{i \in C_+(n)} G'_i \to G'_{\check{i}}$$

.

Now if $j = s$ then we are done as $i_s < \check{i}$ and thus the odd girth of $\prod_{i \in C_+(n)} G'_i$ is less than the odd girth of the graph $G'_{\check{i}}$ and we get a contradiction.

If $j < s$ then we proceede similarly. We shall illustrate it on the case $j = s-1$: We have

$$\prod_{i \in C_+(n)} G'_i \to G'_{\check{i}}$$

(and $|V(G_{\check{i}})| < a_s$) Now we apply HCP for graphs G'_i) to the product $\prod_{i \in C_+(n)} G'_i = \prod_{i=1}^{s} G'_i$ and we obtain

$$\prod_{i=1}^{s-1} G'_i \to G'_{\check{i}}$$

.

If $s-1 < \check{\imath}$ we are done as we get a contradiction as above. If $s-1 > \check{\imath}$ we can continue in the same way for $k = s-1, s-2, ..., j$ all the time applying HCP to the product $\prod_{i=1}^{k} G'_i$), finally obtaining

$$\prod_{i=1}^{s-1} G_i \to G'_{\check{\imath}}$$

which is a contradiction with the girth assumption. (If $\check{\imath} < i_1$ then we get a contradiction of $G_{i_1} \to G_{\check{\imath}}$ and odd girth.

Note that every edge of the graph G'_n belongs to an odd cycle of length $\leq 2n+1$. This proves the Theorem 4.1. ♡

Corollary 1 *The coloring poset $\mathcal{C}$ is universal (for countable posets).*

5 Concluding Remarks

The above proof of the universality of finite graphs and homomorphism order can be modified to get the following:

Theorem 5.1 *$\mathcal{C}$ contains universal poset $\mathcal{R}$ ($\mathcal{R}$ for rigid) such that for any two graphs $G, G' \in \mathcal{R}$ there exists at most one homomorphism $G \to G'$.*

Proof. (**a sketch**) We want to proof that all our graphs G_n (in the notation of Theorem 4.1) can be made rigid and there is at most one homomorphism between them. However, the inverse family $S_i, i = 1, 2, \ldots$ can be assumed to be rigid (and this was proved in [16]) and the unicity of homomorphisms $S_j \to S_i$ is claimed by HCP. Also our starting building blocks (i.e. graphs G_i correspnding the case $C_-(n) = C_+(n) = \emptyset$ can be made rigid (providing we take a rigid H then the graph $H \star (I, a, b)$, $I = P_3 \oplus K_{k-3}$, is also rigid. The graphs $G_i \times H$ can be made rigid (by suspension) The unicity of mappings between G_i and G_j is taken care by HCP and rigidity of S_i. This is all using we known techniques and the details will appear elsewhere. ♡

Corollary 2 *Every thin category may be represented by the category of graphs and homomorphisms.*

Our proof of the universality of $\mathcal{C}$ does not use any special (ad hoc) constructions. The building blocks are graphs with high chromatic number and with large odd girth (which can be changed to the large girth), and these building blocks can be chosen arbitrarily at random. (This yields that one can prove the universality of the coloring poset $\mathcal{C}$ for many restricted families of graphs.) In fact presently when proving some steps in the proof of Theorem 5.1 one relies on the probabilitic method.

For example it is not known how to construct uniquelly H-colorable graph with a large girth (see [29, 20].

However it is possible that even very simple families of graphs (such as bounded degree graphs, or even oriented paths; compare [19]) could represent every countable posets. Presently, there are no negative results in this direction.

Another related research is motivated by Müller's Theorem [12] where one tries to control homomorphism into *all* small graphs. This approach to [12] is taken in [20].

Finally, let us remark, that the problem of representing of infinite posets (of arbitrary cardinality) by infinite graphs and their homomorphisms has a positive solution only under certain set-theoretic axioms. Even the existence of a proper class of independent graphs (i.e. a representation of the discrete poset indexed by ordinals) is unknown, say, in ZFC. (See [21] for a disscussion of this.) Our method has also some implications for representation by infinite graphs. However this is a paper on finite combinatorics and we leave it at that.

References

[1] G.Brightwell, P.Winkler: Graph Homomorphisms and Phase Transitions, J. Comb. Th.B 77, (1999), 221-262.

[2] M. Dyer, C. Greenhill: The Complexity of Counting Graph Homomorphisms (submitted)

[3] T.Feder, M.Vardi: Monotonne monadic SNP and constraint satisfaction. In: Proceedings of the 25th ACM STOC, ACM, 1993,pp. 612-622

[4] R. Fraïssée: Theory of Relations, Studies in Logic and the Foundation of mathematics 118, Elsevier, 1986.

[5] D. Greenwell and L. Lovász, Applications of product coloring, Acta Math. Acad. Sci. Hungar. 25(1974), 335-340.

[6] Z. Hedrlín: On universal partly ordered sets and classes, J. Algebra 11 (1969), 503-509.

[7] Z. Hedrlín: Extensions of structures and full embedding of categories, Actes du Congrès Intern. de Math., Dunod, Paris, 1971, pp. 319-322.

[8] Z. Hedrlín, J. Lambek: How comprehensive is the category of semigroups?, J. Algebra 11 (1969), 195 - 212.

[9] P.Hell, J. Nešetřil: On the complexity of H-coloring, J. Comb. Th. B,48(1990),92-110

[10] P.Hell, J. Nešetřil: The core of a graph, Discrete Math. 109 (1992), 117-126

[11] B. Larose, C. Tardif: Hedetniemi's conjecture and the retracts of products of graphs, 15 pages, submitted, 1999.

[12] V. Müller, *On coloring of graphs without short cycles*, Discrete Math., 26(1979), 165-179.

[13] J. Nešetřil: Aspects of Structural Combinatorics, Taiwanese J. Math. 3, 4 (1999), 381 - 424.

[14] J. Nešetřil: The Homomorphism Structure of Classes of Graphs, Combinatorics, Probab. and Comp. 8 (1999), 177-184

[15] J. Nešetřil, A. Pultr: On classes of relations and graphs determined by subobjects and factorobjects, Discrete Math. 22(1978), 287-300

[16] J. Nešetřil, V.Rödl: Chromatically optimal rigid graphs, J. Comb. Th. B, 46(1989),133-141

[17] J. Nešetřil, C.Tardif: Density. In: Contemporary Trends in Discrete Mathematics (R.L. Graham, J. Kratochvíl, J. Nešetřil, F.S. Roberts, eds.), AMS, 1999, pp. 229-237

[18] J. Nešetřil, C.Tardif: Duality Theorems for Finite Structures (characterizing gaps and good characterizatios). KAM-DIMATIA Series, 98-407 (to appear J. Comb. Th. B)

[19] J. Nešetřil, X. Zhu: Path Homomorphisms, Proc. Camb. Phil. Soc., 120 (1996),207-220

[20] J. Nešetřil, X. Zhu: On Sparse Graphs with Given Colorings and Homomorphisms, KAM-DIMATIA Series 2000-456.

[21] A. Pultr, V. Trnková: Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories, North Holland, 1980.

[22] F. S. Roberts: T - colorings of Graphs, Recent Results and Open Problems, Discrete Math. 93(1991),229-245

[23] N. Sauer: Hedetniemi's Conjecture - a survey In: Combinatorics, Graph Theory, Algorithms and Applications (M.Fiedler, J. Kratochvíl, J. Nešetřil, eds.), North Holland (2000)

[24] E. Sopena: Oriented Chromatic Number. In: Combinatorics, Graph Theory, Algorithms and Applications (M.Fiedler, J. Kratochvíl, J. Nešetřil, eds.), North Holland (2000)

[25] C.Tardif, X. Zhu: The level of nonmultiplicativity of graphs (preprint 1999).

[26] P. Vopěnka, A. Pultr, Z. Hedrlín: A rigid relation exists on any set, Comm.Math.Univ.Carol. 6(1965), 149-155.

[27] E. Welzl: Color Families are Dense, J. Theoret. Comp. Sci. 17 (1982), 29-41

[28] X. Zhu: Circular chromatic number: a survey. In: Combinatorics, Graph Theory, Algorithms and Applications (M.Fiedler, J. Kratochvíl, J. Nešetřil, eds.), North Holland (2000)

[29] X. Zhu, Uniquely H-colorable graphs with large girth, J. Graph Theory, **23** (1996), 33-41.