

# Füredi–Hajnal Conjecture Implies Stanley–Wilf Conjecture

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## Abstract

We show that Stanley–Wilf enumerative conjecture on permutations follows easily from Füredi–Hajnal extremal conjecture on 0-1 matrices. We apply the method of deriving an (almost) exponential bound on number of objects from a (almost) linear bound on their sizes, which was discovered by Alon and Friedgut. They proved by it a weaker form of Stanley–Wilf conjecture. Using bipartite graphs, we give a simpler proof of their result.

Покажем, что гипотеза Стэнли и Вилфа о числе перестановок вытекает простым образом из экстремальной гипотезы Фиреды и Хайнала о 0-1 матрицах. Применяем метод вывода (почти) экспоненциальной оценки числа объектов из (почти) линейной оценки их величин открытый Алоном и Фридгутом. Этим методом они доказали гипотезу Стэнли и Вилфа в ослабленной форме. С помощью двудольных графов получим более простое доказательство их результата.

Stanley–Wilf conjecture asserts that the number of  $n$ -permutations not containing a given permutation is exponential in  $n$ . Alon and Friedgut [1] proved that it is true provided we have a linear upper bound on lengths of certain words over an ordered alphabet. They also proved a weaker version of it with an almost exponential upper bound. In the present note we want to inform the reader about this interesting development by reproving the latter result in a simpler way. We use bipartite graphs instead of words. We point out that in 1992 Füredi and Hajnal almost made an extremal conjecture on 0-1 matrices that now can easily be seen to imply Stanley–Wilf conjecture. We prove that both extremal conjectures are logically equivalent.

Two finite sequences  $u = a_1a_2\dots a_k \in \mathbf{N}^*$  and  $v = b_1b_2\dots b_l \in \mathbf{N}^*$  of positive integers are *isomorphic* if  $k = l$  and, for every  $i$  and  $j$ ,  $a_i < a_j$  is equivalent to  $b_i < b_j$ . We say that  $v$  *contains*  $u$  if  $v$  has a subsequence that is isomorphic to  $u$ ; we write  $v \supset_< u$ . Replacing  $<$  by  $=$ , we define in the same way the corresponding isomorphism and containment relation  $\supset_ =$ . For example,  $31225345 \supset_ = 2121$  but  $31225345 \not\supset_ < 2121$ . The length of a sequence  $u$  is  $|u|$  and  $[n]$  is the set  $\{1, 2, \dots, n\}$ .

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Permutations can be represented as elements of  $\mathbf{N}^*$ , e.g.  $p = 32154$ . If  $|p| = k$ , we speak of a  $k$ -permutation. The following conjecture from 1990 (?) is well known among enumerative combinatorialists.

**STANLEY–WILF CONJECTURE.** For each permutation  $p$  there is a constant  $c = c(p)$  such that the number  $S_n(p)$  of  $n$ -permutations  $q$ ,  $q \not\prec p$  satisfies  $S_n(p) < c^n$ .

An evidence for it is the result of Bóna [2] that it holds for all  $p$  of the form  $p = I_1 I_2 \dots I_r$  where  $I_1 < I_2 < \dots < I_r$  and each sequence  $I_i$  is decreasing.

A sequence  $v = b_1 b_2 \dots b_l \in \mathbf{N}^*$  is  $k$ -sparse if  $b_j = b_i, j > i$  implies  $j - i \geq k$ . In other words, in each interval in  $v$  of length  $\leq k$  all terms are distinct.

**ALON–FRIEDGUT CONJECTURE.** For each  $k$ -permutation  $p$  there is a constant  $c = c(p)$  such that if  $v \in [n]^*$  is  $k$ -sparse and  $v \not\prec p$  then  $|v| < cn$ .

In [1] it is shown that if AFC is true then so is SWC. To be precise, AFC is put there in the form of a question rather than a conjecture.

Suppose  $M = (m_{ij})$  and  $N = (n_{ij})$  are  $a \times b$  and  $c \times d$  matrices with entries in  $\{0, 1\}$ .  $M$  contains  $N$  if there are indices  $1 \leq i_1 < \dots < i_c \leq a$  and  $1 \leq j_1 < \dots < j_d \leq b$  such that, for all  $r = 1 \dots c$  and  $s = 1 \dots d$ ,  $m_{i_r, j_s} = 1$  whenever  $n_{r,s} = 1$ . So  $M$  has a (not necessarily consecutive) submatrix of  $N$ 's size that has 1s on all the places where  $N$  has them and maybe on some others.

Füredi and Hajnal [3] investigated the extremal function  $f(m, n; N)$  that is defined as the maximum number of 1s in an  $m \times n$  0-1 matrix  $M$  that does not contain  $N$ ;  $f(n; N) = f(n, n; N)$ . They looked systematically at all 37 substantially distinct  $N$ s with four 1s and no zero row or column. (We leave to the interested reader to prove as an exercise that if  $N$  has at most three 1s then  $f(m, n; N) = O(m + n)$ . For four 1s the situation is much more complicated.) In the concluding section of [3] they ask if for each permutation matrix  $N$  we have  $f(n; N) = O(n)$ : "Is it true that the complexity of all permutation configurations are linear?" We take the liberty to formulate it as a conjecture.

**FÜREDI–HAJNAL CONJECTURE.** For each permutation matrix  $N$  we have  $f(n; N) = O(n)$ .

We show, using the ideas of [1], that FHC implies SWC. We prefer to think of the matter in terms of bipartite graphs. Let  $G$  and  $H$  be two simple bipartite graphs with parts  $[a], [b]'$  and  $[c], [d]'$ . Here  $[b]' = \{1', 2', \dots, b'\}$  and all the parts are linearly ordered in the natural way. We say that  $G$  contains  $H$  if  $H$  is a (ordered!) subgraph of  $G$ , that is there exist increasing injections  $f : [c] \rightarrow [a]$  and  $g : [d]' \rightarrow [b]'$  such that if  $\{i, j'\}$  is an edge of  $H$  then  $\{f(i), g(j')\}$  is an edge of  $G$ . Besides being a sequence and a 0-1 matrix, each permutation  $p = a_1 a_2 \dots a_k$  is also a bipartite graph  $G_p$  with the parts  $[k], [k]'$  and the edges  $\{a_i, i'\}, i \in [k]$ . FHC then says that each bipartite graph on  $[n], [n]'$  not containing  $G_p$  has  $O(n)$  edges.

**Theorem 1** *If FHC is true then so is SWC.*

**Proof.** Let  $p$  be a permutation and  $M(n)$  the set of simple bipartite graphs on  $[n], [n]'$  that do not contain  $G_p$ . We assume FHC — there is a constant  $c$  such that  $|E(G)| < cn$  for each  $G \in M(n)$ . Let  $n > 1$ . For a  $G \in M(n)$  we define a  $G_1$  on  $[m], [m]'$ , where  $m = \lceil n/2 \rceil$ , by

$$\{i, j'\} \in E(G_1) \iff \exists e \in E(G) : e \cap \{2i - 1, 2i\} \neq \emptyset \& e \cap \{2j' - 1, 2j'\} \neq \emptyset.$$

Clearly,  $G_1 \in M(m)$ . Also, one  $G_1$  arises from at most  $15^{|E(G_1)|} < 15^{cm}$  graphs  $G$  because there are 15 possibilities for a nonempty restriction of  $G$  to  $\{2i - 1, 2i\}, \{2j' - 1, 2j'\}$ . Hence,

$$|M(n)| < 15^{c \lceil n/2 \rceil} \cdot |M(\lceil n/2 \rceil)|.$$

Iterating the inequality until  $|M(1)| = 2$ , we obtain an upper bound on  $|M(n)|$  that is exponential in  $n$ . But  $S_n(p) \leq |M(n)|$  because for each  $n$ -permutation  $q, q \not\prec p$  we have  $G_q \in M(n)$ . Thus

$$S_n(p) \leq |M(n)| < 15^{2cn}.$$

□

Alon and Friedgut prove the weaker form of SWC with an almost exponential bound by means of the following result due to Klazar [5]. Suppose  $u \in [k]^*$  is given. If  $v \in [n]^*$  is  $k$ -sparse and  $v \not\prec_{=} u$  — notice that now we use the weaker containment, then

$$|v| < nc^{\alpha(n)^d}, \tag{*}$$

where  $c, d > 1$  are moderate constants depending only on  $u$  and  $\alpha(n)$  is the inverse of the Ackermann function  $A(n)$  known from the recursion theory.

We remind the reader the definition of  $A(n)$  and  $\alpha(n)$ . If  $F_1(n) = 2n, F_2(n) = 2^n$ , and in general  $F_{i+1}(n) = F_i(F_i(\dots F_i(1)\dots))$  with  $n$  iterations of  $F_i$ , then  $A(n) = F_n(n)$  and  $\alpha(n) = \min\{m : A(m) \geq n\}$ . Although  $\alpha(n) \rightarrow \infty$ , in practice  $\alpha(n)$  is bounded:

$$\alpha(n) \leq 4 \text{ for } n < 2^{2^{\dots^2}}$$

where the tower has  $2^{16} = 65536$  2s.

**Theorem 2** *For each fixed permutation  $p$ ,*

$$S_n(p) \leq |M(n)| < 225^{n\beta(n)} \text{ with } \beta(n) = c^{\alpha(n)^d},$$

where  $c$  and  $d$  are moderate constants depending only on  $|p|$ .

**Proof.** We use the notation of the previous proof and set  $k = |p|$ . If instead of  $|E(G_1)| < cm$  the bound  $|E(G_1)| < m\beta(m)$  with an increasing function  $\beta(m)$  is used, we get

$$S_n(p) \leq |M(n)| < 15^{2n\beta(n)} = 225^{n\beta(n)}.$$

Let  $G \in M(n)$ . It remains to derive a good bound  $|E(G)| < n\beta(n)$ . Consider the sequence  $v = L_1L_2 \dots L_n \in [n]^*$ , where  $L_i$  is the list of the neighbours of  $i'$  in  $G$ , in the increasing order. The sequence  $v$  is in general not  $k$ -sparse but it is easy to see that by deleting  $\leq (k-1)(n-1)$  appropriate elements,  $\leq k-1$  from the beginning of each of  $L_2, \dots, L_n$ , we can obtain a  $k$ -sparse subsequence  $w$ . It is also not difficult to see that if  $v \supseteq u(k)$ , where  $u(k) = 12 \dots k12 \dots k \dots 12 \dots k$  has  $k$  segments  $12 \dots k$ , then  $G$  contains  $G_p$ , in fact for any  $k$ -permutation  $p$ . So  $w \not\supseteq u(k)$  and we can apply the aforementioned result:

$$|E(G)| = |v| \leq (k-1)(n-1) + |w| < (k-1)(n-1) + nc^{\alpha(n)^d}.$$

□

Our bound is weaker compared to the bound of Alon and Friedgut in [1]. They use a more complicated induction step in which they decrease  $n$  more than to ours  $\lceil n/2 \rceil$ . As we mentioned, they do not work with graphs but in  $(\mathbf{N}^*, \supset_<)$  and  $(\mathbf{N}^*, \supset_=)$ .

We show that both extremal conjectures are equivalent.

**Theorem 3** *AFC and FHC are mutually equivalent.*

**Proof.** We prove first that AFC implies FHC. If  $v = a_1a_2 \dots a_l \in \mathbf{N}^*$ ,  $1 \leq i \leq j < k \leq l$ , and  $a_j > a_{j+1}$ , we say that  $i$  and  $k$  are in  $v$  separated by fall. Let  $p$  be a permutation and  $G$  a bipartite graph on  $[n], [n]'$  not containing  $G_p$ . Consider the sequence  $v = L_1L_2 \dots L_n \in [n]^*$  of the previous proof. Recall that each  $L_i$  is increasing. It may happen that  $v \supset_< p$  because one  $L_i$  can contribute to the subsequence isomorphic to  $p$  by more than one element. To prevent it, we take a larger permutation  $p' \supset_< p$ ,  $|p'| = k'$  such that each two consecutive elements of the subsequence isomorphic to  $p'$  are in  $p'$  separated by fall. Now  $v \supset_< p'$  is impossible. As we know, by deleting  $< k'n$  elements we can obtain a  $k'$ -sparse subsequence  $w$ . By AFC for  $p'$ , we have a linear bound on  $|w|$ . So  $|E(G)| = |v| < k'n + |w| = O(n)$ .

We prove that FHC implies AFC. If we are content in AFC with *any* bound, the (unconditional) proof is easy. Suppose  $u \in [k]^*$ ,  $v \in [n]^*$  is  $k$ -sparse, and  $v \not\supset_< u$ . Split  $v$  into intervals of length  $k$  and a remainder of length  $< k$ . By the pigeonhole principle, there are at most  $(|u| - 1)n()k$  intervals. Therefore

$$|v| < k((|u| - 1)n()k + 1).$$

Now suppose that  $p$  is a  $k$ -permutation and  $v \in [n]^*$  is a  $k$ -sparse sequence not containing  $p$ . Using FHC, we prove a linear bound on  $|v|$ . There is a constant  $c$  such that  $|E(G)| < cn$  holds for each bipartite graph on  $[n], [n]'$  not containing  $G_p$ . If  $k > c$ , we set  $l = k$  and  $w = v$ . Else we fix a positive integer  $l, l > c$  and take the longest  $l$ -sparse subsequence  $w$  of  $v$ . We show that  $|w|$  is proportional to  $|v|$ . The subsequence  $w$  splits  $v$  into nonempty intervals disjoint to  $w$ . Let  $I$  be one of them. Since no term of  $I$  can be used to extend  $w$ , each of them equals to one of the  $\leq l-1$  terms of  $w$

preceding  $I$  or following it. So there are only  $\leq 2l - 2$  distinct numbers in  $I$  and, since  $I$  is  $k$ -sparse and  $I \not\prec p$ ,  $|I| < k((k - 1)2l - 2)(k + 1) = d(k, l)$ . Thus

$$|v| < (d(k, l) + 1)(|w| + 1).$$

We split  $w$  into  $m$  intervals  $w_i$  of length  $l$  and a remainder  $r$  of length  $< l$ :  $w = w_1 w_2 \dots w_m r$ . We define a bipartite graph  $G$  on  $[n], [m]'$  by

$$\{i, j'\} \in E(G) \iff i \text{ appears in } w_j.$$

Clearly,  $G$  does not contain  $G_p$  and each  $j'$  has degree  $l$ . If  $m \geq n$ , after adding some isolated vertices to  $[n]$   $G$  can be regarded as a graph on  $[m], [m]'$ . But  $G$  has  $lm > cm$  edges, a contradiction. Hence  $m < n$  and we have the bound

$$|v| < (d(k, l) + 1)(|w| + 1) < (d(k, l) + 1)(ln + 1) = O(n).$$

□

### Conclusion and remarks

In (\*) one can set, for  $n > n(u)$ ,  $c = 1000k^3$  and  $d = |u| - 4$ , see [5]. Thus in Theorem 2 one can set, for  $n > n(k)$ ,  $c = 1000k^3$  and  $d = k^2$ . The bound (\*) cannot in general be improved to  $O(n)$ . For example, it is known that if  $v_1, v_2 \in [n]^*$  are 2-sparse and have the maximum length with respect to  $v_1 \not\prec_{=} 12121, v_2 \not\prec_{=} 121212$ , then  $n\alpha(n) \ll |v_1| \ll n\alpha(n)$  and  $n2^{\alpha(n)} \ll |v_2| \ll n2^{\alpha(n)}$ . For more information see the book [8] of Sharir and Agarwal. So it is possible that FHC is false. If FHC is true, to prove it seems to require tools far stronger than those in [3].

The paper [1] teaches us that we understand SWC better by viewing permutations as elements of  $\mathbf{N}^*$ . We saw that we can view them also as bipartite graphs. The situation can be generalized further to the class of ordered hypergraphs, see [6] and [7].

In the end of [4] Gessel mentions the problem to decide if for each permutation  $p$  the sequence  $\{S_n(p)\}_{n \geq 1}$  is P-recursive. The above results are not relevant to the problem and it appears to be very difficult. On the other hand, a strong optimism in this respect was expressed by Zeilberger [9].

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