A note on homomorphism independent families

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This note on the structure of graph homomorphisms is motivated by a result and a question by X. Caicedo ([3]). A family of objects $\{A_i \mid i \in J\}$ is said to be *independent* if there is no homomorphism $A_i \to A_j$ for distinct $i, j \in J$. We establish *infinite independent families of finite objects* (further, IIFF) for many classes of graphs thus extending the results of Caicedo and Bonato ([2]) connected with problems concerning finitely generated Horn classes: A class $\mathcal{K}$ of graphs is a *finitely generated Horn class* (briefly, an FGH class) if there is a finite set $\mathcal{A}$ of finite graphs such that $\mathcal{K} = \mathcal{SP}(\mathcal{A})$, where $SP(\mathcal{A})$ denotes the class of all subgraphs of products (see 2.1 below) of members of $\mathcal{A}$. Dropping the condition of finiteness of $\mathcal{A}$ we get the notion of *Universal Horn Class* (briefly, UH). In [3] Caicedo proved that there are continuum many UH-classes, and Bonato ([2]) proved (thus answering a question from [3]) that there are continuum many UH-classes of H-colourable graphs (that is, graphs G for which there is a homomorphism $G \to H$) for any non-bipartite graph H. Both these results use ad hoc methods using the Dedekind-MacNeille completion and a density theorem. Here we present a more direct approach based on the following result of the authors ([12]):

Proposition: *Let H be a fixed finite graph. Then the class*

$$\to H = \{G \mid \text{there is a homomorphism } G \to H\}$$

is an FGH-class.

Thus, in order to establish continuum many UH-classes in a class $\mathcal{K}$ it suffices to construct an IIFF of graphs in $\mathcal{K}$ (this was also observed in [3]). Namely, given an infinite IIFF $\mathcal{H}$, any subset $\mathcal{H}' \subseteq \mathcal{H}$ creates a UH class $\to \mathcal{H}' = \{\to H \mid H \in \mathcal{H}'\}$.

This is naturally linked to two areas studied extensively in the literature: to embeddings of categories and algebraic universality (from where we will use the so called arrow construction – see [15], or e.g. [6], [7]), and to density problems. The arrow construction will be recalled in Section 1, the relation to density in Section 2. Section 3 contains some remarks.

1 Arrow construction

1.1. Recall from [5] that the class **Graph** of directed graphs (sets with binary relations) together with graph homomorphisms (relation-preserving maps) constitutes a so called algebraically universal category. That is, each category of algebras – *which includes any category whose class of objects is a set* – can be represented as a full subcategory of **Graph**) (moreover, if there are not too many measurable cardinals, *any concrete* category is a full subcategory of **Graph**; for details see [15]).

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In fact, to see that there is a countable homomorphism independent class in **Graph**, moreover one consisting of finite graphs, is obvious: It is enough to set

$$G_k = (\{0,1,\dots,k\},\{(i,i+1) \mid i = 0,1,\dots,k-1\} \cup \{(0,k)\}).$$

1.2. Given a (undirected) graph $A = (V(A), E(A))$ with two distinguished vertices r, p, and an directed graph (X, R) (where $R \subseteq X \times X$), construct a new graph

$$(A, r, p) * (X, R) = (V(A) \times R / \sim, \mathcal{E})$$

where $\sim$ is the equivalence relation identifying

- for any x, all the (r, x, y) with (r, x, z), y, z arbitrary,
- for any y, all the (p, x, y) with (p, z, y), x, z arbitrary, and
- for any y, all the (r, x, y) with (p, y, z), x, z arbitrary,

and where $\{[a, x, y], [b, x, y]\} \in \mathcal{E}$ whenever $\{a, b\} \in E(A)$ ($[\dots]$ indicates the equivalence class.

The idea of the construction is obvious: One takes the directed graph (X, R) and replaces a directed edge ("arrow") (x, y) by a separate copy of A, glueing the r into x, and p into y.

Let $\mathcal{C}$ be a class of graphs. (A, r, p) is called *an arrow for* $\mathcal{C}$ if

- for any (X, R), each graph homomorphism $h : A \to (A, r, p) * (X, R)$ is given by $h(a) = [a, x, y]$ for a fixed $(x, y) \in R$,
- each $(A, r, p) * (X, R)$ is in $\mathcal{C}$.

By the condition (1) we immediately see that if (A, r, p) is an arrow, the homomorphisms $(A, r, p) * (X, R) \to (A, r, p) * (Y, S)$ are in a natural one-one correspondence with the homomorphisms $(X, R) \to (Y, S)$ in **Graph**.

Consequently,

if there is an arrow for $\mathcal{C}$ *then* $\mathcal{C}$ *is algebraically universal in the sense mentioned above, and in particular it has arbitrarily large homomorphism independent families.*

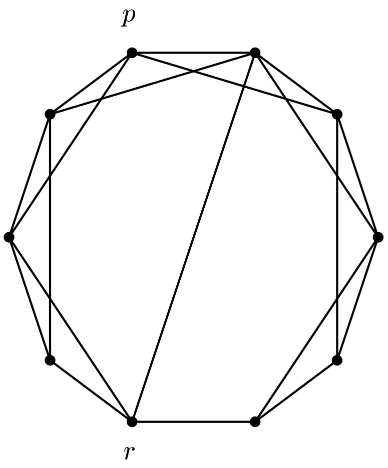
Moreover,

if there is a finite arrow then there is an infinite homomorphism independent family of finite objects from $\mathcal{C}$.

1.3. An arrow $G(n)$ for the class $\mathbf{Ch}_n$ of (precisely) n-chromatic undirected graphs, $n \geq 3$, can be defined as follows:

$$V(G(n)) = \{0, 1, \dots, 3n\}, \quad r = 0,\ p = 2n-2,$$
$$E(G(n)) = \{\{i, j\} \mid 0 < |i - j| < n\} \cup \{\{0, 3n\}\} \cup \{\{0, 2n-1\}\}.$$

(The proof that each $G(n)$ is an arrow appeared, in essence, in [7] where the graphs were originally constructed for other purposes; see also [15].) See the following picture for $G(3)$.



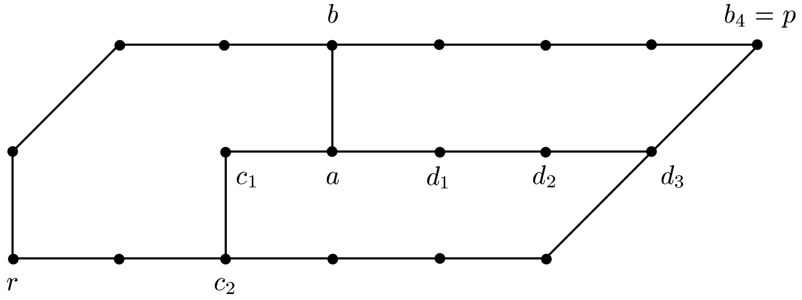
Thus,

Ch_n *contains a homomorphism independent family of any size, and a countable one consisting of finite objects.*

1.4. Similarly, modifying the construction from [6] we can construct an arrow for the class LC_n of undirected graphs with the shortest length of a cycle at least n. Take $n \geq 7$ odd and set

$$\begin{aligned} V(H(n)) =& \{a, b, c_1, c_2, \ldots, c_{n-2}, b_4, \ldots, b_{n-2}, d_1, \ldots, d_{n-3}\}, \quad r = c_4,\ p = b_4, \\ E(H(n)) =& \{\{a,b\}, \{a,d_1\}, \{a,c_1\}, \{c_{n-2},b\}, \{b_{n-2},b\}, \{d_3,b_4\}, \{d_{n-3},c_2\}\} \cup \\ & \cup \{\{b_i, b_{i+1}\} \mid i = 4, \ldots, n-3\} \cup \{\{c_i, c_{i+1}\} \mid i = 1, \ldots, n-3\} \cup \\ & \cup \{\{d_i, d_{i+1}\} \mid i = 1, \ldots, n-4\} \end{aligned}$$

See the following picture for $H(9)$.



The proof that $H(n)$ is an arrow appeared, first, in [6] for the case of $n = 7$. However, what one uses there is just that the cycles are odd of an equal length, and that they are long enough to allow to put the nodes of the exceptional degree (a, b, c_2 and d_3) into distinct distances ($d(a,b) = 1$, $d(a,c_2) = 2$, $d(a,d_3) = 3$, the other distances > 3), which one can have for any $n \geq 7$.

Thus,

LC_n *contains a homomorphism independent family of any size, and a countable one consisting of finite objects.*

1.5. Denote by Onto_H the class of all graphs G with a homomorphic image isomorphic to H. It follows from [1] that for any finite non-bipartite graph H there exists a finite arrow (A, r, p) such that H is a homomorphic image of A. Thus we have also that

Onto_H *contains a homomorphism independent family of any size and a countable one consisting of finite objects.*

This will be also proved by another method in Section 2.

1.6. Let us investigate in more detail the case of directed graphs left open in [2].

Given an directed graph H denote by $\rightarrow H$ the class of all directed graphs which are homomorphic to H. Further, define the height $h(H)$ of H as the minimum length of the directed path $P_n = (\{0, 1, \cdots, n\}, \{(i, i+1) \mid i = 0, \cdots, n-1\})$ such that there is a homomorphism $H \to P_n$. If there is no such path, we put $h(H) = \infty$. The following is the main results proved in Hell and Nešetřil [9] (a graph H is said to be a *core*, or *minimal*, if every homomorphism $H \to H$ is an automorphism):

Theorem: *For a finite directed core graph H the following statements are equivalent:*

- $\rightarrow H$ *is algebraically universal,*
- $\rightarrow H$ *contains two homomorphically independent graphs,*
- $h(H) > 3$.

This yields a complete characterization of the classes $\rightarrow H$ with IIFF:

Corollary: $\rightarrow H$ *has IIFF iff* $h(H) > 3$.

1.7. It should be noted that in the previous (and other) examples it is crucial that there is a *finite* arrow. An interesting category in this context is that of partially ordered sets and strictly monotone mappings, $\mathsf{Poset}_<$. This category is algebraically universal ([15]) and hence contains arbitrarily large morphism independent families. There is, however, no independent family of finite objects. We have

Proposition: *Let P, Q be finite posets. Then there is either a strictly monotone map $f : P \to Q$ or a strictly monotone map $f : Q \to P$.*

Proof: For $x \in P$ define $\nu(x)$ inductively as follows: If x has no predecessor, set $\nu(x) = 0$, else $\nu(x) = \max\{\nu(y) \mid y < x\} + 1$. Further, set $\nu(P) = \max\{\nu(x) \mid x \in P\}$. Let $\nu(P) \le \nu(Q) = n$. Choose a chain

$$a_0 < a_1 < \cdots < a_n$$

in Q and define $f : P \to Q$ by setting $f(x) = a_{\nu(x)}$. □

2 Ideal classes of finite graphs

As noted in [2], the problem of the existence of an IIFF is related to density of classes of graphs (see [16]). A short proof of density of the class of undirected graphs was given by Nešetřil and Perles (see [11]) and another one in [14] (where in fact a complete characteristics of density of classes of relational structures was presented).

However, the density is really not necessary and these proofs yield the IIFF's directly. We will present here a proof based on [11]) (see also [13]).

2.1. Recall that the (categorical) product in **Graph** is given by the formula

$$V(G_1 \times G_2) = V(G_1) \times V(G_2),$$
$$\{(x_1,x_2),(y_1,y_2)\} \in E(G_1 \times G_2) \quad \text{iff} \quad \{x_i,y_i)\} \in E(G_i) \quad \text{for both} \quad i = 1,2.$$

The projection homomorphisms sending (x_1,x_2) to x_j will be denoted by

$$p_j : G_1 \times G_2 \to G_j.$$

A class $\mathcal{C}$ of finite (undirected) graphs is said to be *ideal* if

- $G \in \mathcal{C}$ & $\mathcal{F} \in$ **Graph** finite $\Rightarrow$ $\mathcal{G} \times \mathcal{F} \in \mathcal{C}$, and
- $\mathcal{C}$ contains a graph with an odd cycle.

Thus for instance

- if H is not bipartite, the class of all H-colourable graphs (see [8]), that is, of the G such that there is a homomorphism $G \to H$, is ideal; or
- the class $H\!\not\to$ of all finite G such that there is no homomorphism $H \not\to G$ is ideal (for any H non-bipartite)

2.2. Denote by $E(k,l)$ the class of all finite graphs G such that the chromatic number $\chi(G)$ is greater than k and at the same time the length $\lambda(G)$ of the shortest odd cycle in G is greater than l. It is a combinatorial classics proved by Erdős in [4] and going back to Tutte and Zykov that

$$\forall k,l, \quad E(k,l) \neq \emptyset.$$

Note that $E(k,l)$ is an ideal class for any k,l.

2.3. We have

Theorem: *Let $\mathcal{C}$ be an ideal class in* **Graph**. *Then $\mathcal{C}$ contains an infinite homomorphism independent family of graphs.*

Proof: Choose an $H \in \mathcal{C}$, H not bipartite, and set $l_0 = \lambda(H)$. We will construct a homomorphism independent family $\{G_1,G_2,\cdots\} \subseteq \mathcal{C}$ inductively. First, choose a $G_1 \in \mathcal{C}$ such that $l_1 = \lambda(G_1) > l_0$ (such a $G_1 \in \mathcal{C}$ exists: consider the product $C_m \times H$ where C_m is the cycle of length m). Now let $G_1,\cdots,G_n$, $n \geq 1$ be chosen so that

- there is no homomorphism $h : G_i \to G_j$ with $i \neq j,\ \ i,j \leq n$,
- $\lambda(G_i) > l_0$ for all i, $1 \leq i \leq n$.

Set

$$k = (\max_{i=1,\ldots,n} |V(G_i)|)^{|V(H)|}, \quad l = \max_{i=1,\ldots,n} \lambda(G_i)$$

and choose a graph F,

$$F \in E(k,l).$$

Set

$$G_{n+1} = F \times H.$$

For $j \leq n$ there is no homomorphism $h : G_j \to G_{n+1}$ as else we had a homomorphism $p_1 h : G_j \to F$ which would yield an odd cycle of length $\leq \lambda(G_j)$ in F.

Now suppose there is, for some $1 \leq j \leq n$ a homomorphism

$$h : G_{n+1} = F \times H \to G_j.$$

Consider the mappings

$$h_x = (y \mapsto h(x,y)) : V(H) \to V(G_j)$$

(generally – and typicaly – not homomorphisms $H \to G_j$). Thus we can define a mapping

$$(x \mapsto h_x) : V(F) \to V(G_j)^{V(H)}.$$

Since $\chi(F) > |V(G_j)^{V(H)}|$ there exists an edge $\{x_1, x_2\} \in E(F)$ such that $h_{x_1} = h_{x_2}$. Set

$$f = h_{x_1} = h_{x_2}.$$

Now $f : H \to G_j$ is a graph homomorphism: If $\{y_1, y_2\} \in E(H)$ we have

$$\{f(y_1), f(y_2)\} = \{f_{x_1}(y_1), f_{x_2}(y_2)\} = \{h(x_1, y_1), h(x_2, y_2)\} \in E(G_j).$$

This contradicts the fact that $l_0 < \lambda(G_j)$. □

3 Remarks

The existence of the IIFF's in Section 1 was connected with the universality of the resulting categories. The construction in Section 2 was different: the IIFF was constructed without any relation to universality. It was not made clear, however, how much weaker the existence of an IIFF is, as compared with the universality. This will be done now by means of a few examples.

3.1. Unary algebras: Consider the class **Alg**(1) of unary algebras (X, α) (which is, of course, a subclass of that of directed graphs defined by first order axioms). This category is not universal (in fact, it even does not contain any non-trivial rigid object – that is, each non-trivial object has a non-identity endomorphism: the α is an endomorphism of (X, α), and if it is the identity then each map $f : X \to X$ is an endomorphism). But there is an IIFF: Consider the system

$$C_p = (\{0, 1, \ldots, p-1\}, \alpha = (j \to j+1 \bmod p)), \quad p \ \text{prime}.$$

3.2. Unary algebras with a constant: Even the existence of an IIFF constituted by rigid objects is still far from universality. Take the class **Alg**$(1,0)$ of the algebras (X, α, x_0) with one binary and one unary operation. We have the IIFF of rigid objects

$$(C_p, 0), \quad p \ \text{prime}$$

(C_p from 3.1). But **Alg**$(1,0)$ is far from being universal (see [5]). In fact, for instance, any group representable in **Alg**$(1,0)$ as an endomorphism semigroup is abelian.

3.3. Also note that if

$$G_1, G_2, \ldots$$

is an IIFF in $\mathcal{C}$ and if $\mathcal{C}$ is closed under products then

$$G_1 \times G_1, G_2 \times G_2, \ldots$$

is an IIFF, and in this latter one there cannot be any rigid object. Now being a square is a property which can be expressed in a wide system of categories (including that of undirected graphs) intrinsically by the existence of one binary and one unary operation satisfying certain equations (see [10] for characterization of products; this can be easily modified to characterizing squares). This observation gives, for some first order classes, first order extensions (with the definition of homomorphisms preserved) with IIFF's but at most one rigid object.

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