Construction of Sparse Graphs with Prescribed Circular Colorings

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Abstract

This paper constructively proves the following result: Suppose $k \geq 3d-2$, $(k,d) = 1$, A is a finite set and $f_1, f_2, \cdots, f_n$ are mappings from A to $\{0, 1, \cdots, k-1\}$. Then for any integer l, there is a graph $G = (V, E)$ of girth at least l with $A \subset V$, such that G has exactly n (k,d)-colorings (up to a permutation of the colors) $g_1, g_2, \cdots, g_n$, and each g_i is an extension of f_i. This result generalizes a result of V. Müller who proved this for k-colorings. Note that for $n = 1$, the method presented in this paper gives a construction of uniquely (k,d)-colorable graphs. A probabilistic proof of the existence of uniquely (k,d)-colorable graphs was given in [17].

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1 Introduction

Suppose $G = (V, E)$ is a graph and k, d are positive integers with $k \geq 2d$ and $(k, d) = 1$. A (k, d)-coloring of G is a mapping $f : V \to \{0, 1, \cdots, k-1\}$ such that for every edge xy of G, $|f(x) - f(y)|_k \geq d$. Here

$$|f(x) - f(y)|_k = \min\{|f(x) - f(y)|,\ k - |f(x) - f(y)|\}$$

is called the *modulo k distance between $f(x)$ and $f(y)$*. We say G is *(k, d)-colorable* if G has a (k, d)-coloring. The *circular chromatic number* $\chi_c(G)$ of G is defined as

$$\chi_c(G) = \inf\{k/d : G \text{ is } (k, d)\text{-colorable}\}.$$

Note that a $(k, 1)$-coloring is simply an ordinary k-coloring of G. Therefore $\chi_c(G) \leq \chi(G)$ for any graph G. On the other hand, it was shown in [1, 15, 21] that for any graph G, $\chi_c(G) > \chi(G) - 1$, and hence $\chi(G) = \lceil \chi_c(G) \rceil$. So $\chi_c(G)$ is a refinement of $\chi(G)$. The circular chromatic number was also known as the star chromatic number, and was first introduced by Vince in 1988 [15]. This concept has attracted considerable attention in the past decade. Readers are referred to [21] for a comprehensive survey on research concerning this parameter.

This paper investigates circular coloring of graphs of large girth. The existence of graphs of large girth and large chromatic number is established by the following landmark result:

Theorem 1.1 (Erdős [2]) *Let k and l be positive integers. Then there exists a graph G with the following properties:*

- *$\chi(G) = k$;*
- *$g(G) \geq l$, where $g(G)$ denotes the girth of G.*

The proof of Theorem 1.1 in [2] uses probabilistic method. Later on, several constructive proofs of Theorem 1.1 are found [5, 7, 13].

Given integers $k \geq 2d$. Let G^k_d be the graph with vertices $\{0, 1, \cdots, k-1\}$ in which $i \sim j$ if and only if $|i - j|_k \geq d$. It is easy to see [15] that $\chi_c(G^k_d) = k/d$. Moreover, a (k, d)-coloring f of a graph is simply a homomorphism (i.e., edge-preserving mapping) from G to G^k_d. So a graph G is (k, d)-colorable if and only if G admits a homomorphism to G^k_d.

As a consequence of the observation above, for every rational $r \geq 2$, there is a graph G with $\chi_c(G) = r$. A natural question is whether or not there exist graphs G of large girth with $\chi_c(G) = r$. This question was answered in affirmative in [17]. Indeed, a stronger result was proved in [17]. Given graphs G and H. We say G is *H-colorable* if there is a homomorphism from G to H. We say G is *uniquely H-colorable* if there is an onto homomorphism f from G to H, and moreover, for any homomorphism g from G to H there is an automorphism σ of H such that $g = \sigma \circ f$.

The following result was proved in [17]:

Theorem 1.2 ([17]) *Let l be an integer and let H be a graph which is a core (i.e., H does not admit a homomorphism to any of its proper subgraphs). Then there is a graph G of girth at least l such that G is uniquely H-colorable.*

It was proved in [17] that uniquely G_d^k-colorable graphs have circular chromatic number k/d. Therefore we have the following corollary:

Corollary 1.1 *For any integer l and for any fraction $k/d \geq 2$, there is a graph G of girth at least l with $\chi_c(G) = k/d$.*

Theorem 1.2 was proved in [17] by probabilistic method. The problem of finding a constructive proof of Corollary 1.1 was posed in [17]. In this paper, we shall constructively prove the following much stronger result:

Theorem 1.3 *Let k, d be positive integers such that $k \geq 3d - 2$ and $(k, d) = 1$. Suppose A is a finite set and $f_1, f_2, \cdots, f_n$ are mappings from A to $\{0, 1, \cdots, k-1\}$. Then for any integer l, there is a graph $G = (V, E)$ with $A \subset V$ for which the following are true:*

- *G has girth at least l;*
- *for each $i \in \{1, 2, \cdots, n\}$, there is exactly one homomorphism g_i from G to G_d^k such that $g_i|_A = f_i$ (i.e., each f_i can be uniquely extended to a (k, d)-coloring of G);*
- *if g is a homomorphism from G to G_d^k then there is an index $i \in \{1, 2, \cdots, n\}$ and an automorphism σ of G_d^k such that $g = \sigma \circ g_i$ (i.e., besides the extensions of f_i's, there are no other homomorphisms from G to G_d^k).*

Observe that if $n = 1$ (A is an arbitrary set and f_1 is an arbitrary mapping from A to $\{0, 1, \cdots, k-1\}$), then we obtain Theorem 1.2, i.e., there is a sparse graph which is uniquely (k, d)-colorable.

Theorem 1.3 generalizes a result of V. Müller [9] who proved this for k-colorings.

In Theorem 1.3, we required that $k \geq 3d-2$. Although the result is also true for $2d \leq k < 3d-2$ (which can be proved by the probabilistic method [14]), however, the present proof does not work for $2d \leq k \leq 3d-3$. A constructive proof of this case remains unknown. (In [14], we gave a construction for any given odd girth.)

2 The construction of indicators

The remaining of the paper is devoted to the proof of Theorem 1.3. We assume that k, d are fixed positive integers with $\gcd(k,d)=1$ and $k \geq 3d-2$.

We view the vertices $\{0, 1, \cdots, k-1\}$ of G_d^k as elements of the group Z_k, and it is within this group, we take the sum (and subtraction) of two elements. For $x, y \in Z_k$, the circular distance between x and y is defined as $|x-y|_k = \min\{|x-y|, k-|x-y|\}$.

Definition 2.1 *Suppose I is a graph and a, b are two vertices of I. The triple $(I; a, b)$ is called a non-edge indicator of G_d^k if I admits a homomorphism to G_d^k, and for any homomorphism f from I to G_d^k, $f(a)$ and $f(b)$ are non-adjacent in G_d^k, i.e., $f(a)f(b)$ is a non-edge of G_d^k.*

First we shall show that there exist non-edge indicators $(I; a, b)$ of G_d^k which have large girth and large distance between a and b.

Lemma 2.1 *For any integer l, there exists a non-edge indicator $(I; a, b)$ of G_d^k such that I has girth at least l, and the distance between a and b is at least $l-1$.*

Proof. It is easy to construct a graph G of girth at least l which does not admit a homomorphism to G_d^k. For example, we may construct a graph of girth at least l and chromatic number at least $k/d+1$. Then it is easy to see that G does not admit a homomorphism to G_d^k. (If G admits a homomorphism to G_d^k, then G is (k,d)-colorable, and hence $\chi(G) < \chi_c(G)+1 \leq k/d+1$.) We may assume that G is critical in the sense that $G-e$ admits a homomorphism to G_d^k for any e of G. (If G is not critical, then by deleting some edges of G, we shall obtain a graph G' which does not admit a homomorphism to G_d^k and which is critical). Let $e=ab$ be an edge of G, and let $I=G-e$. Then I admits a homomorphism to G_d^k, and for any homomorphism f from I to G_d^k, $f(a)f(b)$ is a non-edge of G_d^k, for otherwise f would be a homomorphism from G to G_d^k. Since G has girth at least l, I has girth at least l and the distance between a and b is at least $l-1$. ∎

In the following, we shall assume that l is a fixed integer, and all the non-edge indicators $(I; a, b)$ will have girth at least l and distance between a and b at least $l-1$.

Suppose $(I;a,b)$ is a non-edge indicator of G_d^k. Let

$$M(I) = \max\{|f(a)-f(b)|_k : f \text{ is a homomorphism from } I \text{ to } G_d^k\}.$$

It follows from the definition that for any non-edge indicator $(I;a,b)$ of G_d^k we have $0 \leq M(I) \leq d-1$.

We shall show that there is a non-edge indicator of G_d^k for which $M(I)=0$, i.e., for any homomorphism f from I to G_d^k we have $f(a)=f(b)$.

Theorem 2.1 *There is a non-edge indicator $(I;a,b)$ of G_d^k with $M(I)=0$.*

Proof. Assume to the contrary that for every non-edge indicator $(I;a,b)$ of G_d^k we have $M(I)>0$. Let $(I;a,b)$ be a non-edge indicator of G_d^k with $M(I)$ minimum among all non-edge indicators of G_d^k. Let s be the least integer such that $sM(I) \geq d$. Note that $2 \leq s \leq d$. We shall derive a contradiction by constructing another non-edge indicator $(I';a',b')$ of G_d^k with $M(I') < M(I)$.

We consider two cases.

Case 1: $s \geq 3$. Take $s+2$ copies of $(I;a,b)$, and denote the jth copy by $(I_j;a_j,b_j)$. Identify b_j with a_{j+1} for $j=1,2,\cdots,s-1$, also identify b_{s-1} with a_{s+1}, and identify b_{s+1} with a_{s+2}. Add two edges, one joining a_1 to b_s, the other joining a_2 to b_{s+2}.

Let I' be the resulting graph, and let $a'=b_s$, $b'=b_{s+1}$. The graph I' is as indicated in Fig. 1 below.

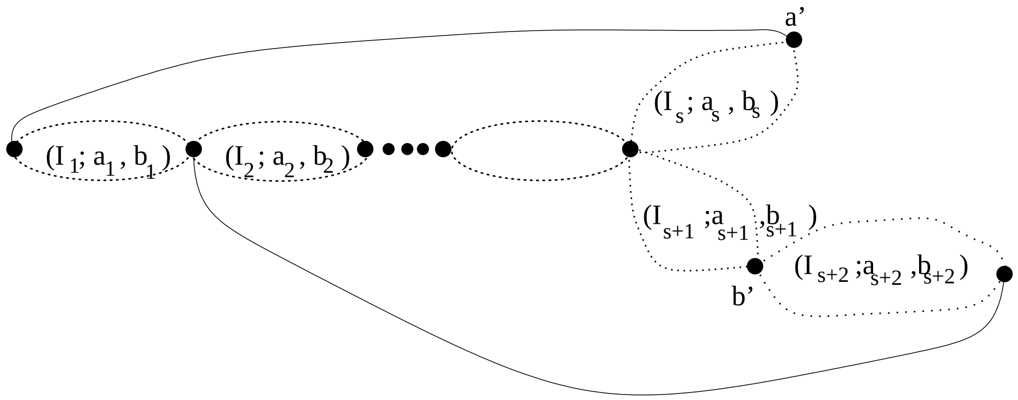

Figure 1: The construction of non-edge indicator $(I';a',b')$ for Case 1

We shall show that the triple $(I';a',b')$ is a non-edge indicator of G_d^k with $M(I') < M(I)$.

It is obvious that I' has girth at least l and the distance between a' and b' is at least $l-1$. First we show that I' admits a homomorphism to G_d^k. By the definition of $M(I)$ and by symmetry of G_d^k, for any $i \in V(G_d^k)$ there is a homomorphism from I to G_d^k such that $f(a) = i$ and $f(b) = i + M(I)$. Therefore there is a mapping f from I' to G_d^k which maps a_1 to 0, b_j to $jM(I)$ for $j = 1, 2, \cdots, s$, and maps b_{s+1} to $sM(I)$, b_{s+2} to $(s+1)M(I)$ such that the restriction of f to each of the copies of I is a homomorphism. Since $k \geq 3d - 2$, $1 \leq M(I) \leq d - 1$, we conclude that $d \leq sM(I) \leq k - d$ (recall that s is the least integer such that $sM(I) \geq d$). Therefore 0 is adjacent to $sM(I)$, and $M(I)$ is adjacent to $(s+1)M(I)$ in G_d^k. Hence f is indeed a homomorphism from I' to G_d^k.

Next we shall show that for any homomorphism f from I' to G_d^k we have $|f(a') - f(b')|_k < M(I)$. This would imply that $(I'; a', b')$ is a non-edge indicator of G_d^k with $M(I') < M(I)$, contrary to the choice of $(I; a, b)$.

Assume to the contrary that there is a homomorphism f from I' to G_d^k with $|f(a') - f(b')|_k \geq M(I)$.

Let $\vec{Q}$ be the orientation of the complement of G_d^k in which ij is a directed edge if and only if $j = i + \ell \bmod k$ for some $1 \leq \ell \leq d - 1$. Given an edge ij of the complement of G_d^k, we say ij is a *forward edge* of Q if ij is a directed edge of Q, and say ij is a *backward edge* of Q if ji is a directed edge of Q.

For each $j \in \{1, 2, \cdots, s\}$, consider the images $f(a_j)$ and $f(b_j)$. Since $(I; a, b)$ is a non-edge indicator, it follows that for each $j \in \{1, 2, \cdots, s\}$, either $f(a_j) = f(b_j)$ or $f(a_j)f(b_j)$ is a non-edge of G_d^k. In case $f(a_j)f(b_j)$ is a non-edge of G_d^k, $f(a_j)f(b_j)$ is an edge of the complement of G_d^k. So in this case $f(a_j)f(b_j)$ is either a forward edge or a backward edge of Q. Since a_1b_s is an edge of I', it follows that $|f(a_1) - f(b_s)|_k \geq d$. Since $a_{j+1} = b_j$ for $j = 1, 2, \cdots, s-1$, we know that

$$\begin{aligned} |f(a_1) - f(b_s)|_k &\leq |f(a_1) - f(b_1)|_k + |f(a_2) - f(b_2)|_k + \cdots + |f(a_s) - f(b_s)|_k \\ &\leq sM(I). \end{aligned}$$

As s is the minimum integer such that $sM(I) \geq d$, we conclude that $|f(a_j) - f(b_j)|_k \neq 0$ for $j = 1, 2, \cdots, s$. Moreover, all these edges $f(a_j)f(b_j)$ are either all forward edges of Q or all backward edges of Q. Without loss of generality, we assume that $f(a_j)f(b_j)$ are all forward edges of Q. Using the same argument, we can show that all the pairs

$$f(a_2)f(b_2), f(a_3)f(b_3), \cdots, f(a_{s-1})f(b_{s-1}), f(a_{s+1})f(b_{s+1}), f(a_{s+2})f(b_{s+2})$$

are edges of the complement of G_d^k, and they are either all forward edges of Q or all backward edges of Q. Since $s \geq 3$ and since $f(a_2)f(b_2)$ is assumed to be a forward edge of Q, it follows that all the pairs above are forward edges of Q. Thus $f(b_s) = f(a_s) + t \bmod k$ for some $0 < t \leq M(I)$ and $f(b_{s+1}) = f(a_{s+1}) + t' \bmod k$ for some $0 < t \leq M(I)$. Since a_s and a_{s+1} are identified into a single vertex, we conclude that

$$|f(a') - f(b')|_k = |f(b_s) - f(b_{s+1})|_k = |t - t'|_k < M(I).$$

This completes the proof of Case 1.

Case 2: $s = 2$.

Take $2l+1$ copies of $(I; a, b)$ (where l is the girth required). The jth copy is denoted by $(I_j; a_j, b_j)$. Identify b_j with a_{j+1} for $j = 1, 2, \cdots, l, l+2, l+3, \cdots, 2l$. Identify b_1 with a_{l+2}, b_{2l+1} with b_{l+1}.

Add edges joining a_j to b_{j+1} for $j = 1, 2, \cdots, l, l+2, l+3, \cdots, 2l$; and an edge joining a_1 to b_{l+2}. Let I' be the resulting graph, let $a' = a_{l+1}, b' = a_{2l+1}$. The graph I' is as indicated in Fig. 2 below.

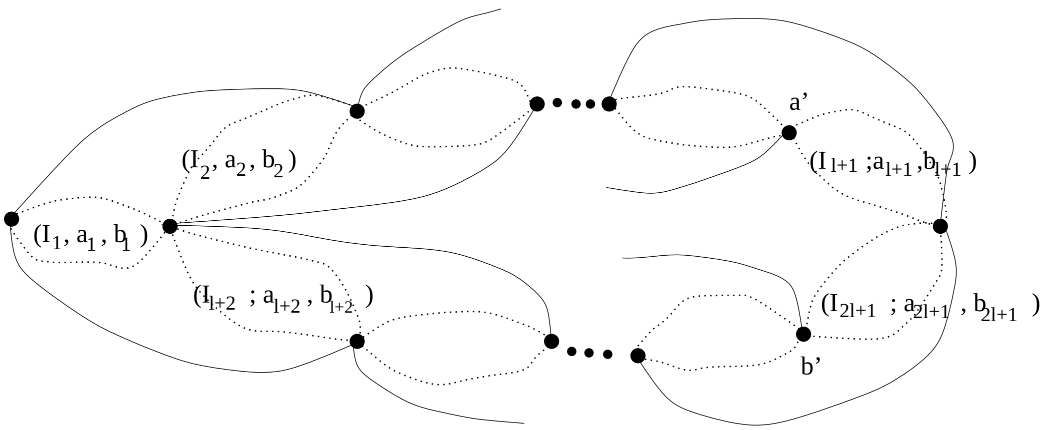

Figure 2: The construction of non-edge indicator $(I'; a', b')$ for Case 2

We shall show that the triple $(I; a', b')$ is a non-edge indicator of G_d^k with $M(I') < M(I)$.

It is easy to check that I' has girth at least l, the distance between a' and b' is at least $l-1$ and there exists a homomorphism h from I' to G_d^k. Now we shall show that if f is a homomorphism from I' to G_d^k then $|f(a') - f(b')|_k < M(I)$. The argument is also similar to the proof of Case 1.

By using the fact that for $j = 1, 2, \cdots, l, l+2, l+3, \cdots, 2l$, $f(a_j)f(b_{j+1})$ are edges of G_d^k (hence $|f(a_j) - f(b_{j+1})|_k \geq d$), and that $f(a_1)f(b_{l+2})$ is also an edge of G_d^k, and by the same argument as in the proof of Case 1, we conclude that

$$f(a_1)f(b_1), f(a_2)f(b_2), \cdots, f(a_{l+1})f(b_{l+1})$$

are either all forward edges of Q or backward edges of Q. Without loss of generality, we assume that all these pairs are all forward edges of Q or backward edges of Q. The same argument shows that

$$f(a_1)f(b_1), f(a_{l+2})f(b_{l+2}), f(a_{l+3})f(b_{l+3}), \cdots, f(a_{2l+1})f(b_{2l+1})$$

are either all forward edges of Q or backward edges of Q. Since $f(a_1)f(b_1)$ is already assumed to be a forward edge of Q, we conclude that $f(a_{2l+1})f(b_{2l+1})$ is also a forward edge of Q. Thus $f(a_{2l+1}) = f(b_{2l+1}) - t \bmod k$ for some $0 < t \leq M(I)$ and $f(a_{l+1}) = f(b_{l+1}) - t'$ for some $0 < t' \leq M(I)$. As b_{l+1} and b_{2l+1} are identified into a single vertex, we conclude that

$$|f(a') - f(b')|_k = |f(a_{l+1}) - f(a_{2l+1})|_k = |t' - t|_k < M(I).$$

This completes the proof of Case 2 as well as the proof of Theorem 2.1 ∎

Definition 2.2 *A triple $(H; a, b)$ is called a k/d-*superedge *if the following are true:*

- *For any homomorphism f from H to G_d^k, $f(a)f(b)$ is an edge of G_d^k.*

- *For any edge $e = ij$ of G_d^k, there is a homomorphism from H to G_d^k such that $f(a) = i$ and $f(b) = j$.*

Theorem 2.2 *For any $k \geq 3d - 2$, and for any integer l there is a k/d-superedge $(H; a, b)$ such that H has girth at least l and the distance between a and b is at least l.*

Proof. Let $(I; a, a')$ be a non-edge indicator of G_d^k with $M(I) = 0$. Let H be obtained from I by adding one vertex b and an edge joining b to a'. Then it is straightforward to verify that the triple $(H; a, b)$ is a k/d-superedge. ∎

Suppose $(H; a, b)$ is a k/d-superedge. Given a graph G and an edge $e = xy$ of G, we shall denote by $G[e \leftarrow (H; a, b)]$ the graph obtained from the disjoint union of G and H by deleting the edge e and identifying x with a and y with b. This operation is called *replacing the edge e with the k/d-superedge* $(H; a, b)$.

Theorem 2.3 *Suppose $(H; a, b)$ is a k/d-superedge. If G is a graph with $\chi_c(G) = k/d$ and $e = xy$ is an edge of G. Then $G[e \leftarrow (H; a, b)]$ also has circular chromatic number k/d.*

Proof. By the definition of k/d-superedge, any (k, d)-coloring f of G can be extended to a (k, d) coloring of $G[e \leftarrow (H; a, b)]$. Thus $\chi_c(G[e \leftarrow (H; a, b)]) \leq k/d$.

It remains to show that $\chi_c(G[e \leftarrow (H; a, b)])$ is not strictly less than k/d. Assume to the contrary that $\chi_c(G[e \leftarrow (H; a, b)]) = k'/d' < k/d$. Let c be an (k', d')-coloring of $G[e \leftarrow (H; a, b)]$. Since $\chi_c(G) > k'/d'$ we conclude that $|c(x) - c(y)|_{k'} \leq d' - 1$.

Let $c' : V(G[e \leftarrow (H; a, b)]) \to \{0, 1, \cdots, k'd - 1\}$ be defined as $c'(u) = c(u)d$. Then $|c'(x) - c'(y)|_{k'd} \leq (d' - 1)d$. Without loss of generality, we may assume that

$c'(x) = c(x) = 0$ and $c'(y) \leq (d'-1)d$. Let $c'' : V(H) \to \{0, 1, \cdots, k-1\}$ be defined as $c''(u) = \lfloor c'(u)/d' \rfloor$. For any edge uv of H, $dd' \leq |c'(u) - c'(v)| \leq k'd - dd'$. Hence

$$d \leq |c''(u) - c''(v)| \leq \lceil k'd/d' \rceil - d \leq k - d.$$

So c'' is a (k, d)-coloring of H. However, $|c''(a) - c''(b)| < d$, contrary to the assumption that $(H; a, b)$ is a k/d-superedge. ∎

Corollary 2.1 *For any $k \geq 3d-2$, and for any integer l, there is a graph G of girth at least l such that $\chi_c(G) = k/d$.*

Proof. Let $(H; a, b)$ be a k/d-superedge whose girth is at least l and the distance between a and b is at least $l-1$. Replace each edge of G^k_d with the k/d-superedge $(H; a, b)$. Then the resulting graph G has girth at least l and $\chi_c(G) = k/d$. ∎

3 Projectivity of the graphs G^k_d

Suppose G and H are graphs. The *categorical product* $G \times H$ of G and H has vertices $V(G) \times V(H)$ in which $(g, h)(g', h')$ is an edge if and only if gg' is an edge of G and hh' is an edge of H. We shall denote the product $G \times G$ by G^2, and let $G^n = G^{n-1} \times G$.

It is easy to verify that the mapping $p_i : G^n \to G$ defined as $p_i(x_1, x_2, \cdots, x_n) = x_i$ is a homomorphism. Such a homomorphism is called a *projection*.

Definition 3.1 *A graph G is called* projective *if for any integer $n \geq 2$, any homomorphism $f : G^n \to G$ satisfying $f(x, x, \cdots, x) = x$ is a projection.*

Projectivity of complete graphs was proved by Müller [9]. General projective graphs have been studied by Larose and Tardif [6] and they gave sufficient conditions for a graph to be projective. These results imply that the graphs G^k_d are projective. However, the arguments in [6] rely on some other results (from algebra), and the proofs are complicated. For the completeness of this paper, we give a direct (and perhaps easier) proof of the projectivity of the graphs G^k_d.

Theorem 3.1 *Suppose $k \geq 2d$ and $(k, d) = 1$. The graph G^k_d is projective.*

Proof. Let $n \geq 2$ be an integer, and let $f : (G^k_d)^n \to G^k_d$ be a homomorphism such that $f(i, i, \cdots, i) = i$. (Recall the vertices of G^k_d are $0, 1, \cdots, k-1$.) We shall prove that f is a projection.

Consider the image of $f(i,i,\cdots,i,i+1)$. Since $(i,i,\cdots,i,i+1)$ is adjacent to

$$\begin{array}{l}(i+d+1,i+d+1,\quad\cdots,\quad i+d+1,i+d+1),\\(i+d+2,i+d+2,\quad\cdots,\quad i+d+2,i+d+2),\\\qquad\cdots\cdots,\\(i+k-d,i+k-d,\quad\cdots,\quad i+k-d,i+k-d)\end{array}$$

(where the summation are modulo k), it follows that $f(i,i,\cdots,i,i+1)$ is adjacent to all the vertices $i+d+1,i+d+2,\cdots,i+k-d$. The only vertices of G_d^k adjacent to all these vertices are $i,i+1$. Therefore $f(i,i,\cdots,i,i+1)\in\{i,i+1\}$.

If $f(0,0,\cdots,0,1)=0$, then since $f(k-d,k-d,\cdots,k-d,k-d+1)$ must be adjacent to 0, we have $f(k-d,k-d,\cdots,k-d,k-d+1)=k-d$. Repeat this argument, we conclude that $f(k-sd,k-sd,\cdots,k-sd,k-sd+1)=k-sd$ for all s. As $(k,d)=1$, we conclude that $f(i,i,\cdots,i,i+1)=i$ for all i. Therefore either $f(i,i,\cdots,i,i+1)=i$ for all i, or $f(i,i,\cdots,i,i+1)=i+1$ for all i.

Now we shall prove that for any i,j, $f(i,i,\cdots,i,i+j)=i+\alpha_j$, where $\alpha_1=0$ or 1, and for $j\geq 2$, $\alpha_j=\alpha_{j-1}$ or $\alpha_{j-1}+1$. (The summation is carried out modulo k). We prove this by induction on j. We have already shown that it is true for $j=1$. Since $(i,i,\cdots,i,i+j)$ is adjacent to

$$\begin{array}{l}(i+d+1,i+d+1,\quad\cdots,\quad i+d+1,i+j+d),\\(i+d+2,i+d+2,\quad\cdots,\quad i+d+2,i+j+d+1),\\\qquad\cdots\cdots,\\(i+k-d,i+k-d,\quad\cdots,\quad i+k-d,i+j+k-d-1),\end{array}$$

and by induction hypothesis,

$$\begin{array}{rcl}f(i+d+1,i+d+1,\cdots,i+d+1,i+j+d)&=&i+d+1+\alpha_{j-1},\\f(i+d+2,i+d+2,\cdots,i+d+2,i+j+d+1)&=&i+d+2+\alpha_{j-1},\\&&\cdots\cdots,\\f(i+k-d,i+k-d,\cdots,i+k-d,i+j+k-d-1)&=&i+k-d+\alpha_{j-1},\end{array}$$

we conclude that $f(i,i,\cdots,i,i+j)$ is adjacent to

$$i+d+1+\alpha_{j-1},i+d+2+\alpha_{j-1},\cdots,i+k-d+\alpha_{j-1}.$$

It follows that

$$f(i,i,\cdots,i,i+j)\in\{i+\alpha_{j-1},i+1+\alpha_{j-1}\}.$$

By the same argument as above, we can show that either $f(i,i,\cdots,i,i+j)=i+\alpha_{j-1}$ for all i, or $f(i,i,\cdots,i,i+j)=i+\alpha_{j-1}+1$ for all i. In the former case, let $\alpha_j=\alpha_{j-1}$, and in the latter case, let $\alpha_j=\alpha_{j-1}+1$, and we are done.

Now the above argument can be carried out to $j = k$. Then we have $i + \alpha_k = f(i, i, \cdots, i, i+k) = f(i, i, \cdots, i, i) = i$ (as $i + k \equiv i \pmod{k}$). So $\alpha_k \bmod k = 0$. As $0 \le \alpha_1 \le 1$ and $\alpha_{j-1} \le \alpha_j \le \alpha_{j-1} + 1$, we conclude that either $\alpha_j = 0$ for all j, or $\alpha_j = j$ for all j.

If $n = 2$, then in the former case, we have $f = p_1$ and in the latter case we have $f = p_2$.

Assume that $n \ge 3$. First we consider the case that $\alpha_j = 0$ for all j. For $i = 0, 1, \cdots, k-1$, let $g_i : (G_d^k)^{n-1} \to G_d^k$ be defined as $g_i(x_1, x_2, \cdots, x_{n-1}) = f(x_1, x_2, \cdots, x_{n-1}, i)$. Then $g_i(x, x, \cdots, x) = x$, by the induction hypothesis, g_i is a projection.

Suppose $g_0 = p_j$. We prove that $g_d = p_j$. Assume to the contrary that $g_d = p_{j'}$ for some $j' \ne j$. Let $u \in (G_d^k)^n$ be the vertex whose j'th coordinate is $k - d$, the other coordinates are 0, let $v \in (G_d^k)^n$ be the vertex whose j'th coordinate is 0, the other coordinates are d. Then $f(u) = 0$ and $f(v) = 0$ but uv is an edge of $(G_d^k)^n$.

Similarly we have $g_{2d} = p_j, g_{3d} = p_j, \cdots$. Hence $g_i = p_j$ for all i. So $f = p_j$.

Now we consider the case that $\alpha_j = j$ for all j. Let $g : (G_d^k)^{n-1} \to G_d^k$ be defined as $g(x_1, x_2, \cdots, x_{n-1}) = f(x_1, x_1, x_2, \cdots, x_{n-1})$. By induction hypothesis, g is a projection, and hence $g = p_{n-1}$. We shall prove that for any vertices x, y and $x_2, \cdots, x_{n-1}$, $f(x, y, x_2, \cdots, x_{n-1}) = x_{n-1}$. We prove this by induction on the even distance $ed(x, y)$ between x and y (which is defined as the length of the shortest even length x-y-path. Note that every two vertices of G_d^k are connected by an even length path, as G_d^k is nonbipartite and connected).

If $ed(x, y) = 0$ then this follows from the fact that $g = p_{n-1}$. Assume that $d(x, y) = 2k + 2$, and $x = v_0, v_1, \cdots, v_{2k}, v_{2k+1}, v_{2k+2} = y$ is an x-y-path. Let x_i' be any neighbour of x_i for $i = 2, 3, \cdots, n-1$. Then $(x, y, x_2, \cdots, x_{n-1})$ is adjacent to $(v_1, v_{2k+1}, x_2', \cdots, x_{n-1}')$. By the induction hypothesis, $f(v_1, v_{2k+1}, x_1', x_2', \cdots, x_{n-1}') = x_{n-1}'$. Let $z = f(x, y, x_2, \cdots, x_{n-1})$. Then $x_{n-1}' \sim z$. $x_{n-1}' \in N(z)$ for all $x_{n-1}' \in N(x_{n-1})$. As x_{n-1}' is an arbitrary neighbour of x_{n-1}, we conclude that $N(x_{n-1}) \subseteq N(z)$. If $z \ne x_{n-1}$, then by fixing every other vertices of G_d^k, and sending x_{n-1} to z, we would obtain a homomorphism from G_d^k to a proper subgraph of G_d^k, contrary to the fact that G_d^k is a core. Thus we conclude that $z = x_{n-1}$ and hence $f = p_n$. ∎

4 Construction of sparse graphs

In this section we prove Theorem 1.3 by constructing a graph with the prescribed properties. The proof uses a similar strategy as Müller's proof [9]. In the following, $(H; x, y)$ is a fixed k/d-superedge which has girth at least l and the distance between x and y is at least $l - 1$; and $(I; a, b)$ is a non-edge-indicator with $M(I) = 0$ which has girth at least l and the distance between a and b is at least $l-1$.

Lemma 4.1 *Suppose $g, n, m \in N$. Let $M = \{1, 2, \cdots, m\}$, $K = \{0, 1, \cdots, k-1\}$ and $C = K^n \times M$. There exists a graph $G_1 = (V, E)$ such that the following are true:*

- *$C \subset V$;*
- *G_1 has girth at least l;*
- *$\chi_c(G_1) = k/d$;*
- *for each $i \in \{1, 2, \cdots, n\}$ there is a (k, d)-coloring f of G_1 such that $f((r_1, r_2, \cdots, r_n, m')) = r_i$;*
- *if f is a (k, d)-coloring of G_1 then there is an index $i \in \{1, 2, \cdots, n\}$ and an automorphism σ of G_d^k such that for every $(r_1, r_2, \cdots, r_n, m') \in C$, $\sigma \circ f((r_1, r_2, \cdots, r_n, m')) = r_i$;*
- *for $x, y \in C$, $x \neq y$, $d_{G_1}(x, y) \geq l - 1$.*

Proof. Take the product graph $(G_d^k)^n$, and replace each edge of $(G_d^k)^n$ by the superedge $(H; x, y)$. Denote the resulting graph by $X = (V', E')$. With an abuse of notation, we shall view the set $K^n = \{(r_1, r_2, \cdots, r_n) : r_i \in \{0, 1, \cdots, k-1\}\}$ as a subset of V'. Each pair of vertices of K^n has distance at least $l-1$ in the graph X (as all the edges have been replaced by the superedge $(H; x, y)$). By the definition of k/d-superedge, if f is a homomorphism from X to G_d^k, then the restriction of f to K^n is a homomorphism of $(G_d^k)^n$ to G_d^k. Since every homomorphism of $(G_d^k)^n$ to G_d^k is a projection, we conclude that the restriction to K^n of any homomorphism f from X to G_d^k is a projection, up to an automorphism of G_d^k. Thus if f is a homomorphism from X to G_d^k, then there is an index i and an automorphism σ of G_d^k such that $f(r_1, r_2, \cdots, r_n) = \sigma(r_i)$, for all $(r_1, r_2, \cdots, r_n) \in K^n$. Conversely, it also follows from the definition of k/d-superedge that each projection of $(G_d^k)^n$ to G_d^k can be extended to a homomorphism from X to G_d^k.

Take the set $C = K^n \times M$. For $r_1, r_2, \cdots, r_n \in K$ and for $j, j' \in M$ such that $j \neq j'$, connect $(r_1, r_2, \cdots, r_n, j)$ and $(r_1, r_2, \cdots, r_n, j')$ with $(I; a, b)$ (i.e., add a copy of I, identify $(r_1, r_2, \cdots, r_n, j)$ with a and $(r_1, r_2, \cdots, r_n, j')$ with b). Denote the resulting graph by $Y = (V'', E'')$. Again we view C as a subset of V''.

Now take the disjoint union of X and Y, identify $(r_1, r_2, \cdots, r_n)$ of X with $(r_1, r_2, \cdots, r_n, 1)$ of Y. Denote the resulting graph by G_1. It is obvious that G_1 has girth at least l (as H and I have girth at least l, the distance between a and b, x and y are at least $l-1$). If f is a homomorphism from G_1 to G_d^k, then there is an index i and an automorphism σ of G_d^k such that $f(r_1, r_2, \cdots, r_n, 1) = \sigma(r_i)$, for all $(r_1, r_2, \cdots, r_n, 1) \in C$. By the choice of $(I; a, b)$, we conclude that $f(r_1, r_2, \cdots, r_n, j) = \sigma(r_i)$, for all $(r_1, r_2, \cdots, r_n, j) \in C$. So the graph G_1 has the properties listed in Lemma 4.1. ∎

The graph G_1 is almost what we wanted. It contains no short cycles, and has circular chromatic number k/d. Moreover, if restricted to the set C there are exactly n (k, d)-colorings which are the n projections. These n projections can be replaced by any n prescribed k-colorings on a given finite set A as follows: For any given set A with n prescribed k-colorings $f_1, f_2, \cdots, f_n$ of the set A, by choosing m large enough (say $m \geq |A|$) there is a one to one mapping ϕ from A into C such that $\phi(x) = (f_1(x), f_2(x), \cdots, f_n(x), m(x))$. By using this embedding, then the projections p_i coincide with f_i on A.

The only problem with G_1 is that each of these n (k, d)-colorings on C may have many different extensions to the whole graph G_1. In other words, to obtain a homomorphism from G_1 to G_d^k, the images of the vertices in C are fixed (to one of the projections). However, the images of vertices in $V(G_1) - C$ are not fixed. Our goal is to modify the graph G_1 so that the images of the vertices in $V(G_1) - C$ are also fixed.

Suppose $x \in V(G_1) - C$, if we add edges between x and vertices of C with images $d, d+1, \cdots, k-d$, the only possible image of x is 0. So by adding edges between C and $V(G_1) - C$, we can fix the images of vertices of $V(G_1) - C$. This is exactly what we shall do: add edges between vertices with fixed images to vertices whose images are not fixed yet. However, we need to be careful that by adding these edges, no short cycles will occur. This would be impossible if the number of vertices with fixed images is small compared to the number of vertices whose images are not fixed. The first things we shall do is to increase the number of vertices with fixed images without increasing the number of vertices whose images are not fixed.

Lemma 4.2 *Let $S = \{1, 2, \cdots, s\}, K = \{0, 1, \cdots, k-1\}, M = \{1, 2, \cdots, m\}$. Suppose $m > (k-d+1)(2(k-d+1))^l$. Then there is a graph G_2 with vertex set $S \times K^n \times M$ which has the following properties:*

1. *G_2 has girth at least l;*
2. *each vertex has degree at most $2(k-d+1)$;*

3. *if $s_1 \geq 2$ and $x = (s_1, r_1, r_2, \cdots, r_n, m_1) \in V(G_2)$ then for each $d \leq i \leq k-d$, there exists exactly one vertex $y = (s_1 - 1, r_1 + i, r_2 + i, \cdots, r_n + i, m(i))$ of G_2 such that y is adjacent to x (here the sum $r_j + i$ is modulo k sum);*

4. *if $x \sim y$, $x = (s_1, r_1, r_2, \cdots, r_n, m_1)$ and $y = (s_2, r'_1, r'_2, \cdots, r'_n, m_2)$ then $|s_1 - s_2| = 1$ and there is an integer $d \leq i \leq k-d$ such that $r'_j = r_j + i$;*

Proof. We order the vertices of G_2 lexicographically: if $x = (s_1, r_1, r_2, \cdots, r_n, m_1)$ and $y = (s_2, r'_1, r'_2, \cdots, r'_n, m_2)$ then $x < y$ if $s_1 < s_2$ or $s_1 = s_2$ and there exists $i \in \{1, 2, \cdots, n\}$ such that $r_j = r'_j$ for $j < i$ and $r_i < r'_i$ or $s_1 = s_2$ and $r_j = r'_j$ for all j and $m_1 < m_2$.

For $x \in V(G_2)$, let $U_x = \{y \in V(G_2) : y < x\}$. We shall construct the graph G_2 inductively. For $x = (2, 0, 0, \cdots, 0, 1)$, $G_2|U_x$ is the empty graph. Suppose $x = (s_1, r_1, r_2, \cdots, r_n, m_1)$ and $G_x = G_2|U_x$ (which has the properties listed above) has been constructed, we shall construct $G_x \cup \{x\}$ as follows:

For each $d \leq i \leq k-d$, we need to find a vertex $y(i) = (s_1 - 1, r_1 + i, r_2 + i, \cdots, r_n + i, m(i))$ and add an edge between x and $y(i)$, so that the resulting graph still has properties (1) and (2).

Suppose we have already found $y(1), y(2), \cdots, y(i-1)$ and added the edges $xy(1), xy(2), \cdots, xy(i-1)$ to $G_x \cup \{x\}$. Denote this graph by G^i. Since each vertex of G_2 has degree at most $2(k-d+1)$, we have

$$|\{y \in U_x : d_{G^i}(x, y) < l\}| \leq (2(k-d+1))^l.$$

Let $U' = \{y \in V(G_x) : y = (s_1 - 1, r_1 + i, r_2 + i, \cdots, r_n + i, m') \text{ and } d_{G^i}(x, y) \geq l\}$. Then $|U'| \geq m - (2(k-d+1))^l$. It suffices to find a vertex $y \in U'$ such that $d_{G^i}(y) < 2(k-d+1)$. Then by adding the edge xy, the resulting graph still has girth at least l and each vertex of degree at most $2(k-d+1)$. Suppose such a vertex does not exists, i.e., each $y \in U'$ has degree at least $2(k-d+1)$. Hence

$$\begin{aligned} q &= \Sigma_{m' \in M} d_{G^i}((s_1 - 1, r_1 + i, r_2 + i, \cdots, r_n + i, m')) \\ &\geq 2(k-d+1)(m - (2(k-d+1))^l). \end{aligned}$$

On the other hand it follows from the properties (3) and (4) of the graph G^i that $q \leq 2m(k-d)$. Since $m > (k-d+1)(2(k-d+1))^l$, we have

$$\begin{aligned} q &\leq 2m(k-d) \\ &= 2m(k-d+1) - 2m \\ &< 2m(k-d+1) - 2(k-d+1)(2(k-d+1))^l \\ &= 2(k-d+1)(m - (2(k-d+1))^l), \end{aligned}$$

which is a contradiction.

Therefore the required vertex $y \in U'$ does exist, and the graph $G_x \cup \{x\}$ can be constructed. This completes the proof of Lemma 4.2. ∎

Lemma 4.3 *The graph G_2 constructed in Lemma 4.2 is (k,d)-colorable, and for any (k,d)-coloring f of G_2, if $f((1, r_1, r_2, \cdots, r_n, m')) = r_i$ for any $(r_1, r_2, \cdots, r_n, m') \in K^n \times M$, then $f((s', r_1, r_2, \cdots, r_n, m')) = r_i$ for all $(s', r_1, r_2, \cdots, r_n, m') \in S \times K^n \times M$.*

Proof. It follows from the construction that the mapping f defined as $f((s', r_1, r_2, \cdots, r_n, m')) = r_i$ is a homomorphism from G_2 to G_d^k. So G_2 is (k,d)-colorable. Assume that f is a (k,d)-coloring for which $f((1, r_1, r_2, \cdots, r_n, m')) = r_i$ for any $(r_1, r_2, \cdots, r_n, m') \in K^n \times M$, then it follows from the definition of the graph G_2 that f has a unique extension (as a (k,d)-coloring) to the set $\{2\} \times K^n \times M$, namely $f((2, r_1, r_2, \cdots, r_n, m')) = r_i$. Inductively, we can prove that f has a unique extension (as a (k,d)-coloring) to the set $S \times K^n \times M$, namely $f((s', r_1, r_2, \cdots, r_n, m')) = r_i$. ∎

Proof of Theorem 1.3:

To obtain the graph required by Theorem 1.3, we take the disjoint union of G_1 and G_2, identify $(r_1, r_2, \cdots, r_n, m')$ of G_1 with $(1, r_1, r_2, \cdots, r_n, m')$ of G_2, for all $(r_1, r_2, \cdots, r_n, m') \in K^n \times M$. Then the resulting graph G' has circular chromatic number k/d, and there are exactly n (k,d)-colorings of the vertices in $S \times K^n \times M$, namely the projections, which can be extended to (k,d)-colorings of G'. However, each of these n (k,d)-colorings of $S \times K^n \times M$ may have many extensions. For each of these n (k,d)-colorings of $S \times K^n \times M$, we arbitrarily fix one extension to G'. Suppose $f_1, f_2, \cdots, f_n$ are the n (k,d)-colorings we have chosen for G'. Now we shall add edges to G' so that G' has no other (k,d)-colorings. All the added edges are between $(V(G_1) - C)$ and vertices of G_2 of the form $(l \cdot t, r_1, r_2, \cdots, r_n, 1)$. (Recall that G_2 has vertices $S \times K^n \times M$ where $S = \{1, 2, \cdots, s\}$). To ensure the resulting graph has no short cycles, we shall choose $s \geq k \cdot l \cdot |V(G_1)|$ in the construction of G_2.

We order the vertices of $V(G_1) - C$ as $x_1, x_2, \cdots, x_p$. The added edges are the following: for each $x_t \in (V(G_1) - C)$, for $i = d, d+1, \cdots, k-d$, add an edge connect x_t to $(l \cdot ((k-d)t + i), f_1(x_t) + i, f_2(x_t) + i, \cdots, f_n(x_t) + i, 1)$. (The additions in $f_j(x_t) + i$ are modulo k).

It is straightforward to verify that the resulting graph G has girth at least l, and $f_1, f_2, \cdots, f_n$ are the only (k,d)-colorings of G. As observed earlier, for any finite set A, for any n mappings $g_1, g_2, \cdots, g_n$ from A to $\{0, 1, \cdots, k-1\}$, by choosing M large enough, we can embed A into the set $K^n \times M$ so that the mappings g_i coincides with the mappings f_i. This completes the proof of Theorem 1.3.

5 A remark

A homomorphism from a graph G to a graph H is also called an *H-coloring* of G. In this language, a (k,d)-coloring of G is equivalent to a G_d^k-coloring of G. Instead of considering (k,d)-colorings of a graph G, we may consider H-colorings of G for any H. A natural question is whether or not Theorem 1.3 holds for H-coloring. This question was answered in [14] by the following theorem:

Theorem 5.1 *For any finite set A, any projective core graph H and for any n mappings $f_1, f_2, \cdots, f_n$ from A to $V(H)$, there is a sparse graph $G = (V, E)$ with $A \subset V$ such that G has exactly n H-colorings (up to an automorphism of H) $g_1, g_2, \cdots, g_n$ such that each g_i is an extension of f_i.*

The proof given in [14] is probabilistic.

However by following the proof of this paper, it is easy to see that we have a constructive proof for a given H if the following are true and can be proved constructively:

1. H is projective;

2. for any vertex x of H, there is a triple $(I; a, b)$, where I is a sparse graph, a, b are two specified vertices of I far away from each other, such that there is a homomorphism f from I to H with $f(a) = x$. Moreover, for any homomorphism from I to H with $f(a) = x$ we have $f(b) = x$.

In Sections 2 and 4, we have shown constructively that for $k \geq 3d - 2$, the graph G_d^k has property (2) listed above (Property (1) always has a constructive proof, if it is true, [6]). There are many classes of graphs H are shown to be projective in [6]. If H is also a core, then Theorem 5.1 holds for H. There are some other graphs H (besides G_d^k) for which we have a constructive proof of having Property (2). In particular, we can prove this for the Petersen graph and its complement.

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