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# A rigid graph for every set

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**Abstract**

A graph $G$ is called *rigid* if the identical mapping $V(G) \to V(G)$ is the only homomorphism $G \to G$. In this note we give a simple construction of a rigid oriented graph on every set.

## 1 Construction

Let $X$ be an infinite set and assume that $X$ is an ordinal $X = \{\beta; \beta < \alpha\}$. Let $X' = \{\beta'; \beta' < \alpha'\}$ be a disjoint copy of $X$. Further let $\{a, b, c, a', b', c'\}$ be six vertices disjoint with $X$ and $X'$. For every ordinal $\beta < \alpha$ with countable cofinality let us choose a sequence $\{\beta_n\}$ such that $\sup \beta_n = \beta$. We define the oriented graph $(V, E)$ by the following set of arcs:

$(0, a), (a, b), (b, c), (c, 0), (0, b)$ and $(0', a'), (a', b'), (b', c'), (c', 0'), (a', c')$;
$(\beta, \gamma)$ and $(\beta', \gamma')$ for all $\beta < \gamma < \alpha$;
$(\beta, \beta')$ for all $\beta < \alpha$;
for every ordinal $\beta < \alpha$ with countable cofinality let $\beta$ be joined with $\beta'_n$ by an oriented path of length $n + 2$. (All these paths are supposed to be vertex disjoint.)

Let $V$ be the set of all vertices thus obtained. Clearly $V$ is a countable union of sets of cardinality $\leq \alpha$ and thus $V$ and $X$ are in 1-1 correspondence.

**Theorem 1.1** *The oriented graph $G = (V, E)$ is rigid.*

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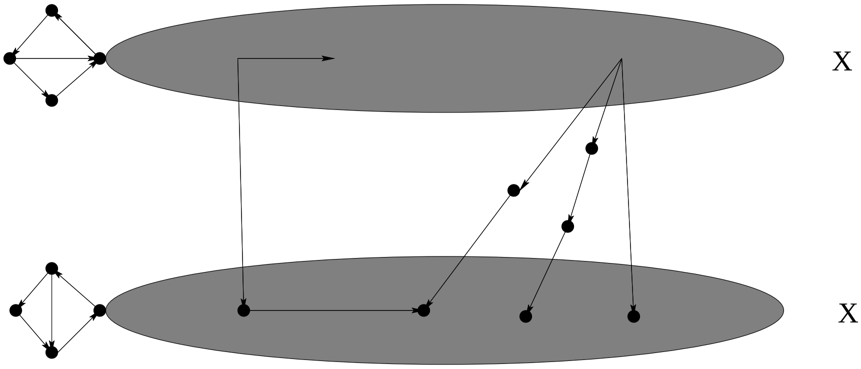

## 2 Proof

Let $f : V \to V$ be a homomorphism. As $X$ and $X'$ are transitive orientations of complete graphs, $f$ restricted to both $X$ and $X'$ is an injection. The graph $G$ is acyclic with the exception of vertices $\{a, b, c, 0\}$ and $\{a', b', c', 0'\}$. The graphs induced by these sets are isomorphic, but there is not homomorphism between them which would preserve vertices 0 and $0'$. However the mapping $f$ restricted to $\{a, b, c, 0\}$ satisfies either $f(0) = 0$ or $f(0) = 0'$ (as vertices $0, 0'$ are distinguished by large clique which includes them). This shows that both $f(0') = 0$ and $f(0) = 0'$ are impossible ant thus $f(0) = 0, f(0') = 0'$. Consequently $f$ restricted to the set $\{a, a', b, b', c, c'\}$ is the identity. It follows that $f$ maps $X$ to $X$ and $X'$ to $X'$. As pairs $(\beta, \beta')$ are the only arrows between $X$ and $X'$ we have that $(f(\beta))' = f((\beta)')$ for every $\beta < \alpha$. Let $\beta(0)$ be the smallest $\beta$ for which $f(\beta) \neq \beta$. Then necessarily $f(\beta(0)) > \beta(0)$ (as if $f(\beta(0)) < \beta(0)$ then $f(f(\beta(0))) < f(\beta(0))$ which violates that $\beta(0)$ was minimal). Thus $\beta(0) < f(\beta(0))$. We put $f(\beta(0)) = \beta(1)$ and $\beta(n) = f^n(\beta(0))$ (the $n$-times iterated mapping $f$). Let $\beta = \sup\beta(n)$. $\beta$ is also the limit of the sequence $\{\beta_n\}$. However as the sequences $\beta(n)$ and $\beta_n$ are interlacing and as $f$ maps the set $\{\beta(n)\}$ into a subset we get by monotonicity that that the sets $\{\beta(n)\}$ and $\{f(\beta_n)\}$ are interlacing again and thus by the definition of the graph $G$ we get $f(\beta) = \beta$ (as $\beta$ is the only vertex joined to the set $\{\beta_n'\}$ by directed paths). But then $f(\beta_n) = \beta_n$ for every $n$ (as in our situation the mapping $f$ has to preseve the length of paths between $\beta$ and $\beta_n'$. This is a contradiction as if we choose $m$ and $n$ such that $\beta(m) < \beta_n < \beta(m+1)$ then $f(\beta_n) > \beta(m+1)$, a final contradiction.

## 3 Remarks

The existence of a rigid graph on every set is an important result which lies in the heart of several combinatorial and non-combinatorial embeddings (see [3]). It has been proved by Vopěnka, Hedrlín and Pultr [4] and reproved in [1] and [3]. All these proofs are modification of the original proof in that they either use embeddings for relational systems with countably many relations (as in [1]) or depend more on the ordinal arithmetic (as in [3],p. 63-65). This is not necessary as shown by our direct construction.

Perhaps the importance of this result justifies yet another simpler proof. Let us remark that all the proofs (including the present one) are using "fixing" ordinals with countable cofinality (this original idea of [4] is credited to Vopěnka in [3]).

Finally let us remark that this is an infinite problem as all finite directed paths are rigid. These simplest finite rigid graphs serve as building blocks of our construction. It is important that one can prove the existence of a rigid graph on every set in ZFC. This is in a sharp (and surprising) contrast with difficulties when one wants to construct a proper class of *mutually rigid* graphs (see [3] for a disscussion of this).

If we want to construct a rigid undirected graph (or graphs with some special properties) we can replace each edge by one suitable finite undirected rigid graph. As can be expected the same is true if we want to get an acyclic or well founded relation (we use e.g. rigid balanced orientation of a path), see [3] and [2].

## References

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