On Sparse Graphs with Given Colorings and Homomorphisms

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Abstract

We prove that for every graph F there exists a sparse graph G (i.e., graph of large girth) and a homomorphism $c : G \to F$ such that each homomorphism g from G to a small *pointed* graph H is of the form $g = f \circ c$, where f is a homomorphism from F to H. This generalizes (for $H = K_k$) the existence of uniquely k colorable (and uniquely H-colorable) sparse graphs. By using the notion of projective graphs, (introduced by C. Tardif and B. Larose), we also prove the existense of sparse graphs with prescribed H-colorings on a given set, for projective core graphs H. This generalizes a celebrated result of V. Müller which proves this for k-colorings. We also completely characterize graphs H for which the analogy of Müller's theorem holds.

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1 Introduction

Given a graph G, we denote by $\chi(G)$ its chromatic number, by $g(G)$ its girth, and by $\mathrm{oddg}(G)$ its odd girth. (Thus $g(G)$ is the shortest length of a cycle in G while $\mathrm{oddg}(G)$ is the shortest length of an odd cycle in G.)

Given graphs G and H, a *homomorphism* from G to H is a vertex-mapping $f : V(G) \to V(H)$ such that $\{x, y\} \in E(G)$ implies $\{f(x), f(y)\} \in E(H)$. If there exists a homomorphism from G to H, then we say G is *homomorphic to* H, and write $G \to H$. It is easy to see that the question whether there exists a homomorphism from G to K_n (the complete graph with n vertices) is equivalent to the question whether G is n-colorable. In this spirit, we also say that G is H*-colorable* if G is homomorphic to H, and a homomorphism from G to H is also called an H*-coloring* of G.

For H-colorings the role of complete graphs is played by cores: a graph G is called a *core* if any homomorphism $G \to G$ is an automorphism. We shall need one more definition: A graph H is said to be *pointed for* G (or shortly G-pointed) if for any two homomorphisms $g, g' : G \to H$ which satisfy $g(x) = g'(x)$ for all $x \neq x_0$ (for some fixed vertex $x_0 \in V(G)$) holds also $g(x_0) = g'(x_0)$. Note that any core graph H is H-pointed.

The following is the main result of this paper:

Theorem 1.1 *For every graph F and every choice of positive integers k and l there exists a graph G together with a homomorphism $c : G \to F$ with the following properties:*

i. $g(G) > l$;

ii. For every graph H with at most k vertices and there exists a homomorphism $g : G \to H$ if and only if there exists a homomorphism $f : F \to H$.

iii. For every F-pointed graph H with at most k vertices and for every homomorphism $g : G \to H$ there exists a unique homomorphism $f : F \to H$ such that $g = f \circ c$.

The conditions *ii.* and *iii.* may be expressed by the following diagram:

It is easy to give an example which shows that the statement analogous to *iii.* can not be true for all (i.e. not necessarily pointed) small graphs H: Given two homomorphisms $f', f'' : F \to H$ satisfying $f'(x) = f''(x)$ for all $x \neq x_0$ and $f'(x_0) \neq f''(x_0)$ the set $c^{-1}(x_0)$ may be splitted in two set A and B and we can define $g : G \to H$ by $f' \circ c(x)$ for all x with $c(x) \neq x_0$ and $g(x) = f' \circ c(x)$ for all $x \in A$ and $g(x) = f'' \circ c(x)$ for all $x \in B$. The homomorphism g cannot be written as $g = f \circ c$ for a homomorphism $f : F \to H$.

$$\begin{tikzcd} F \arrow[r, "f"] & H \\ G \arrow[u, "c"] \arrow[ur, "g"'] & \end{tikzcd}$$

Theorem 1.1 is a generalization of the study of graphs with large chromatic number and large girth. The existence of such graphs is established by the following landmark result (and landmark proof):

Theorem 1.2 (Erdős [3]) *let k and l be positive integers. Then there exists a graph G with the following properties:*

i. $\chi(G) > k$;

ii. $g(G) > l$.

Theorem 1.1 may look like a technical lemma, however, it has several interesting corollaries which prove structural extensions of Erdős theorem.

We say a graph G is uniquely H-colorable if there is an onto homomorphism c from G to H, and any other homomorphism from G to H is the composition $\sigma \circ c$ of c with an automorphism σ of H.

The problem of the existence of uniquely k-colorable graphs with large girth has an interesting history: [10] settled triangle-free case (i.e. $l = 3$) and this was improved by Greenwell and Lovász [4] to a given odd girth. Meanwhile, in 1973, Erdős claimed the general case by probabilistic method (see [8]) and the construction was provided in the full generality by Müller [8, 9].

Müller's proof used a constructive proof of Theorem 1.2. (Later a non-constructive proof has been published by Bollobás and Sauer [2].) Finally, Zhu [13] proved the following result which is a particular case (choose $F = H$) of our Theorem 1.1

Corollary 1.1 *For every core H and positive integer l there exists uniquely H-colorable graph G with girth $\geq l$.*

In fact we prove that G is *strongly uniquely H-colorable* in the sense that any homomorphism $G \to H'$ to any small pointed graph H' is induced by a homomorphism $F \to H'$.

A constructive proof of this result is presently open. This we state as

Problem 1.1 *Find a constructive proof of Corollary 1.1.*

The existence of sparse uniquely k-colorable graphs was generalized in [8, 9] in another direction, where the following remarkable strengthening of the unique colorability was proved:

Theorem 1.3 (Müller [8, 9]) *Let k, l, t be positive integers, $k > 2$. Let A be a finite set and let $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_t$ be distinct partitions of the set A each into at most k classes. Then there exists a k-chromatic graph $G = (V, E)$ such that the following holds:*

i. $g(G) > l$;

ii. A is a subset of V;

iii. G has just t colorings $\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_t$ by k-colors such that each of the coloring $\mathcal{B}_i$ restricted to the set A coincides with the coloring $\mathcal{A}_i$, $i = 1, 2, \cdots, t$.

It seems that this result is little known (although it is included in Jensen - Toft book [5]) and even the existence of construction of uniquely colorable graphs without short cycles was recently quoted as a problem.

In this paper we approach Müller's Theorem more generally and thus perhaps find a proper setting for it.

We now demonstrate that Müller's theorem 1.3 corresponds to the case $F = K_k^t, H = K_k$ of Theorem 1.1. For this purpose, we need the following particular property of complete graphs (which is established in [9]):

The only homomorphisms $f : K_k^t \to K_k$ satisfying $f(x, x, \cdots, x) = x$ are projections.

In a recent paper by B. Larose and C. Tardif [6, 7], this property was studied (in relationship to Hedetniemi's product conjecture) and called projectivity: A graph H is said to be *t-projective* if every homomorphism $f : H^t \to H$ which satisfies $f(x, x, \cdots, x) = x$ for every $x \in V(H)$ is a projection. A graph is *projective* if it is t-projective for every t (i.e. for projective graphs, up to an automorphism, the only homomorphisms $H^t \to H$ are projections).

(By a *product* we mean here *categorical product* (sometimes called direct product) defined as follows: Suppose G and H are simple finite graphs. The direct product $G \times H$ of G and H has vertex set $V(G \times H) = \{(x, y) : x \in V(G) \text{ and } y \in V(H)\}$ and edge set $E(G \times H) = \{\{(x, y)(x', y')\} : \{x, x'\} \in E(G) \text{ and } \{y, y'\} \in E(H)\}$.)

Thus the above mentioned result of Müller can be stated by saying that complete graphs are projective. Larose and Tardif proved some sufficient conditions for a graph to be projective. It is easy to derive from these conditions that many classes of graphs are projective, including Kneser graphs, graphs G_k^d, etc.. Our companion paper [12] contains a short proof of the projectivity of complete graphs and graphs G_k^d.

The notion of projective graphs leads to the following (which extends Müller's theorem to non-complete graphs):

Corollary 1.2 *Let H be projective core graph with k vertices. Let A be a finite set and let $f_1, f_2, \cdots, f_t$ be distinct mappings $A \to V(H)$. Then there exists a graph $G = (V, E)$ such that the following holds:*

i. A is a subset of V;

ii. For every $i = 1, 2, \cdots, t$ there exists unique homomorphism $g_i : G \to H$ such that g_i restricted to the set A coincides with the mapping f_i;

iii. For every homomorphism $f : G \to H$ there exists $i, 1 \le i \le t$ and an automorphism h of H such that $h \circ f_i = f$;

iv. G has girth $> l$.

Proof. Consider the graph $F = H^t \times K_N$ where K_N is the complete graph with N vertices, $N > max\{k^{kt}, |A|\}$. We apply 1.1 for the graphs F and H. Thus there exists a graph G and a homomorphism $c : G \to F$ (we preserve the notation of Theorem 1.1) such that any homomorphism $g : G \to H$ there exists a homomorphism $f : F \to H$ such that $g = f \circ c$. Now , up to automorphisms of H, all the homomorphisms $F \to H$ are induced by t projections $\pi_1, \pi_2, \cdots, \pi_t : H^t \to H$. In other words, every homomorphism $f : F \to H$ for which $f(x, \cdots, x, a) = x$ is of the form $f(x, a) = \pi_i(x)$ for every vertex (x, a) of F and some $i, 1 \le i \le t$. Hence, up to an automorphism of H, there are exactly t homomorpihsms from G to H: $\pi_i \circ c$, $i = 1, 2, \cdots, t$. (Here we use $N > k^{kt}$, see section 3 where this will be explained in a greater detail.) Now consider mappings $f_1, f_2, \cdots, f_t$ together with an injective mapping $f_0 : A \to V(K_N)$. Then the corresponding mapping $\phi = (f_0, f_1, f_2, \cdots, f_t) : A \to V(F)$ is injective and thus we can identify A with its image of $\phi(A)$. Then each of the t homomorphisms $\pi_i \circ c$ is an extension of the mapping f_i. ∎

Theorem 1.1 has also the following corollary known as *Sparse Incomparability Lemma* [11], [13]:

Corollary 1.3 *For every pair H, H' of graphs such that H' is H-colorable and H fails to be H'-colorable there exists a graph G with the following properties:*

i. $g(G) \ge l$

ii. G is H-colorable and G fails to be H'-colorable.

(To obtain Corollary 1.3 we put $F = H$, $k = |V(H')|$ in Theorem 1.1.)

The paper is organized as follows: In Section 2 we prove Theorem 1.1. As the proof is non-constructive we include in Section 3 a simple proof of a weaker statement with girth replaced by the odd girth. In Section 4 we prove that the analogy of Müller's theorem holds for projective core graphs only (this is stated as Theorem 3.2) and add a few remarks and problems.

2 Proof of Theorem 1.1

Our proof uses probabilistic method and most of the calculations are fairly standard. But it is an indication of the proper setting of Theorem 1.1 that the proof is perhaps easier than the proofs of particular cases, [2, 13].

Suppose that the graph F has a vertices and that the vertices are $\{1, 2, \cdots, a\}$, and the edge set $E(F)$ has cardinality q. Let $V_1, V_2, \cdots, V_a$ be disjoint n-sets. Let G_0 be the graph with vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_k$, and $\{x, y\} \in E(G_0)$ if and only if $x \in V_i, y \in V_j$ and $(i, j) \in E(F)$. Then G_0 has qn^2 edges. Let $\mathcal{G}$ be the set of all subgraphs G of G_0 with $m = \lfloor qn^{1+\epsilon} \rfloor$ edges, where $0 < \epsilon < 1/l$. Put also $\delta = \min\{\epsilon l, 1/k\}$. Then $|\mathcal{G}| = \binom{qn^2}{m}$.

In the following, n is assumed to be sufficiently large. We consider $\mathcal{G}$ as a probability space with each member occurring with the same probability $1/|\mathcal{G}|$. This is asymptotically the same thing as the random graph $\mathbf{G}$ where we choose edges from the set $E(G_0)$ independently with the probability $n^{-1+\epsilon}$.

We shall make use of the fact that most graphs in $\mathcal{G}$ have few short cycles which are pairwise vertex disjoint. On the other hand, the edges are "dense" in some sense. These results are stated as Claims 1-4:

Claim 1 The expected number of cycles of length $\leq l$ in a graph $G \in \mathcal{G}$ is bounded by n^δ and thus asymptotically almost all graphs from $\mathcal{G}$ have at most n^δ cycles of length $\leq l$.

Claim 2 The expected number of pairs of cycles of length $\leq l$ in a graph $G \in \mathcal{G}$ which intersect in at least one vertex is bounded by n^δ and thus asymptotically almost all graphs from $\mathcal{G}$ have at most n^δ cycles of length $\leq l$, and these cycles are all vertex disjoint.

Claim 3 A set $A \subset V$ is said to be *large* if there are i, j, $1 \leq i < j \leq k$, $\{i, j\} \in E(F)$, such that $|A \cap V_i| \geq \delta n$ and also $|A \cap V_j| \geq \delta n$. We call an edge $\{i, j\}$ of F a *good edges* of A if $|A \cap V_i| \geq \delta n$ and $|A \cap V_j| \geq \delta n$. For a large set A denote by $|\mathbf{G}/A|$ the minimum number of edges of $\mathbf{G}$ which lie in the set $\{\{x, y\};\ x \in V_i,\ y \in V_j\}$ for a good edge of A.

Then the probability $Prob[A \text{ large implies } |\mathbf{G}/A| \geq n] = 1 - o(1)$.

Thus asymptotically almost all graphs from $\mathcal{G}$ have the property that all good edges (of F) of any large set induce at least n edges (of $\mathbf{G}$).

Claim 4 Almost all graphs from $\mathcal{G}$ do not contain two non empty sets $A \subset V_{i_0}, B \subset V_{j_0}, 1 \leq i_0 < j_0 \leq a$, $|A| + |B| \geq \delta n$ such that the set $A \cup B$ contains at most $\min\{|A|, |B|, n^\delta\}$ edges and these edges form a matching (i.e. a set of mutually disjoint edges).

For the completeness let us include at least short proofs of Claims 3 and 4:

Proof. (of Claim 3)

We first estimate probability

$$\alpha = Prob[A \text{ large implies } |\mathbf{G}/A| \geq n].$$

We have

$$1 - \alpha \leq \sum_{A \text{ large}} Prob[|\mathbf{G}/A|] < n] \leq 2^{kn} \cdot \binom{qn^2}{n} \cdot (1-p)^{\delta^2 n^2 - n}.$$

Now bounding very roughly

$$\binom{qn^2}{n} \leq \binom{k^2 n^2}{n} \leq k^{2n} n^{2n} < e^{cn \log_2 n}$$

and

$$(1-p)^{\delta^2 n^2 - n} \leq e^{-p(\delta^2 n^2 - n)}$$

we obtain

$$1 - \alpha < e^{cn \log_2 n - c' n^{1+\epsilon}}$$

for some positive constants c and c' which are independent on n.

Thus we get $Prob[A \text{ large implies } |\mathbf{G}/A| \geq n] = 1 - o(1)$. ∎

Proof. (of Claim 4)

This is similar, this time we give a counting version of the proof. We shall show that very few graphs in $\mathcal{G}$ contain subgraphs induced by $A \cup B$, $|A \cup B| = \delta n$ with at most $\min\{|A|, |B|, n^\delta\}$ edges (even without matching condition). Towards this end for $b \leq n^\delta, s \leq \min\{b, n^\delta\}$ we denote by $P(b, s)$ the expected number of pairs $A \subset V_i, B \subset V_j$ such that $\{i, j\} \in E(H)$, $|A| + |B| = \delta n$, $|A| = b$ and there are exactly s edges between W_i and W_j.

Then

$$P(b,s) < 2q\binom{n}{\delta n - b}\binom{n}{b}\binom{b(\delta n - b)}{s}\binom{qn^2 - b(\delta n - b)}{m-s}\binom{qn^2}{m}^{-1}.$$

We have $b < \frac{\delta n}{2}$, and $b(\delta n - b) \geq \frac{\delta b n}{2}$ thus

$$\binom{qn^2 - b(\delta n - b)}{m-s}\binom{qn^2}{m}^{-1} < \binom{qn^2 - \delta bn/2}{m}\binom{qn^2}{m}^{-1} < e^{-\frac{bn^\delta}{2}}.$$

(Here we used the inequality $\binom{a-x}{b}\binom{a}{b}^{-1} \leq \left(\frac{a-b}{a}\right)^x < e^{-bx/a}$.)

Therefore

$$P(b,s) < 2qn^{\delta n - b}n^b(bn)^s e^{-\frac{bn^\delta}{2}} < 2q(bn)^s \exp\left(-\frac{bn^\delta}{2} + \delta n \mathrm{log} n\right) < n^{2s}e^{-\frac{bn^\delta}{2}}.$$

Let $P(b)$ be the sum of all $P(b,s)$ for which $s \leq \min\{b, n^\delta\}$.

If $b < n^\delta$, then $s \leq b$, and hence

$$P(b) < \exp\left(-\frac{bn^\delta}{2} + 3b\mathrm{log} n\right) < e^{-\frac{bn^\delta}{3}} < e^{-n^{\delta/2}}.$$

If $b \geq n^\delta$, then $s \leq n^\delta$, and hence

$$P(b) < \exp\left(-\frac{n^\delta}{2k} + 3n^\delta \mathrm{log} n\right) < e^{-\frac{n^\delta}{3k}} < e^{-n^\delta}.$$

Therefore

$$\sum_{1 \leq b \leq \frac{n}{2}} P(b) < e^{-n^{\delta/3}}.$$

This establishes proof of Claim 4. ∎

(Proofs of all these claims are by now a folklore and appear in various forms in the literature and mostly go back to Erdős.)

Thus let G' be an instance of the graph from $\mathcal{G}$ with all the properties claimed in Claims 1-4 for majority of graphs from $\mathcal{G}$.

Explicitly, let $G' \in \mathcal{G}$ be any graph with the following properties:

The graph G' contains at most n^δ cycles of length $\leq l$ and all these cycles are vertex disjoint.

Consequently, there exists a set M of edges of G' which forms a matching in G' of size at most n^δ such that the graph $G' - M = (V(G'), E(G') - M)$ has no cycles of length $\leq l$. Put $G = G' - M = (V, E)$. We prove that the graph G is our desired graph. It is clear that G has girth $> l$

The graph G has further properties which follow from Claims 3 and 4. For every good pair $i < j$ in any large set A there is an edge of G with one vertex in $A \cap V_i$ and the other in $A \cap V_j$ (this follows from Claim 3). Further, it follows from the Claim 4 that even for every pair of non empty subsets A, B, $|A| + |B| \geq \delta n$ there exists an edge of G with its end vertices in the set $A \cap B$.

Define the mapping $c : V \to V(F)$ by $c(x) = i$ iff $x \in V_i, i = 1, \cdots, a$. Clearly c is a homomorphism $G \to F$.

Let H be a fixed graph with at most k vertices and let $g : G \to H$ be a homomorphism.

We define a mapping $\phi : V(F) \to V(H)$ as follows: For each $i \in V(F)$, there exists a vertex $x \in V(H)$ such that $|V_i \cap g^{-1}(x)| \geq n/k \geq \delta n$ by pigeonhole principle. We let $\phi(i)$ be any (fixed) x with $|V_i \cap g^{-1}(x)| \geq n/k \geq \delta n$. (If there are more than one x satisfy the condition, we arbitrarily choose one.)

We prove that ϕ is a homomorphism $F \to H$. Thus let $\{i, j\}$ be an edge of F. Put $A_i = g^{-1}(\phi(i))$ and $A_j = g^{-1}(\phi(j))$. As there exists an edge e of G with its ends in $A_i \cup A_j$ (this follows from Claim 3) we have that $\phi(i)$ and $\phi(j)$ form an edge of H.

This proves part ii. of the Theorem 1.1. Thus from now on let H be a pointed graph.

It follows that we can further assume that $|V_i \cap g^{-1}(j)| < \delta n$ for all $j \neq i$. For otherwise there are i, j such that $j \neq i$ with $|V_i \cap g^{-1}(j)| \geq \delta n$. Then we can define another mapping ϕ' which agrees with ϕ on every other vertex, and $\phi'(i) = j$. However this is a contradiction as H is a F -pointed graph.

The set $V_i \cap g^{-1}(\phi(i))$ will be denoted by W_i, $i = 1, \cdots, a$.

Thus the homomorphism ϕ is uniquely determined by the homomorphism g. It remains to prove that $\phi \circ c = g$.

Assume to the contrary that $\phi \circ c \neq g$.

Thus there exists $x \in V(H), x \neq x_0$ and $i_0 \in V(F)$ such that $\phi(i_0) = x_0 \neq x$ while the set $W = g^{-1}(x) \cap V_{i_0}$ is non-empty.

Now for any $j \in V(F)$ with $\{j, i_0\} \in E(F)$ the set $W_j \cup W$ has at least δn vertices and thus (using Claim 4) there exists an edge $e \in E(G)$ with its ends in W_j and W. And this happens (by the same argument) if and only if there exists an edge $e \in E(G)$ with with its ends in W_j and W_{i_0}. Hence the mapping ϕ' defined by $\phi'(i) = \phi(i)$ for all $i \neq i_0$, $\phi'(i_0) = x$ is a homomorphism $F \to H$ and $\phi' \neq \phi$. This is contradicting the property that H is F-pointed.

Thus $g = \phi \circ c$.

This completes the proof of Theorem 1.1

3 Odd Girth, constructible sets and products

Here we prove the analogy of Theorem 1.1 for graphs of a given odd girth by giving a construction. The reason why we prove this weaker statement is that for general graphs H such a construction is not available (and this is stated as Problem 1; in a companion paper [12] we give the constructive proof of Theorem 1.1 for graphs G_k^d). Moreover we characterize all graphs H for which the Müller type theorem 1.1 is valid.

Theorem 3.1 *For every graph F and every choice of positive integers k and l there exists a graph G_0 such that the graph $G = F \times G_0$ together with the projection $c : F \times G_0 \to F$ has the following properties:*

i. $\mathrm{oddg}(G) > l$*;*

ii. For every graph H with at most k vertices, there exists a homomorphism $g : G \to H$ if and only if there exists a homomorphism $f : F \to H$.

iii. For every F-pointed core graph H with at most k vertices and for every homomorphism $g : G \to H$ there exists a unique homomorphism $f : F \to H$ such that $g = f \circ c$.

.

$$\begin{tikzcd}
F \arrow[r, "f"] & H \\
F \times G_0 \arrow[u, "c"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Proof

Let G_0 be a graph with odd girth $> l$ and chromatic number $> k^{|V(F)|}$ (a construction is provided e.g. by iterated shift graphs). Put $G = F \times G_0$. It is well known (and easy to see) that $\mathrm{oddg}(F \times G_0) = \max\{\mathrm{oddg}(F), \mathrm{oddg}(G_0)\}$.

Thus let H be a graph with at most k vertices and let $g : G \to H$ be a homomorphism.

For every $y \in V(G_0)$ define the mapping $f_y : V(F) \to V(H)$ by $f_y(x) = g((x, y))$. Note that the mapping f_y need not be a homomorphism $F \to H$ but as $\chi(G_0) > k^{|V(F)|}$ there exists an edge $\{y, y'\} \in E(G_0)$ such that $f_y = f'_y$. However in this case the mapping $f = f_y = f'_y$ is a homomorphism $F \to H$: given $\{x, x'\} \in E(F)$ we have $\{(x, y), (x', y')\} \in E(F \times G_0)$ and thus $\{g(x, y), g(x', y')\} = \{f_y(x), f'_y(x')\} = \{f_y(x), f_y(x')\} = \{f(x), f(x')\} \in E(H)$.

This proves *ii.*

So suppose that in addition the graph H is a F-pointed core. Under this assumption the validity of *iii.* follows readily from the following (which generalizes [4, 11, 15]):

Claim If f_y (defined above) is a homomorphism $F \to H$ for some $y \in V(G_0)$ and that $\{y, z\} \in E(G_0)$, then $f_z = f_y$.

Assume to the contrary that $f_z(x_0) \neq f_y(x_0)$ for a vertex $x_0 \in V(F)$. Define mapping $g : V(F) \to V(H)$ as $g(x) = f_y(x)$ for $x \neq x_0$ and $g(x_0) = f_z(x_0)$. Then g is a homomorphism $F \to H$ (It suffices to check edges of F incident with x_0: If $\{x, x_0\} \in E(F)$ then $\{(x, y), (x_0, z)\} \in E(F \times G_0)$ and thus $\{f_y(x), f_z(x_0)\} = \{g(x), g(x_0)\} \in E(H)$.) However this is in contrary with the fact that H is F-pointed.

Corollary 3.1 *Let H be a projective core graph with k vertices. Let A be a finite set and let $f_1, f_2, \cdots, f_t$ be distinct mappings $A \to V(H)$. Then there exists a graph G_0 such that the graph $G = H^t \times G_0$ contains a subset A' together with 1-1 correspondence $\iota : A' \to A$ with the following properties:*

i. For every $i = 1, 2, \cdots, t$ there exists unique homomorphism $g_i : G \to H$ such that g_i restricted to the set A' coincides with the mapping $f_i \circ \iota$;

ii. For every homomorphism $f : G \to H$ there exists $i, 1 \leq i \leq t$ such that $f_i \circ \iota = f$;

iii. G has odd girth $> l$.

One can ask to what extend is the projectivity a necessary condition for a validity of Corollaries 1.2,3.1.

Recent work of Claude Tardif and Benoit Larose allows us to characterize all graphs H for which an analogy of Müller's Theorem is valid. This is non-trivial and it is based on the following notions which are introduced in [6, 7] and go back to [1]:

A set C of vertices of a graph H is said to be *constructible* if there exists a graph G, vertices $x_0, x_1, \ldots, x_n$ of G and vertices $y_1, \ldots, y_n$ of H such that C is the set of all $g(x_0)$ where g is any homomorphism from G to H such that $g(x_i) = y_i$ for all $i = 1, \ldots, n$.

Larose and Tardif [6] proved a remarkable result which states that the graph H is projective if and only if every subset of its vertex set is constructible and this is equivalent to that every two element subset of $V(H)$ is constructible.

Perhaps surprisingly, the constructibility of 2-element sets is equivalent with the validity of Müller's theorem. We state this as:

Theorem 3.2 *For a core graph H, the following statements are equivalent:*

I. *For any choice of a finite set A and distinct mappings $f_1, f_2, \cdots, f_t : A \to V(H)$ there exists a graph $G = (V, E)$ such that the following holds:*

i. A is a subset of V;

ii. For every $i = 1, 2, \cdots, t$ there exists unique homomorphism $g_i : G \to H$ such that g_i restricted to the set A coincides with the mapping f_i;

iii. For every homomorphism $f : G \to H$ there exists $i, 1 \leq i \leq t$ and a homomorphism $h : H \to H$ such that $h \circ f_i = f$;

iv. G has girth $> l$.

II. *The graph H is projective;*

III. *Every 2-subset of $V(H)$ is constructible.*

Proof. The equivalence of **II**. and **III**. was established in [6], **II**. implies **I**. by Corollary 1.2. We prove **I**. implies **III**.:

Assume H has vertex set $V = \{v_1, v_2, \cdots, v_n\}$. Let $\{v_i, v_j\}$ be a 2-element subset of V. We need to show that $\{v_i, v_j\}$ is constructible.

Consider i. for $A = V^2$ with projections $f_1 = \pi_1$ and $f_2 = \pi_2$. Let G be a graph satisfying **I**.. Let $x_0 = (v_i, v_j)$ and for $i = 1, 2, \cdots, n$, let $x_i = (v_i, v_i)$ and $y_i = v_i$. Then for any homomorphism $f : G \to H$ with $f(x_i) = y_i$ for $i = 1, 2, \cdots, n$, and $f(x_0) \in \{v_i, v_j\}$. Furthermore, there is a homomorphism f from G to H, namely the extension of f_1 to G, which satisfies $f(x_i) = y_i$ for $i = 1, 2, \cdots, n$ and $f(x_0) = v_i$; and there is a homomorphism f from G to H, namely the extension of f_2 to G, which satisfies $f(x_i) = y_i$ for $i = 1, 2, \cdots, n$ and $f(x_0) = v_j$. Thus $\{v_i, v_j\}$ is a constructible set. ∎

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