On the dimer problem in 3-dimensional lattices: a preliminary version

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Abstract

We present a new expression for the enumeration of perfect matchings of the 3-dimensional cubic lattice. Perfect matchings are also called dimer arrangements and their enumeration is a well-known problem in statistical physics. As a consequence this provides a new expression for the enumeration of the edge-cuts of the cubic lattice, known also as 3-dimensional Ising problem.

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1 Introduction

In this paper we present a new expression for the dimer problem, i.e. for the enumeration of perfect matchings (dimer arangements) of the 3-dimensional cubic lattice. The dimer problem for 2-dimensional lattices appears in calculations of the thermodynamic properties of a system of diatomic molecules-dimers. It was solved by Kasteleyn ([7]) and by Temperley, Fisher ([6]). The same problem for 3-dimensional lattices remains an important open problem of statistical physics (see [8] for references). As a consequence we give a new expression for the 3-dimensional Ising problem.

A graph is a pair $G = (V, E)$ where V is a set of *vertices* and E is a set of unordered pairs of elements of V, called *edges*. A graph with some regularity properties is often called a *lattice*. We associate with each edge e of G a variable x_e and we let $x = (x_e : e \in E)$. For each $M \subset E$, let $x(M)$ denote the product of the variables of the edges of M.

An *orientation* of a graph $G = (V, E)$ is a *digraph* $D = (V, A)$ obtained from G by assigning an orientation to each edge of G, i.e., by ordering the elements of each edge of G. The elements of A are called *arcs*.

A subset of edges $P \subset E$ is called *perfect matching* or *dimer arrangement* if each vertex belongs to exactly one element of A. The dimer problem may be formulated as the problem to find polynomial $\mathcal{P}(G, x)$ which equals the sum of $x(P)$ over all perfect matchings P of G. It was first considered by Roberts ([9]) in 1935, and by Fowler and Rushbrook ([2]). Many fundamental observations about the dimer and monomer-dimer problem in general graphs are given by Heilmann, Lieb ([4, 5]).

Let m be odd positive integer and k even positive integer. The cubic lattice $Q_{m,m,k}$ is the following graph: It has vertices $V_{xyz}(Q_{mmk})$, $x, y = 1, ..., m$, $z = 1, ..., k$, and the following edges:

1. The edges $v_{xyz}(Q_{mmk}) = \{V_{xyz}(Q_{mmk}), V_{xy(z+1)}(Q_{mmk})\}$, $z = 1, ..., k-1$, called *vertical*,

2. The edges $w_{xyz}(Q_{mmk}) = \{V_{xyz}(Q_{mmk}), V_{x(y+1)z}(Q_{mmk})\}$, $y = 1, ..., m-1$, called *width*,

3. The edges $h_{xyz}(Q_{mmk}) = \{V_{xyz}(Q_{mmk}), V_{(x+1)yz}(Q_{mmk})\}$, $x = 1, ...m-1$, called *horisontal*.

Let us denote a vertical subpath $(V_{xy1}(Q_{mmk}), ..., V_{xyk}(Q_{mmk}))$ by $V_{xy}(Q_{mmk})$.

Q_{mmk} is a bipartite graph, which means that its vertices may be partitioned into two sets Z_1, Z_2 such that if e is an edge of Q_{mmk} then $|e \cap Z_1| = |e \cap Z_2| = 1$. Moreover, we have also that $|Z_1| = |Z_2| = mmk/2$. Let $\mathcal{Z}$ be square $(Z_1 \times Z_2)$ matrix defined by $\mathcal{Z}_{ij} = x_{ij}$ if ij is an edge of Q_{mmk} and $\mathcal{Z}_{ij} = 0$ otherwise.

We will consider matrix $\mathcal{Z}$ with its rows and columns ordered in aggreement with the order $(V_{11}(Q_{mmk})V_{12}(Q_{mmk})....V_{1m}(Q_{mmk})V_{21}(Q_{mmk}).....V_{mm}(Q_{mmk}))$. We will assume that $V_{111}(Q_{mmk}) \in Z_1$.

Note that $\mathcal{P}(Q_{mmk}, x)$ equals the permanent of $\mathcal{Z}$. In this paper we show that $\mathcal{P}(Q_{mmk}, x)$ may be computed from the average of determinants of certain signings of $\mathcal{Z}$, where a *signing* of a matrix is obtained by multiplying some of its entries by -1.

A result of similar flavour was proved by Heilmann and Lieb [4, 5]: $\mathcal{P}(Q_{mmk}, x)$ equals the average of $(det(Z))^2$ over all signings Z of $\mathcal{Z}$. A difference with our expression is that we replace the average of a multiquadratic function by the average of a multilinear function, with a restricted range though. This may help to solve the dimer problem and the Ising problem in three dimensions.

The proof of our result is involved: we embed the three-dimensional cubic lattice to a 2-dimensional surface, use a Theorem of Galluccio and Loebl ([3]) to express $\mathcal{P}(Q_{mmk}, x)$ as a linear combination of Pfaffians of matrices associated with orientations of Q_{mmk} and finally characterize the coefficients of this linear combination.

1.1 Statement of the main result.

If D is an orientation of Q_{mmk} and $e \in A(D)$ then we say that e is oriented according to the *natural ordering* if its orientation is in aggreement with the ordering $(V_{11}(Q_{mmk})V_{12}(Q_{mmk})....V_{1m}(Q_{mmk})V_{21}(Q_{mmk}).....V_{mm}(Q_{mmk}))$.

An orientation D is called *admissible* if the following edges are oriented in D in aggreement with the natural ordering: the edges of $V_{ij}(Q_{mmk})$, $i, j = 1, ...m$, the edges $w_{xyk}(Q_{mmk})$, $x \leq m$ and y odd, $w_{xy1}(Q_{mmk})$, $x \leq m$ and y even, h_{xmk}, x odd and h_{x11}, x even.

Let D be an admissible orientation of Q_{mmk}. We define orientation $\bar{D}$ as follows:

1. For each $x \leq m$, and $(y, z) \in \{(i, j); i \text{ odd}, j < k\}$ do the following: let $n(xyz)$ be the number of arcs w_{xab}, $a \leq y$ odd and $k > b \geq z$ oriented in D against the natural ordering. If $n(xyz)$ odd then we orient w_{xyz} in $\bar{D}$ against the natural ordering, else according to the natural ordering.

2. For each $x < m$ odd, and $(y, z) \neq (m, k)\}$ do the following: let $n(xyz)$ be the number of arcs h_{abc}, $a \leq x$ odd and $(m, k) > (b, c) \geq (y, z)$ oriented in D against the natural ordering. If $n(xyz)$ odd then we orient h_{xyz} in $\bar{D}$ against the natural ordering, else according to the natural ordering.

3. All the remaining arcs are directed in $\bar{D}$ in the same way as in D.

Let D be an admissible orientation of Q_{mmk}. Then we let $sgn(D) = h + \sum_{x=1}^{m} w(x)$, where $w(x) = |\{(yz); z > 1, y$ even and both $w_{xyz}, w_{x,(y-1),(z-1)}$ are oriented in $\bar{D}$ against the natural ordering $\}|$; $h = |\{(xyz); (y, z) \neq (1, 1), x$ even and both $w_{xyz}, w_{(x-1),y',z'}$, where (y', z') is the predecessor of (y,z) in the lexicographic order, are oriented in $\bar{D}$ against the natural ordering $\}|$.

Let D be an orientation of Q_{mmk}. Let us associate a signing $A'(D)$ of $\mathcal{Z}$ with it such that $A'(D)_{ij} = x_{ij}$ if $(ji) \in E(D)$, $A'(D)_{ij} = -x_{ij}$ if $(ij) \in E(D)$, and $A'(D)_{ij} = 0$ otherwise.

Matrix $A'(D)$ is called *admissible* if D is admissible. If A is an admissible matrix, i.e. $A = A'(D)$ for some admissible orientation D then we let $sgn(A) = sgn(D)$.

Now we state the main results of the paper. Let $C_r = 1/2(m-1)[m(k-1)+mk-1]$.

Theorem 1.1 *$\mathcal{P}(Q_{mmk}) = 2^{-C_r}\sum(-1)^{sgn(A)}det(A)$ where the sum is over all admissible matrices.*

Let $\alpha = \prod_{e\in M} x_e$ where M is unique perfect matching of the collection of paths $V_{11}(Q_{mmk}) \cup ... \cup V_{mm}(Q_{mmk})$.

Theorem 1.2 *$\mathcal{P}(Q_{mmk}, x) = -2^{C_r}\alpha + \beta(2^{C_r}+1)$, where β equals the average of $det(A)$, A admissible matrix with $sgn(A)$ even.*

1.2 Basic definitions.

Let G be a graph. We associate with each edge e of G a variable x_e and we let $x = (x_e : e \in E)$. For each $M \subset E$, let $x(M)$ denote the product of the variables of the edges of M.

A graph $G' = (V', E')$ is called a *subgraph* of a graph $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. A *perfect matching* of a graph is a set of disjoint edges, whose union equals the set of the vertices.

Let $\{v_1, e_1, v_2, e_2, ..., v_i, e_i, v_{i+1}, ..., e_n, v_{n+1}\}$ be a sequence such that each v_j is a vertex of a graph G, each e_j is an edge of G and $e_j = v_jv_{j+1}$, and $v_i \neq v_j$ for $i < j$ except if $i = 1$ and $j = n + 1$. If also $v_1 \neq v_{n+1}$ then P is called *a path* of G. If $v_1 = v_{n+1}$ then P is called *a cycle* of G. In both cases the *length* of P equals n. When no confusion arises we shall also denote paths by simply listing their edges, namely $P = (e_1, e_2, \dots, e_n)$.

A graph $G = (V, E)$ is *connected* if it has a path between any pair of vertices, and it is 2-*connected* if the graph $G_v = (V - \{v\}, \{e \in E; v \notin e\})$ is connected for each vertex v of G. Each maximal 2-connected subgraph of G is called a 2-*connected component* of G.

An *orientation* of a graph $G = (V, E)$ is a *digraph* $D = (V, A)$ obtained from G by assigning an orientation of each edge of G, i.e., by ordering the elements of each edge of G. The elements of A are called *arcs.*

Definition 1.3 *The generating function of the perfect matchings of G is the polynomial $\mathcal{P}(G, x)$ which equals the sum of $x(P)$ over all perfect matchings P of G.*

Definition 1.4 *Let $G = (V, E)$ be a graph with $2n$ vertices and D an orientation of G. Denote by $A(D)$ the skew-symmetric matrix with the rows and the columns indexed by V, where $a_{vw} = -x_{v,w}$ in case (v, w) is an arc of D, $a_{vw} = x_{v,w}$ in case (w, v) is an arc of D, and $a_{vw} = 0$ otherwise.*

The Pfaffian of the skew-symmetric matrix $A(D)$ is defined as

$$Pf(A(D)) = \sum_P s^*(P)a_{i_1j_1}\cdots a_{i_nj_n}$$

where $P = \{\{i_1 j_1\}, \cdots, \{i_n j_n\}\}$ *is a partition of the set* $\{1, \ldots, 2n\}$ *into pairs,* $i_k < j_k$ *for* $k = 1, \ldots, n$, *and* $s^*(P)$ *equals the sign of the permutation* $i_1 j_1 \ldots i_n j_n$ *of* $12 \ldots (2n)$.

Each nonzero term of the expansion of the Pfaffian of $A(D)$ *equals* $x(P)$ *or* $-x(P)$ *where* P *is a perfect matching of* G.

The Pfaffian is a determinant-type expression. Note the following classic result of Cayley (see [1]).

Theorem 1.5 *Let* G *be a graph and let* D *be an orientation of* G. *Then*

$$Pf^2(A(D)) = det(A(D)).$$

Definition 1.6 *Let* A *be a matrix whose elements are variables with positive or negative sign, and zeroes. We denote by* $S(A)$ *the matrix of the same size as* A *such that* $S(A)_{ij} = 1$ *iff* A_{ij} *has positive sign,* $S(A)_{ij} = -1$ *iff* A_{ij} *has negative sign,* $S(A)_{ij} = 0$ *iff* $A_{ij} = 0$.

An *embedding* of a graph on a surface is defined in a natural way: the vertices are embedded as points, and each edge is embedded as a continuous non-self-intersecting curve connecting the embeddings of its endvertices. The interiors of the embeddings of the edges are pairwise disjoint and the interiors of the curves embedding edges do not contain points embedding vertices.

A graph is called *planar* if it may be embedded on the plane. A *plane graph* is a planar graph together with its planar embedding. The embedding of a plane graph partitions the plane into connected regions called *faces*. The (unique) unbounded face is called *outer face* and the bounded faces are called *inner faces*.

Plane graphs with some regularities are sometimes called *2-dimensional lattices*.

Let G be a plane graph. A subgraph of G consisting of the vertices and the edges embedded on the boundary of a face will also be called *a face*. If a plane graph is 2-connected then each face is a cycle.

The *genus* g of a graph G is that of the orientable surface $\mathcal{S} \subset I\!R^3$ of minimal genus on which G may be embedded. Any orientable surface of genus g has a *polygonal representation* obtained by "cutting up" the g handles of its space model. In what follows we base our working definition of a surface on this concept.

Next section is devoted to description of a theorem of Galluccio and Loebl ([3]) which will be a basis of our reasoning.

2 Generalised g-graphs.

Definition 2.1 *A surface* S_g *of genus* g *consists of a base* B_0 *and* $2g$ *bridges* B^i_j, $i = 1, ..., g$ *and* $j = 1, 2$, *where*

i) B_0 *is a convex* $4g$*-gon with vertices* $a_1, ..., a_{4g}$ *numbered clockwise;*

ii) B_1^i, $i = 1, \ldots, g$, *is a 4-gon with vertices* $x_1^i, x_2^i, x_3^i, x_4^i$ *numbered clockwise. It is glued with* B_0 *so that the edge* $[x_1^i, x_2^i]$ *of* B_1^i *is identified with the edge* $[a_{4(i-1)+1}, a_{4(i-1)+2}]$ *of* B_0 *and the edge* $[x_3^i, x_4^i]$ *of* B_1^i *is identified with the edge* $[a_{4(i-1)+3}, a_{4(i-1)+4}]$ *of* B_0*;*

iii) B_2^i, $i = 1, \ldots, g$, *is a 4-gon with vertices* $y_1^i, y_2^i, y_3^i, y_4^i$ *numbered clockwise. It is glued with* B_0 *so that the edge* $[y_1^i, y_2^i]$ *of* B_2^i *is identified with the edge* $[a_{4(i-1)+2}, a_{4(i-1)+3}]$ *of* B_0 *and the edge* $[y_3^i, y_4^i]$ *of* B_2^i *is identified with the edge* $[a_{4(i-1)+4}, a_{4(i-1)+5(mod 4g)}]$ *of* B_0*.*

Observe that in Definition 2.1 we denote by $[a, b]$ edges of polygons and not edges of graphs. The usual representation in the space of an orientable surface $\mathcal{S}$ of genus g may be then obtained from its polygonal representation S_g by the following operation: for each bridge B, glue together the two segments which B shares with the boundary of B_0, and delete B.

Definition 2.2 *A graph G is called a g-graph if it may be embedded on S_g so that all the vertices belong to the base B_0, and the embedding of each edge uses at most one bridge. The set of the edges embedded entirely on the base will be denoted by E_0 and the set of the edges embedded on the bridge B_j^i will be denoted by E_j^i, $i = 1, \ldots, g$, $j = 1, 2$. We also let $G_0 = (V, E_0)$ and $G_j^i = (V, E_0 \cup E_j^i)$. Moreover the following conditions need to be satisfied too.*

1. *the outer face of $G_0 = (V, E_0)$ is a cycle, and it is embedded on the boundary of B_0,*

2. *if $e \in E_1^i$ then e is embedded entirely on B_1^i and one endvertex of e belongs to $[x_1^i, x_2^i]$ and the other one belongs to $[x_3^i, x_4^i]$. Similarly, if $e \in E_2^i$ then e is embedded entirely on B_2^i and one endvertex of e belongs to $[y_1^i, y_2^i]$ and the other one belongs to $[y_3^i, y_4^i]$.*

3. *each vertex is incident with at most one edge which does not belong to E_0,*

From now on, we shall consider g-graphs together with a fixed embedding on S_g. Given a g-graph G, we denote by C_0 the cycle which forms the outer face of E_0.

Definition 2.3 *Let G be a g-graph and let $G_j^i = (V, E_0 \cup E_j^i)$. If we draw $B_0 \cup B_j^i$ on the plane as follows: B_0 is unchanged, and the edge $[x_1^i, x_4^i]$ ($[y_1^i, y_4^i]$ respectively) of B_j^i is drawn so that it belongs to the external boundary of $B_0 \cup B_j^i$, we obtain a planar embedding of G_j^i. This embedding will be called planar projection of E_j^i outside B_0.*

Definition 2.4 *Let $G = (V, E)$ be a g-graph. An orientation D_0 of G_0 such that each inner face of each 2-connected component of G_0 is clockwise odd in D_0 is called a basic orientation of G_0.*

Note that a basic orientation always exists for a planar graph. Kasteleyn [7] proved that if D is a basic orientation of a planar graph G then the contributions of all perfect matchings of G have the same sign in $Pf(A(D))$.

From now on we shall fix a basic orientation D_0 for each g-graph.

Definition 2.5 *Let $G = (V, E)$ be a g-graph and D_0 a basic orientation of G_0. We define the orientation D^i_j of each G^i_j as follows: We consider G^i_j embedded on the plane by the planar projection of E^i_j outside B_0 (see Definition 2.3), and complete the basic orientation D_0 of G_0 to an orientation of G^i_j so that each inner face of each 2-connected component of G^i_j is clockwise odd.*
The orientation $-D^i_j$ is defined by reversing the orientation D^i_j of G^i_j.

Observe that after fixing a basic orientation D_0, the orientation D^i_j is uniquelly determined for each i, j.

Definition 2.6 *Let G be a g-graph, $g \geq 1$. An orientation D of G which equals the basic orientation D_0 on G_0 and which equals D^i_j or $-D^i_j$ on E^i_j is called relevant. We define its type $r(D) \in \{+1, -1\}^{2g}$ as follows: For $i = 0, \ldots, g-1$ and $j = 1, 2$, $r(D)_{2i+j}$ equals $+1$ or -1 according to the sign of D^{i+1}_j in D.*

Definition 2.7 *Let G be a g-graph and D a relevant orientation of G. Let $r(D) = (r_1, ..., r_{2g})$. We let $c(r(D))$ equal the product of c_i, $i = 0, ..., g-1$, where $c_i = c(r_{2i+1}, r_{2i+2})$ and $c(1, 1) = c(1, -1) = c(-1, 1) = 1/2$ and $c(-1, -1) = -1/2$.*

Observe that $c(r(D)) = (-1)^n 2^{-g}$, where $n = |\{i; r_{2i+1} = r_{2i+2} = -1\}|$.

The following theorem is proved in Galluccio, Loebl [3].

Theorem 2.8 *Let G be a g-graph with a perfect matching $M_0 \subset E_0$. If we order the vertices of G so that the sign of the contribution of M_0 to the Pfaffian $Pf(A(D_0)$ is positive then*

$$\mathcal{P}(G, x) = \sum_{i=1}^{4^g} c(r(D_i)) Pf(A(D_i))$$

where D_i, $i = 1, \ldots, 4^g$, are the relevant orientations of G.

We need a generalisation of the notion of a g-graph.

Definition 2.9 *Any graph obtained by the following construction will be called generalised g-graph.*

1. *Let $g = g_1, ..., g_n$ be a partition of g into positive integers.*
2. *Let S_{g_i} be a surface of genus g_i, $i = 1, ..., n$. Let us denote the basis and the bridges of S_{g_i} by B^i_0 and $B^i_{j,k}$, $i = 1, ..., n$, $j = 1, ..., g_i$ and $j = 1, 2$.*

3. *For $i = 1, ..., n$ let H_i be a g_i-graph with the property that the subgraph of H_i embedded on B_0^i is a cycle, embedded on the boundary of B_0^i. Let us denote it by C^i.*

4. *Let G_0 be a 2-connected plane graph and let $F_1, ..., F_n$ be a subset of faces of G_0. Let K^i be the cycle boundering F_i, $i = 1, ..., n$. Let each K^i be isomorphic to C^i.*

5. *Then G is obtained by glueing the H_i's into G_0 so that each K^i is identified with C_i.*

For each generalised g-graph G we can define 4^g relevant orientations $D_1, ..., D_{4^g}$ with respect to a fixed basic orientation of G_0, and coefficients $c(r(D_i))$, $i = 1, ..., n$ in the same way as for a g-graph. The following theorem can be proved in the same way as Theorem 2.8.

Theorem 2.10 *Let G be a generalised g-graph with a perfect matching M_0 of G_0. Let D_0 be a basic orientation of G_0. If we order the vertices of G so that the sign of the contribution of M_0 to the Pfaffian $Pf(A(D_0))$ is positive then*

$$\mathcal{P}(G, x) = 2^{-g} \sum_{i=1}^{4^g} c(r(D_i)) Pf(A(D_i))$$

where D_i, $i = 1, \ldots, 4^g$, are the relevant orientations of G.

3 Cubic lattices as generalised g-graphs.

In this section we will describe how to draw $3-$dimensional cubic lattices as generalised g-graphs.

The Cubic lattice.

Let m, n be odd positive integers such that $k = (n-1)/2$ is even. The cubic lattice $Q = Q_{m,m,n}$ is the following graph: It has vertices $V_{xyz}(Q) = V_{xyz}$, $x, y = 1, ..., m$, $z = 1, ..., n$, and the following edges:

1. The edges $\{V_{xyz}, V_{xy(z+1)}\} = v_{xyz}(Q) = v_{xyz}$, $z = 1, ..., n-1$, called *vertical*,

2. The edges $\{V_{xyz}, V_{x(y+1)z}\} = w_{xyz}(Q) = w_{xyz}$, $y = 1, ..., m-1$, called *width*,

3. The edges $\{V_{xyz}, V_{(x+1)yz}\} = h_{xyz}(Q) = h_{xyz}$, $x = 1, ...m-1$, called *horisontal.*

Let us denote a vertical subpath $(V_{xy1}, ..., V_{xyn})$ by $V_{xy}(Q) = V_{xy}$ and let $\bar{V}_{xy}$ denote V_{xy} traversed in the oposite direction.

Let $H_x(Q) = H_x = \{h_{xyz}; z = 1, ..., n, y = 1, ..., m\}$ and $W_{xy}(Q) = W_{xy} = \{w_{xyz}; z = 1, ...n\}$.

Next we describe a drawing of Q on the plane.

First draw the paths V_{xy} along a cycle in the following order:

$$V_{11}, \bar{V}_{12}, V_{13}, ...V_{1m}, \bar{V}_{2m}, V_{2(m-1)}, ..., \bar{V}_{21}, V_{31}, ..., V_{mm}.$$

Next, draw the horisontal edges outside of this cycle, and the width edges inside this cycle.

For each $x = 1, ..., m-1$ the curves representing edges of H_x are pairwise disjoint and for $x = 2, ..., m-2$ the edges of H_x intersect edges of H_{x-1} and H_{x+1}.

For each $x = 1, ..., m$ and $y = 1, ..., (m-1)$ the curves representing edges of W_{xy} are pairwise disjoint and for $y = 2, ..., m-2$ the edges of W_{xy} intersect edges of $W_{x(y-1)}$ and $W_{x(y+1)}$.

The curve representing an edge e will be denoted by $C(e)$.

Now we modify Q into a generalised g-graph Q'.

For each $x = 1, ..., m$ and y even perform the following construction:

1. Without loss of generality assume that in the drawing of Q to the plane, the vertices of V_{xy} appear in the same order as in V_{xy}, not as in $\bar{V}_{xy}$.

2. Let $Aux_1 = \{w_{xyz}; z \text{ odd }\}$. For each edge e of Aux_1 introduce a new vertex to each intersection of $C(e)$ with the curves representing the edges of $W_{x(y-1)} \cup W$, where $W = W_{x(y+1)}$ in case $y < m-1$ and $W = \emptyset$ otherwise. By this operation, each $e \in Aux_1$ is replaced by a path. Call each edge of this path *auxiliary*.

3. Let $Aux_2 = \{w_{x(y-1)n}\} \cup A$, where $A = \{w_{x,(y+1)n}\}$ in case $y < m-1$ and $A = \emptyset$ otherwise. For each edge e of Aux_2 introduce a new vertex to each intersection of $C(e)$ with the curves representing the edges of W_{xy}. Hence each $e \in Aux_2$ is replaced by a path. Call each edge of this path *auxiliary*.

4. The edges $v_{xy1}, v_{xy(n-1)}$ and also $v_{x(y+1)1}, v_{x(y+1)(n-1)}$ in case $y < m-1$ will also be called *auxiliary*.

5. We introduce a new variable a and let $x_e = a$ for each auxiliary edge e.

6. The edges w_{xyz}, $z < n-1$ even will be called *relevant*. If $y < m-1$ then the relevant edges are subdivided by two vertices (added in 3.) into three edges of Q'. The middle one will be called *special* and the other two *long*.

   If $y = m-1$ then the relevant edge w_{xyz} is subdivided by one vertex into two edges of Q'. The one incident to V_{xm} will be called *special* and the other one *long*.

   If e is a relevant edge, and f the corresponding special edge, then we let $x_e = x_f$. We let $x_h = 1$ for each long edge h.

7. The edges obtained from $w_{xy(n-1)}$ will all be called *special*. We assign the variables to them arbitrarily so that their product equals $x_{w_{xy(n-1)}}$.

8. The edges of $W_{x(y-1)} \cup W$ also got subdivided by new vertices introduced in step 2 and step 3.

9. We delete all edges of the paths obtained from $w_{x(y-1)z}$ and $w_{x(y+1)z}$, $z < n$ odd.

10. Each edge $w_{x(y-1)z}, w_{x(y+1)z}$, z even, is subdivided by new vertices into a path. The edge of this path incident with V_{xy} or $V_{x(y+1)}$ will be *special*. The edge e of this path such that $C(e)$ leaves the area boundered by curves representing edges of $W_{xy} \cup V_{xy} \cup V_{x(y+1)}$ will also be *special*. We assign the variables to the edges of the path arbitrarily so that their product equals the variable of the original edge.

10. All vertical edges which are not auxiliary (see 4.) will be called *special*.

This finishes the construction for the width edges. We perform an analogous construction with the horisontal edges of Q.

Finally, let Aux denote the set of all auxiliary edges. Then $Q' - Aux$ is a subdivision of Q_{mmk}. We conclude the construction of Q' by subdividing some special edges so that the graph $Q' - Aux$ is an even subdivision of Q_{mmk}.

This finishes the construction of Q'.

Some properties of Q'.

1. Each edge e of Q' such that $C(e)$ doesnot intersect any curve representing other edge in an interior point is auxiliary or special. Let us denote the plane subgraph of Q' formed by the auxiliary and special edges by Q^p.

2. Any other edge of Q' is drawn on a face of Q^p. Moreover, the edges drawn on a face of Q^p may be drawn onto a pair of bridges above this face, where one of the bridges contains one long edge, and the other bridge contains the remaining edges. Hence, we may view Q' as a generalised g-graph with the planar part equal to Q^p.

3 The special edges form an acyclic subgraph of Q'. Hence any orientation of the special edges may be extended into a basic orientation of Q^p. We will choose basic orientation D^p of Q^p with the following properties:

    1. D^p is in aggreement with the ordering $(V_{11}V_{12}....V_{1m}V_{21}.....V_{mm})$,
    2. Q^p has a perfect matching M whose sign in $Pf(A(D^p))$ is positive,
    3. The orientation of edges on a bridge has positive sign if and only if it is in aggreement with the ordering $(V_{11}V_{12}....V_{1m}V_{21}.....V_{mm})$.

4. If we let $a = 0$ then $\mathcal{P}(Q') = \mathcal{P}(Q_{mmk})$.

We have described how to view Q_{mmk}, m odd and k even, as a generalised g-graph Q'. Now we can use Theorem 2.10 for Q' to compute $\mathcal{P}(Q_{mmk})$.

The relevant orientations of Q'.

Each relevant edge of Q corresponds to unique edge of Q_{mmk}; this unique edge will also be called *relevant*. Hence the relevant edges of Q_{mmk} are: $w_{xyz}(Q_{mmk})$,

$x = 1, ..., m$, y even and $z \leq k-1$ and $h_{xyz}(Q_{mmk})$, x even and $(yz) \in \{(ij); i \leq m, j \leq k, (ij) \neq (mk)\}$. Hence there are $1/2(k-1)m(m-1)$ relevant width edges and $1/2(m-1)(mk-1)$ relevant horisontal edges in Q_{mmk}.

We let $\mathcal{R}$ be the set of relevant edges of Q_{mmk} and $C_r = 1/2(m-1)[m(k-1) + mk - 1]$ denote the number of relevant edges of Q_{mmk}.

The set of special edges of Q' corresponds to the set $\mathcal{S}$ of edges of Q_{mmk} which consists of the edges of $V_{ij}(Q_{mmk})$, $i, j = 1, ...m$, the edges $w_{xyk}(Q_{mmk})$, $x \leq m$ and y odd, $w_{xy1}(Q_{mmk})$, $x \leq m$ and y even, h_{xmk}, x odd and h_{x11}, x even. The orientation D^p corresponds to the orientation $\mathcal{S}^d$ of $\mathcal{S}$ which is in aggreement with the ordering $(V_{11}(Q_{mmk})V_{12}(Q_{mmk})....V_{1m}(Q_{mmk})V_{21}(Q_{mmk}).....V_{mm}(Q_{mmk}))$.

Each relevant orientation D' of Q' is determined by the fixed basic orientation of Q^p, and by a pair of signs for each pair of bridges. Each pair of bridges is associated with a long edge of Q'. Hence these signs may be given by specifying $(d^1_{D'}(e), d^2_{D'}(e)) \in \{+-\}^2$, for each long edge e, where $d^1_{D'}(e)$ denotes the sign of the bridge containing e, and $d^2_{D'}(e)$ denotes the sign of the other bridge.

The long edges of Q' are associated with relevant edges of Q, and hence also with relevant edges of Q_{mmk}.

The relevant edges $w_{x(m-1)z}(Q_{mmk})$ and $h_{(m-1)yz}(Q_{mmk})$ have only one long edge of Q' associated with it. If e is such an edge, we will call it *border edge* and we denote by e_1 the corresponding long edge. We let $d_{D'}(e) = (d^1_{D'}(e_1), d^2_{D'}(e_1), +, +)$.

Let $C_b = m(k-1) + mk - 1$ denote the number of border edges.

Each relevant non-border edge e has two long edges e_1, e_2 associated with it. We will assume that e_1 is incident with vertex $V_{xyz}(Q')$ whose $xy-$coordinate is smaller than the $xy-$coordinate of the vertex incident to e_2. We let $d_{D'}(e) = (d^1_{D'}(e_1), d^2_{D'}(e_1), d^1_{D'}(e_2), d^2_{D'}(e_2)))$.

A *relevant vector* is any element r of $[\{+,-\}^4]^{\mathcal{R}}$ such that $r(e)_3 = r(e)_4 = 1$ for each border edge e of Q_{mmk}. Hence there are $4^{2C_r - C_b}$ relevant vectors.

There is one-to-one correspondence between relevant orientations of Q' and relevant vectors. If D' is a relevant orientation of Q' then let $r(D')$ denote the corresponding relevant vector; if r is a relevant vector, then let $D'(r)$ denote the corresponding relevant orientation of Q'.

Clearly, the sign $m(r)$ of a relevant vector is defined according to Theorem 2.10 as follows: $m(r) = |\{(e, i); i = 0, 1, r(e)_{2i+1} = r(e)_{2i+2} = -1\}|$.

Each orientation D' of Q' naturally determines orientation D of Q_{mmk}. If D' is relevant then D will also be called *relevant*. Each relevant orientation of Q_{mmk} contains $\mathcal{S}^d$. If $D' = D'(r)$ for some relevant vector r then we let $D = D(r)$. Note that possibly $D(r) = D(r')$ for $r \neq r'$. Next we clarify this.

Definition 3.1 *We define an equivalence $*$ on the relevant vectors as follows. $r * s$ if the following holds: there is exactly one relevant non-border edge e such that $r(e) \neq s(e)$. Moreover, $r(e)_1 \neq s(e)_1$, $r(e)_3 \neq s(e)_3$, $r(e)_2 = s(e)_2 \neq r(e)_4 = s(e)_4$.*

Proposition 3.2 *If $r * s$ then $D(r) = D(s)$ and $(-1)^{m(r)} \neq (-1)^{m(s)}$.*

Definition 3.3 *A relevant vector is called stable if it forms a one-element class w.r.t. equivalence $*$.*

Corollary 3.4

$$\mathcal{P}(Q_{mmk}) = 2^{-2C_r+C_b}\sum(-1)^{m(r)}Pfaf(A(D(r)))$$

where the sum is over all stable vectors r.

Proposition 3.5 *Let r, s be stable vectors such that $D(r) = D(s)$. Then $(-1)^{m(r)} = (-1)^{m(s)}$. There are exactly $2^{C_r-C_b}$ stable vectors which define the same orientation of Q_{mmk}.*

Definition 3.6 *If r, s are stable vectors we write $r**s$ if $D(r) = D(s)$.*

Next we characterize $A(D(r))$ for stable vectors r.

Definition 3.7 *A square $(m^2k \times m^2k)$ matrix A is called stable if $A = A(D)$ for some orientation D of Q_{mmk} which extends $\mathcal{S}^d$.*

Remark. If A is a stable matrix then its sign-structure $S(A)$ has the following shape.

$$\begin{matrix} B_1 & -H_1 & 0 & 0 & 0 \\ H_1 & B_2 & -H_2 & 0 & 0 \\ 0 & H_2 & B_3 & -H_3 & 0 \\ 0 & 0 & H_3 & B_4 & -H_{m-1} \\ 0 & 0 & 0 & H_{m-1} & B_m \end{matrix}$$

where each H_i is a signing of the $(mk \times mk)$ identity matrix such that $(H_{2r})_{1,1} = (H_{2r-1})_{km,km} = 1$, $r = 1, ..., (m-1)/2$.

Moreover each B_i, $i \leq m$ has the following form:

$$\begin{matrix} V & -W_1^i & 0 & 0 & 0 \\ W_1^i & V & -W_2^i & 0 & 0 \\ 0 & W_2^i & V & -W_3^i & 0 \\ 0 & 0 & W_3^i & V & -W_{(m-1)}^i \\ 0 & 0 & 0 & W_{(m-1)}^i & V \end{matrix}$$

where V is the incidence matrix of a directed path on k vertices, i.e. it has the following form:

$$\begin{matrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{matrix}.$$

Each W_j^i, $i, j = 1, ..., m$, is a signing of the $(k \times k)$ identity matrix such that $(W_{(2r)}^i)_{1,1} = (W_{(2r-1)}^i)_{k,k} = 1$, $r = 1, ..., (m-1)/2$.

Proposition 3.8 *If r is a stable vector then $A(D(r))$ is a stable matrix. If A is a stable matrix then there is uniquelly determined class $C(A)$ of equivalence $**$ such that $A = A((D(r))$ for each $r \in C(A)$.*

Hence, given a stable matrix A, let us define its sign $m(A)$ to be equal to the sign $m(r)$ of a stable vector of $C(A)$.

Next, we characterise $m(A)$ for a stable matrix A.

Definition 3.9 *Let A be a $(a \times b)$ matrix of $1, -1$ entries only. We define $(a \times b)$ matrices T', T and number $s'(A)$ as follows.*

1. *$T_{ij} = T'_{ij} = A_{ij}$ for i even and $j = 1, ..., b$,*
2. *If i odd then $T'_{i1} = A_{i1}$ and $T'_{ij} = A_{ij}T'_{i(j-1)}$, $j > 1$,*
3. *$T_{1j} = T'_{1j}$ and $T_{ij} = T'_{(i-2)j}T'_{ij}$ for $i > 1$ odd and $j = 1, ..., b$.*
4. *$s'(A) = |\{T_{ij} : i \text{ even and } T_{ij} = T_{(i-1)j} = -1\}|$.*

Proposition 3.10 *Let A be a $(a \times b)$ matrix of $1, -1$ entries only. Then $T_{ij} = A_{ij}$ if i is even and $T_{ij} = A_{11}A_{12}...A_{1j}A_{31}A_{32}...A_{3j}...A_{i1}A_{i2}...A_{ij}$.*

Proposition 3.11 *Let A be a stable matrix as defined in 3.7. Then*

$$m(A) = s'(H) + \sum_{r=1}^{m-1} s'(W_r),$$

where s' is defined in 3.9 and H and W_r are the following matrices.

H is $((m-1) \times (mk-1))$ matrix such that for $j = 1, ..., mk-1$, $H_{ij} = (H_i)_{(mk-j),(mk-j)}$ if i is odd and $H_{ij} = (H_i)_{(mk-j+1),(mk-j+1)}$ if i is even.

For each $r = 1, ..., m$, W_r is $((m-1) \times (k-1))$ matrix such that for $j = 1, ..., k-1$, $(W_r)_{ij} = (W_i^r)_{(k-j),(k-j)}$ if i is odd and $(W_r)_{ij} = (W_i^r)_{(k-j+1),(k-j+1)}$ if i is even.

Proposition 3.11 may be described in the language of orientations as follows:

Definition 3.12 *Let D be an orientation of Q_{mmk} which extends $\mathcal{S}^d$. We define orientation $\bar{D}$ as follows:*

1. *For each $x \leq m$, and $(y, z) \in \{(i, j); i \text{ odd}, j < k\}$ do the following: let $n(xyz)$ be the number of arcs w_{xab}, $a \leq y$ odd and $k > b \geq z$ oriented in D against the natural ordering. If $n(xyz)$ odd then we orient w_{xyz} in $\bar{D}$ against the natural ordering, else according to the natural ordering.*

2. *For each $x < m$ odd, and $(y, z) \neq (m, k)\}$ do the following: let $n(xyz)$ be the number of arcs h_{abc}, $a \leq x$ odd and $(m, k) > (b, c) \geq (y, z)$ oriented in D against the natural ordering. If $n(xyz)$ odd then we orient h_{xyz} in $\bar{D}$ against the natural ordering, else according to the natural ordering.*

3. *All the remaining arcs orient in $\bar{D}$ in the same way as in D.*

Proposition 3.13 *Let D be an orientation of Q_{mmk} which extends $\mathcal{S}^d$. Then $m(A(D)) = h + \sum_{x=1}^{m} w(x)$, where $w(x) = |\{(yz); z > 1, y$ even and both $w_{xyz}, w_{x,(y-1),(z-1)}$ are oriented against the natural ordering $\}|$; $h = |\{(xyz); (y,z) \neq (1,1), x$ even and both $w_{xyz}, w_{(x-1),y',z'}$ where (y',z') is the predecessor of (y,z) in the lexicographic order, are oriented against the natural ordering $\}|$.*

Now we can formulate the main result of the paper.

Theorem 3.14

$$\mathcal{P}(Q_{mmk}, x) = 2^{-2C_r+C_b+C_r-C_b} \sum (-1)^{m(A)} Pfaf(A)$$

over all stable matrices A. The number $m(A)$ is characterised in 3.9, 3.10, 3.11.

Proposition 3.15 *There are 2^{2C_r} stable matrices. There are $2^{C_r-1}(2^{C_r}+1)$ stable matrices with positive sign.*

Proof. The first statement follows directly from the definition of a stable matrix. For Q_{132} there are 4 stable matrices, from which 3 have positive sign. For Q_{152} there are 4^2 stable matrices from which $3 \times 3 + 1 = 10$ have positive sign and $2(3 \times 1) = 6$ have negative sign. For $Q_{1(2a+1)2}$ there are 4^a stable matrices, and the difference between the number of stable matrices of positive sign and stable matrices of negative sign is 2^a. For Q_{mmk} the difference between the number of stable matrices of positive sign and those of negative sign equals $2^{(m-1)/2m(k-1)+(m-1)/2(mk-1)} = 2^{C_r}$. From this Proposition follows. □

4 From Pfaffians to determinants.

In the introduction we let $\mathcal{Z}$ be square $(Z_1 \times Z_2)$ matrix defined by $\mathcal{Z}_{ij} = x_{ij}$ if ij is an edge of Q_{mmk} and $\mathcal{Z}_{ij} = 0$ otherwise. We will consider matrix $\mathcal{Z}$ with its rows and columns ordered in aggreement with the order

$(V_{11}(Q_{mmk}), V_{12}(Q_{mmk}), \ldots., V_{1m}(Q_{mmk}), V_{21}(Q_{mmk}), \ldots.., V_{mm}(Q_{mmk}))$.

We will assume without loss of generality that $V_{111}(Q_{mmk}) \in Z_2$.

Matrix $S(\mathcal{Z})$ may be described as follows:

$$\begin{matrix} Q_1 & I_1 & 0 & 0 & 0 \\ I_2 & Q_2 & I_3 & 0 & 0 \\ 0 & I_4 & Q_3 & I_5 & 0 \\ 0 & 0 & I_6 & Q_4 & I_7 \\ 0 & 0 & 0 & I_{2(m-1)} & Q_m \end{matrix}$$

where every I_i is $(mk/2 \times mk/2)$ identity matrix and each Q_i has the following form:

$$\begin{matrix} R_1^i & Y_1^i & 0 & 0 & 0 \\ Y_2^i & R_2^i & Y_3^i & 0 & 0 \\ 0 & Y_4^i & R_3^i & Y_5^i & 0 \\ 0 & 0 & T_6^i & R_4^i & Y_7^i \\ 0 & 0 & 0 & Y_8^i & R_m^i \end{matrix}$$

where every Y_j^i is $(k/2 \times k/2)$ identity matrix. For $i+j$ odd, R_j^i has the following form:

$$\begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{matrix}$$

and for $i+j$ even R_j^i has the following form:

$$\begin{matrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}.$$

Let D be an orientation of Q_{mmk}. Let us associate yet another matrix with it, denoted by $A'(D)$. $A'(D)$ is a $(Z_1 \times Z_2)$ matrix such that $A'(D)_{ij} = x_{ij}$ if $(ji) \in E(D)$,i.e. $j \in Z_2$, $A'(D)_{ij} = -x_{ij}$ if $(ij) \in E(D)$, i.e. $i \in Z_1$, and $A'(D)_{ij} = 0$ otherwise.

Note that $Pfaf(A(D)) = det(A'(D))$.

Recall that A is called stable if $A = A(D)$ for some orientation D of Q_{mmk} containing $\mathcal{S}^d$. For such orientation D, matrix $A'(D)$ will be called *admissible*. We let $sgn(A'(D)) = m(A(D))$.

Corollary 4.1 $\mathcal{P}(Q_{mmk}, x) = 2^{-C_r} \sum (-1)^{sgn(A)} det(A)$ *where the sum is over all admissible matrices.*

Next we characterize admissible matrices.

Proposition 4.2 *Let A be a signing of $\mathcal{Z}$. A is admissible if and only if $S(A)$ satisfies the following properties:*

1. *For $i+j$ odd, the signing of R_j^i has the following form:*

$$\begin{matrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{matrix}$$

and for $i+j$ even the signing of R_j^i has the following form:

$$\begin{matrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{matrix}.$$

2. $1 = (I_{2j})_{mk/2} = (I_{2(j+1)})_1 = (Y_{2j}^{(2i+1)})_{k/2} = (Y_{2(j+1)}^{(2i+1)})_1$, *for* $j < (m-1)$ *odd and* $i = 0, ..., (m-1)/2$.

3. $-1 = (Y_{2j-1}^{2i})_{k/2} = (Y_{2j+1}^{2i})_1$, *for* $j < (m-1)$ *odd and* $i = 1, ..., (m-1)/2$.

Next we characterize $sgn(A)$ for an admissible matrix A.

Proposition 4.3 *Let* A *be an admissible matrix. Then*
$sgn(A) = s'(M) + \sum_{r=1}^{m-1} s'(N_r)$,
where M *and* N_r *are the following matrices.*

1. M *is* $((m-1) \times (mk-1))$ *matrix such that*
   $M_{ij} = (I_{2i})_{(mk/2-j/2),(mk/2-j/2)}$ *if* i *is odd and* j *is even,*
   $M_{ij} = (I_{2i})_{(mk/2-j/2-1),(mk/2-j/2-1)}$ *if* i *is even and* j *is even.*
   $M_{ij} = -(I_{(2i-1)})_{(mk/2-(j-1)/2),(mk/2-(j-1)/2)}$ *if* j *is odd,*

2. *For each* $r \leq m$ *odd,* N_r *is* $((m-1) \times (k-1))$ *matrix such that*
   $N_{r_{ij}} = (Y_{2i}^r)_{(k/2-j/2),(k/2-j/2)}$ *if* i *is odd and* j *is even,*
   $N_{r_{ij}} = (Y_{2i}^r)_{(k/2-j/2-1),(k/2-j/2-1)}$ *if* i *is even and* j *is even.*
   $N_{r_{ij}} = -(Y_{(2i-1)}^r)_{(k/2-(j-1)/2),(k/2-(j-1)/2)}$ *if* j *is odd,*

3. *For each* $r \leq m$ *even,* N_r *is* $((m-1) \times (k-1))$ *matrix such that*
   $N_{r_{ij}} = -(Y_{(2i-1)}^r)_{(k/2-j/2),(k/2-j/2)}$ *if* i *is odd and* j *is even,*
   $N_{r_{ij}} = -(Y_{(2i-1)}^r)_{(k/2-j/2-1),(k/2-j/2-1)}$ *if* i *is even and* j *is even,*
   $N_{r_{ij}} = (Y_{2i}^r)_{(k/2-(j-1)/2),(k/2-(j-1)/2)}$ *if* j *is odd.*

By Proposition 3.15 there are 2^{2C_r} admissible matrices, and there are $2^{C_r-1}(2^{C_r}+1)$ admissible matrices with positive sign.

Proposition 4.4 *The average of* $det(A)$, A *admissible equals* $\prod(R_j^i)_{pp}$ *over all* i, j, p. *This equals* $\prod_{e \in M} x_e$ *where* M *is unique perfect matching of the collection of paths* $V_{11}(Q_{mmk}) \cup ... \cup V_{mm}(Q_{mmk})$.

Let $\alpha = \prod_{e \in M} x_e$ where M is unique perfect matching of the collection of paths $V_{11}(Q_{mmk}) \cup ... \cup V_{mm}(Q_{mmk})$.

Theorem 4.5 $\mathcal{P}(Q_{mmk}, x) = -2^{C_r}\alpha + \beta(2^{C_r} + 1)$, *where* β *equals the average of* $det(A)$, A *admissible matrix with* $sgn(A)$ *even.*

Proof. $\mathcal{P}(Q_{mmk}, x) = (-2)^{-C_r}[2^{2C_r}\alpha + 2\beta 2^{-C_r}2^{C_r-1}(2^{C_r}+1) = -2^{C_r}\alpha + \beta(2^{C_r}+1)$.
□

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