On oriented path double covers*

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Abstract

In this paper we concentrate on an oriented version of Perfect Path Double Cover (PPDC). An Oriented Perfect Path Double Cover (OPPDC) of a graph G is a collection of oriented paths in the symmetric orientation G_S of G such that each edge of G_S lies in exactly one of the paths and for each vertex v of G there is a unique path which begins in v (and thus the same holds also for terminal vertices of the paths). First we show that the graphs K_3 and K_5 have no OPPDC. Then we study the stucture of a minimal connected graph $G \neq K_3$, $G \neq K_5$ which also has no OPPDC. We show that minimal degree in this graph is at least four.

1 Introduction

Let $G = (V, E)$ be a simple graph. A *path* of length k in G is a sequence $v_1, e_1, v_2, \ldots, e_k, v_{k+1}$ of its vertices and edges where $e_i = \{v_i, v_{i+1}\}$ for $0 \leq i \leq k$ and $v_1, \ldots v_{k+1}$ are distinct vertices. A *circuit* (or a *cycle*) of length k is a sequence $v_1, e_1, v_2, \ldots, e_k, v_{k+1}$ of its vertices and edges where $e_i = \{v_i, v_{i+1}\}$ for $0 \leq i \leq k$, $v_1 = v_{k+1}$ and $v_1, \ldots v_k$ are distinct vertices.

A CYCLE DOUBLE COVER (CDC) of a graph G is a collection of its cycles such that each edge of G lies in exactly two of the cycles. A well known conjecture of P.D.Seymour asserts that every simple bridgeless graph has a CDC.

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Cycle double cover conjecture lies in the very heart of the graph theory. It seems that this elementary problem has a deep topological background and only partial results are known. This problem (in the very short time of its existence) also motivated several related conjectures:

A SMALL CYCLE DOUBLE COVER (SCDC) of a graph on n vertices is a CDC with at most $n-1$ circuits. Bondy conjectures that every simple bridgeless graph has an SCDC.

A PERFECT PATH DOUBLE COVER (PPDC) of a graph G is a collection of its paths such that each edge of G lies in exactly two of the paths and each vertex of G appears precisely twice as an endpoint of the paths.

Li has shown that every simple graph has a PPDC [2].

PPDC for graphs in general is equvalent to SCDC for bridgeless apex graphs (apex graph = graph with a vertex joined to all other vertices). To see this consider a graph $G \setminus v$ where v is a vertex of degree $n-1$ in a bridgeless graph G. G has an SCDC if and only if $G \setminus v$ has a PPDC.

Another group of unsolved related questions are oriented versions of these problems. Denote by G_S the symmetric orientation of G (e.g. $V(G_S) = V(G)$ and $E(G_S) = \{(u,v),(v,u) \,|\, \{u,v\} \in E(G)\}$).

An ORIENTED CYCLE DOUBLE COVER (OCDC) of G is a collection of cycles in G_S of length at least three such that each edge of G_S lies in exactly one of the cycles.

Jaeger conjectured that every bridgeless graph has an oriented cycle double cover. No counterexample to OCDC conjecture is presently known.

In this paper we concentrate on oriented versions of two other conjectures.

An ORIENTED SMALL CYCLE DOUBLE COVER (OSCDC) of a graph G on n vertices is an OCDC with at most $n-1$ elements.

An ORIENTED PERFECT PATH DOUBLE COVER (OPPDC) of a graph G is a collection of paths in G such that each edge of G_S lies in exactly one of the paths and each vertex of G appears just once as a beginning and just once as an end of the paths.

Similarly as above one can se that OPPDC for graphs in general is equivalent to OSCDC for bridgeless apex graphs. Although there is a lot of positive CDC-related results in this area, conjectures OPPDC and OSCDC fail to be true generally. One can easily show that K_3 has no OPPDC and more tediously the same for K_5 (see theorems 2.1 and 2.2). Hence, K_4 and K_6 fail to have an OSCDC.

The paper is organized as follows. In section 2 we consider OPPDC for complete graphs. We prove that K_3 and K_5 have no OPPDC while K_7 and all K_{2n} admit an OPPDC. In section 3 we discuss a structure of a minimal (e.g. with minimal number of edges) connected graph G such that $G \neq K_3$, $G \neq K_5$ and G has no OPPDC. We consenctively show that G has no vertices of degree one, two or three. As a consequence of this we prove that a union of two arbitrary trees has an OPPDC and that all 2-connected graphs on n vertices with at most $2n-1$ edges have an OPPDC (except from K_3).

2 Complete graphs

In this section we are to prove that K_3 and K_5 have no OPPDC. These are presently the only known examples of connected graphs without OPPDC.

It's obvious that if the complete graph K_n has an OPPDC $\mathcal{P}$ then each path in $\mathcal{P}$ has length $n-1$.

Theorem 2.1 *K_3 has no OPPDC.*

Proof: Suppose that K_3 has an OPPDC $\mathcal{P} = \{P_1, P_2, P_3\}$. Without loss of generality we can assume that $V(K_3) = \{1,2,3\}$, $P_1 = 123$ and the path P_2 starts in the vertex 2. Hence, $P_2 = 213$ and the vertex 3 is the end of both P_1 and P_2, contradiction. □

Theorem 2.2 *K_5 has no OPPDC.*

In the proof of this theorem we use following obvious lemma:

Lemma 2.3 *Let $\mathcal{P}$ be an OPPDC of a graph G and $P \in \mathcal{P}$ a path of length four which begins in a and ends in b. Let x, y, z be the internal vertices of P. Then*
(i) $((x,y) \in P$ and $(y,z) \in P) \Rightarrow P = a\ x\ y\ z\ b$
(ii) $((x,y) \in P$ and $(y,z) \notin P) \Rightarrow P = a\ z\ x\ y\ b$
(iii) $((x,y) \notin P$ and $(y,z) \in P) \Rightarrow P = a\ y\ z\ x\ b$
(iv) $((x,y) \notin P$ and $(y,z) \notin P) \Rightarrow ((P = a\ x\ z\ y\ b)$ or $(P = a\ y\ x\ z\ b)$ or $(P = a\ z\ y\ x\ b))$. □

Proof: (of theorem 2.2)
Suppose for contradiction, that $\mathcal{P}$ is an OPPDC of K_5. Let $V(K_5) = \{1,2,3,4,5\}$. Let P_{ij} denote the path of $\mathcal{P}$ beginning with i and ending with j. Without loss of generality we will consider two cases:

A: $\mathcal{P} = \{P_{12}, P_{23}, P_{31}, P_{45}, P_{54}\}$
B: $\mathcal{P} = \{P_{12}, P_{23}, P_{34}, P_{45}, P_{51}\}$

A: Without loss of generality we can assume that $P_{45} = 41235$. According to 2.3(iv) it holds:
a) $P_{54} = 51324$ or b) $P_{54} = 52134$ or c) $P_{54} = 53214$.
a) If $P_{45} = 41235$ and $P_{54} = 51324$ then both P_{31} and P_{23} contain the arc $(2,1)$.
b) If $P_{45} = 41235$ and $P_{54} = 52134$ then both P_{31} and P_{12} contain the arc $(3,2)$.
c) If $P_{45} = 41235$ and $P_{54} = 53214$ then both P_{31} and P_{12} contain the arc $(3,4)$.

B: $\mathcal{P} = \{P_{12}, P_{23}, P_{34}, P_{45}, P_{51}\}$. Set $M = \{(1,2),(2,3),(3,4),(4,5),(5,1)\}$. For each element m of

M there are only two paths of $\mathcal{P}$ such that m can be in the path P. For instance $(1,2) \in P_{34}$ or $(1,2) \in P_{45}, \ldots$. Conversely each path of $\mathcal{P}$ contains at most two elements of M. We consider following cases:

1. Each path in $\mathcal{P}$ contains exactly one arc of M.
   1a) $(1,2) \in P_{34}$ then $(5,1) \in P_{23}$, $(4,5) \in P_{12}$, $(3,4) \in P_{51}$ $(2,3) \in P_{45}$.
   1b) $(1,2) \in P_{45}$ then $(2,3) \in P_{51}$, $(3,4) \in P_{12}$, $(4,5) \in P_{23}$, $(5,1) \in P_{34}$.

2. Exactly one path of $\mathcal{P}$ (we can assume that it is P_{12}) contains two elements of M. Then exactly one paths of $\mathcal{P}$ contains no arc of M.
   2a) $E(P_{23}) \cap M = \emptyset$
   2b) $E(P_{34}) \cap M = \emptyset$
   2c) $E(P_{45}) \cap M = \emptyset$
   2d) $E(P_{51}) \cap M = \emptyset$.

3. Exactly two paths of $\mathcal{P}$ contain by two elements of M. Again without loss of generality one of them is P_{12}. Then the other can't be neither P_{23} nor P_{51}.
   3a) $(3,4),(4,5) \in P_{12},\ (5,1),(1,2) \in P_{34},\ (2,3) \in P_{45}$
   3b) $(3,4),(4,5) \in P_{12},\ (5,1),(1,2) \in P_{34},\ (2,3) \in P_{51}$
   3c) $(3,4),(4,5) \in P_{12},\ (1,2),(2,3) \in P_{45},\ (5,1) \in P_{23}$
   3d) $(3,4),(4,5) \in P_{12},\ (1,2),(2,3) \in P_{45},\ (5,1) \in P_{34}$

Now we are to show that each of the cases 1a),1b), ...,3d) leads to a contradiction.

1a) $(4,5) \notin P_{23}$ and $(5,1) \in P_{23}$. If we use 2.3(iii) for $P = P_{23}$ we get $P_{23} = 25143$.
Further $(5,1) \notin P_{34}$ and $(1,2) \in P_{34}$. If we use 2.3(iii) for $P = P_{34}$ we get $P_{34} = 31254$.
The arc $(2,5) \in P_{23} \cap P_{34}$, a contradiction.

1b) Similarly we can show that the arc $(1,4) \in P_{23} \cap P_{34}$ in this case.

2a) $P_{12} = 13452$ and $E(P_{23}) \cap M = \emptyset$.
Then $(5,1) \in P_{34}$ and $(1,2) \in P_{45}$ and $(2,3) \in P_{51}$.
According to 2.3(ii) for $P = P_{45}$ we get $(1,2) \in P_{45}$ and $(2,3) \notin P_{45} \Rightarrow P_{45} = 43125$.
According to 2.3(ii) for $P = P_{51}$ we get $(2,3) \in P_{51}$ and $(3,4) \notin P_{51} \Rightarrow P_{51} = 54231$.
Thus the arc $(3,1) \in P_{45} \cap P_{51}$, a contradiction.

2b) Analogously we show that the arc $(3,1) \in P_{45} \cap P_{51}$ in this case, a contradiction.
2c) Analogously we show that the arc $(3,1) \in P_{51} \cap P_{34}$, a contradiction.
2d) Analogously we show that the arc $(2,5) \in P_{23} \cap P_{34}$, a contradiction.

3a) $(3,4),(4,5) \in P_{12},\ (5,1),(1,2) \in P_{34},\ (2,3) \in P_{45}$
According to 2.3(i) $P_{12} = 13452$ and $P_{34} = 35124$. According to 2.3(iii) $P_{45} = 42315$.
$((1,5) \in P_{45}$ and $(3,5) \in P_{34}$ and $(4,5) \in P_{12}) \Rightarrow$ the path P_{23} must contain the arc $(2,5)$.
$((1,5) \in P_{45}$ and $(1,2) \in P_{34}$ and $(1,3) \in P_{12}) \Rightarrow$ the path P_{23} must contain the arc $(1,4)$.
It means that $P_{23} = 25143$. This is in contradiction to $P_{34} = 35214$, since $(5,1) \in P_{23} \cap P_{34}$.

3b) Similarly we can show that the arc $(4,5) \in P_{23} \cap P_{12}$ in this case, a contradiction.
3c)Similarly we can show that the arc $(3,4) \in P_{51} \cap P_{12}$ in this case, a contradiction.
3d)Similarly we can show that the arc $(2,3) \in P_{51} \cap P_{45}$ in this case, a contradiction.

Since each of the cases 1a), ..., 3d) leads to contradiction, K_5 has no OPPDC. □

A complete proof of theorem 2.2 can be found in [4]. The graphs K_3 and K_5 are presently the only known examples of connected graphs without an OPPDC. The following two examples show that K_7 and K_{2n} have an OPPDC.

Example 2.4 Let $V(K_7) = \{1,2,3,4,5,6,7\}$
Let us denote $P_1 = 1263547$
$P_2 = 2731465$ $P_3 = 3742516$
$P_4 = 4536721$ $P_5 = 5764132$
$P_6 = 6175243$ $P_7 = 7156234$
One can check that the collection $\mathcal{P} = \{P_1, P_2, \ldots, P_7\}$ is an OPPDC of K_7.

Example 2.5 If G is a graph with all vertices of odd degree, then G has a decomposition in $\frac{n}{2}$ paths, see [3]. The symmetric orientation of these paths forms an OPPDC of the graph G. Thus every graph with all vertices of odd degree has an OPPDC. Consequently every complete graph K_{2n} has an OPPDC. The later statement one can prove directly:
Let $V(K_{2n}) = \{v_0, v_1, \ldots, v_{2n-1}\}$. For $0 \le i \le 2n-1$ set $P_i = v_i, v_{i+1}, v_{i-1}, v_{i+2}, v_{i-2} \ldots, v_{i+n}\}$ where all subscripts are read modulo n. It's easy to verify that $\mathcal{P} = \{P_i \mid 0 \le i \le 2n-1\}$ is an OPPDC of K_{2n}.

3 Minimal degrees

First we make some easy observations about the structure of graphs with OPPDC.

Lemma 3.1 *Let G be a graph with an OPPDC and let G' arise from G by dividing one edge of G. Then G' also has an OPPDC.*

Proof: It's sufficient to divide the path which contains the dividing edge into two paths. The new vertex is the end of the initial part and the beginning of the terminal part. □

It's obvious that a graph has an OPPDC if and only if each of its components has an OPPDC. Thus we can consider only connected graphs. Next we show that a graph G has an OPPDC if each block of G has an OPPDC.

Lemma 3.2 *Let G_1, G_2 be two graphs which have an OPPDC. Suppose that $G_1 \cap G_2 = \{v\}$. Then the union $G_1 \cup G_2$ has an OPPDC.*

Proof: Denote by $\mathcal{P}_i$ an OPPDC of G_i, $i = 1, 2$. Let $P_1 \in \mathcal{P}_1$ be the path which starts in v and $P_2 \in \mathcal{P}_2$ be the path which ends in v. Then the collection $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \{P_1 \cup P_2\} \setminus \{P_1, P_2\}$ is an OPPDC of $G_1 \cup G_2$. □

By applying this lemma we get that if we add a new vertex of degree one to a graph with an OPPDC then the resulting graph also has an OPPDC. Hence every tree has an OPPDC.

It also follows immediately from the lemma 3.2 that if each block of a graph G has an OPPDC then the whole graph has also an OPPDC. The converse of this easy statement is equivalent to the existence of OPPDC for every graph. To see this, consider an arbitrary nonseparable graph G on at least two vertices. Join each vertex of even degree in G to a new vertex v' of degree one. Denote by H the resulting graph. Clearly, G is a block of H. Moreover, each vertex of H is of odd degree and, by example 2.5, H has an OPPDC.

The following theorem deals with vertices of degree two.

Theorem 3.3 *Let G be a simple graph, $G \neq K_3$, and $v \in V(G)$ a vertex of degree two. If $G \setminus v$ has an OPPDC then G has also an OPPDC.*

Proof: Let $N(v) = \{x, y\}$ be the neighbours of the vertex v. Denote by $\mathcal{P}$ an OPPDC of the graph $G \setminus v$. For $u \in V(G \setminus v)$ let P^u (resp. P_u) denote the path of $\mathcal{P}$ begining (resp. ending) with u.

If $P^x \neq P_y$ then by deleting the paths P^x and P_y from $\mathcal{P}$ and adding three new paths $(v, x) \cup P^x$, $P_y \cup (y, v)$ and xvy we get an OPPDC of G. We proceed analogously, if $P^y \neq P_x$. So we can assume that $P^x = P_y$ and $P^y = P_x$. Set $P_1 = P^x = P_y$ and $P_2 = P^y = P_x$. There are two cases to consider:

1. Suppose there exists a path $P \in \mathcal{P} \setminus \{P_1, P_2\}$ such that $x \in P$ or $y \in P$. Without loss of generality we can assume that $x \in P$. Denote by Q the inicial part of P ending with the vertex x and by R the terminal part of P beginning with the vertex x. Surely $y \notin Q$ or $y \notin R$.

If $v \notin Q$ set $Q^* = Q \cup \{(x,v),(v,y)\}$, $R^* = (v,x) \cup R$, $P_1^* = P_1 \cup (y,v)$.
The collection $\mathcal{P} \setminus \{P, P_1\} \cup \{Q^*, R^*, P_1^*\}$ is an OPPDC of G.

If $v \notin R$ set $R^* = \{(y,v),(v,x)\} \cup R$, $Q^* = Q \cup (x,v)$, $P_2^* = (v,y) \cup P_2$.
The collection $\mathcal{P} \setminus \{P, P_2\} \cup \{Q^*, R^*, P_2^*\}$ is an OPPDC of G.

2. Let P_1, P_2 are the only paths that contain x or y. Thus $\deg_{G\setminus v}(x) = 1$ and $\deg_{G\setminus v}(y) = 1$. Denote by x' the only neighbour of x and by y' the only neighbour of y in $G\setminus v$. Since $G\setminus v \neq K_2$, $x' \neq y$.

   (2a) If $x' = y'$ then P_1, P_2 are paths of length two. Let $Q \in \mathcal{P}$ be the path which starts in x'. define the paths P_1^*, Q^*, R, S as follows:
   $P_1^* = P_1 \cup (y,v)$ $\quad$ $R = \{(x,x'),(x',v),(v,y)\}$
   $Q^* = (y,x') \cup Q$ $\quad$ $S = (v,x)$
   Then the collection $\mathcal{P} \setminus \{P_1, P_2, Q\} \cup \{P_1^*, Q^*, R, S\}$ forms an OPPDC of G.

   (2b) If $x' \neq y'$ let $Q \in \mathcal{P}$ be the path which ends in x' and $R \in \mathcal{P}$ the path which ends in y'. Define the paths P_1^*, P_2^*, Q^*, R^* as follows:
   $P_1^* = P_1 \setminus (y',y)$ $\quad$ $Q^* = Q \cup \{(x',x),(x,v),(v,y)\}$
   $P_2^* = P_2 \setminus (x',x)$ $\quad$ $R^* = R \cup \{(y',y),(y,v),(v,x)\}$
   Then the collection $\mathcal{P} \setminus \{P_1, P_2, Q, R\} \cup \{P_1^*, P_2^*, Q^*, R^*, (x,v)\}$ forms an OPPDC of G.

$\square$

In the following two theorems we prove that if a graph with a vertex v of degree three has no OPPDC then there exists a graph on smaller number of edges which also has no OPPDC. In the theorem 3.4 we consider the situation when the neighbours of v induce K_3 in G and in the theorem 3.5 the one when the neighbours of v do not induce K_3 in G.

Theorem 3.4 *Let $G = (V,E)$ be a graph, $v \in V(G)$ a vertex of degree three, $N(v) = \{x,y,z\}$ induce K_3 in G. If $G \setminus v$ has an OPPDC then G has also an OPPDC.*

Proof: Let $\mathcal{P}$ be an OPPDC of $G \setminus v$. Let us denote by $P \in \mathcal{P}$ such a path that $(y,z) \in P$, by $R \in \mathcal{P}$ such a path that $(x,y) \in R$, by $Q \in \mathcal{P}$ such a path that $(z,x) \in Q$. $P = R = Q$ is not possible. Thus we have to consider two cases: $|\{P,R,Q\}| = 2$ and $|\{P,R,Q\}| = 3$.

1. $|\{P,R,Q\}| = 2$. Without loss of generality we can assume that $Q = P \neq R$, e.g. $(x,y) \in R$ and $(y,z),(z,x) \in P$. Surely, z is neither the beginning nor the end of P and z is not both the beginning and the end of R. Hence there is a path $C \in \mathcal{P} \setminus \{P,R\}$ which begins or ends in z.

If z is the beginning of C, set
$P^* = P \setminus \{(z,x)\} \cup \{(z,v),(v,x)\}$, $\quad C^* = (v,z) \cup C$,
$R^* = R \setminus \{(x,y)\} \cup \{(x,v),(v,y)\}$, $\quad S = \{(z,x),(x,y),(y,v)\}$.
$\mathcal{P} \setminus \{C,R,P\} \cup \{C^*,R^*,P^*,S\}$ is an OPPDC of G. We proceed analogously if z is the end of C.

2. $|\{P,R,Q\}| = 3$.
2a) Suppose there is such a path $C \in \mathcal{P}$ that starts or ends in one of the vertices $\{x,y,z\}$. Without loss of generality C starts in x, the other cases can be solved analogously. Set
$P^* = P \setminus \{(y,z)\} \cup \{(y,v),(v,z)\}$, $\quad C^* = (v,x) \cup C$,
$R^* = R \setminus \{(x,y)\} \cup \{(x,v),(v,y)\}$, $\quad S = \{(x,y),(y,z),(z,v)\}$.
$\mathcal{P} \setminus \{C,R,P\} \cup \{C^*,R^*,P^*,S\}$ is an OPPDC of G. .

2b) If there is no such path C, then the vertices x,y,z are exactly all the start and endpoints of the paths P,R,Q. The path R leads (i) from x to y or (ii) from x to z or (iii) from$z$ to y, there are no other possibilities.
(i) If R leads from x to y then of necessity P leads from y to z and Q from z to x. All the paths P,R,Q have length one. If we connect P and Q to one path and add a new path of length zero in z, we get a new OPPDC of $G \setminus v$ which corresponds to the situation solved in the case 1.
(ii) If R leads from x to z then of necessity P leads from y to x and Q from z to y. Set
$P^* = P \setminus \{(y,z)\} \cup \{(x,y),(y,v)\}$, $\quad Q^* = (v,z) \cup Q$,
$R^* = R \setminus \{(x,y)\} \cup \{(x,v),(v,y)\}$, $\quad S = \{(y,z),(z,v),(v,x)\}$.
The collection $\mathcal{P} \setminus \{P,Q,R\} \cup \{P^*,Q^*,R^*,S\}$ forms an OPPDC of G.
(iii) If R leads from z to y then of necessity Q leads from y to x and Q from x to z. Set $\overline{R} = Q \setminus (x,y) \cup (z,x)$, $\quad \overline{P} = R \setminus (y,z) \cup (x,y)$, $\quad \overline{Q} = P \setminus (z,x) \cup (y,z)$. The paths $\overline{R},\overline{P},\overline{Q}$ satisfy all the assumptions of the case (ii).

□

Theorem 3.5 *Let $G = (V,E)$ be a graph, $v \in V(G)$ a vertex of degree three in G. $N(v) = \{x,y,z\}$, $(x,z) \notin E(G)$. If $(G \setminus v) \cup \{(x,z)\}$ has an OPPDC then G also has an OPPDC.*

Proof: Let $\mathcal{P}$ be an OPPDC of $(G \setminus v) \cup \{(x,z)\}$. Denote by $P_1 \in \mathcal{P}$ such a path that $(x,z) \in P_1$ and by $P_2 \in \mathcal{P}$ such a path that $(z,x) \in P_2$. We consider following three cases:
1. There is a path $P_3 \in \mathcal{P} \setminus \{P_1,P_2\}$ which begins in y.
2. There is a path $P_3 \in \mathcal{P} \setminus \{P_1,P_2\}$ which ends in y.
3. There is no such path P_3.
In the case 1 we modify $\mathcal{P}$ as follows:

$P_1^* = P_1 \setminus \{(x,z)\} \cup \{(x,v),(v,z)\}$, $P_3^* = (v,y) \cup P_3$,
$P_2^* = P_2 \setminus \{(z,x)\} \cup \{(z,v),(v,x)\}$, $S = (y,v)$.
In the case 2 we modify $\mathcal{P}$ as follows:
$P_1^* = P_1 \setminus \{(x,z)\} \cup \{(x,v),(v,z)\}$, $P_3^* = P_3 \cup (y,v)$,
$P_2^* = P_2 \setminus \{(z,x)\} \cup \{(z,v),(v,x)\}$, $S = (v,y)$.
In the case 3 we can assume without loss of generality that y is the beginng of P_1 and the end of P_2. Then there must be a path $P_3 \in \mathcal{P} \setminus \{P_1, P_2\}$ that ends in x. Denote by Q the initial part of P_1 ending with the vertex x. Set
$P_1^* = P_1 \setminus \{Q,(x,z)\} \cup \{(y,v),(v,z)\}$, $P_3^* = P_3 \cup (x,v)$,
$P_2^* = P_2 \setminus \{(z,x)\} \cup \{(z,v),(v,x)\}$, $S = (v,y) \cup Q$.

In all three cases the collection $\mathcal{P} \setminus \{P_1,P_2,P_3\} \cup \{P_1^*,P_2^*,P_3^*,S\}$ is an OPPDC of G. □

Corollary 3.6 *If G is a union of two arbitrary trees, $G \neq K_3$, then G has an OPPDC.*

Proof: We proceed by induction on $m = |E(G)|$. Suppose G has no OPPDC. Since G is a union of two trees, there is a vertex $v \in V(G)$ of degree at most three. If $\deg_G(v) = 1$ then by applying 3.2 the graph $G \setminus v$ also has no OPPDC. If $\deg_G(v) = 2$ then by applying 3.3 the graph $G \setminus v$ also has no OPPDC. If $\deg_G(v) = 3$ and the neighbours of v induce K_3 in G then by applying 3.4 the graph $G \setminus v$ also has no OPPDC. If $\deg_G(v) = 3$ and $(x,z) \notin E(G)$, $\{x,z\} \subset N(v)$ then by applying 3.5 the graph $(G \setminus v) \cup \{(x,z)\}$ also has no OPPDC. Thus there is a graph with smaller number of edges $G \setminus v$ or $(G \setminus v) \cup e$ which also has no OPPDC. This graph is also union of two trees and if it isn't isomorphic to K_3 then it has an OPPDC by induction hypothesis, contradiction. If it is isomorphic to K_3 then G is isomorphic to K_4 or to one of the graphs G_1, G_2 on Fig.1, which all easily have an OPPDC, contradiction.

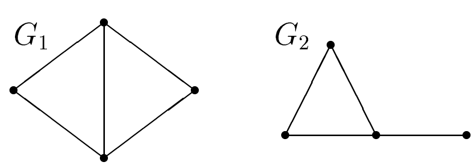

Fig.1

□

Corollary 3.7 *Let G be a (vertex) 2-connected graph, $G \neq K_3$ that arises from a tree on n vertices by adding $k \leq n$ edges. Then G has an OPPDC.*

Proof: We proceed by induction on k. For $k = 0$ or $k = 1$ the statement follows immediately from lemmas 3.1, 3.2 and the fact that both C_4 and G_2 on Fig.1 have an OPPDC. For $k \geq 2$ we count the average degree $PDEG(G)$ in G.

$$PDEG(G) = \frac{2|(E(G)|}{|V(G)|} = \frac{2(k + (n-1))}{n} \leq \frac{2(2n-1)}{n} = 4 - \frac{2}{n} < 4$$

Hence in G there is a vertex of degree two or three. If G has no OPPDC then according to 3.3, 3.4 and 3.5 there is a smaller graph with no OPPDC. This graph arises from a tree on $n-1$ vertices by adding at most $k-1$ edges and thus it has an OPPDC by induction hypothesis, if it isn't isomorphic to K_3, contradiction. If it is isomorphic to K_3, then G is isomorphic to K_4 or G_1 on the Fig.1, a contradiction. □

Theorem 3.8 *n-dimensional hypercube has an OPPDC for each $n \geq 1$.*

Proof: Let H be an n-dimensional hypercube (e.g. $V(H) = \{0,1\}^n$, $\{u, v\} \in E(H)$ iff $u_i \neq v_i$ for unique i). For $v \in V(H)$ let v^* be the opposite vertex to v (e.g. $v_i^* = 1$ iff $v_i = 0$). We show that H has an OPPDC $\mathcal{P}$ such that for every vertex $v \in V(G)$ holds: if v is the beginning of a path $P \in \mathcal{P}$ then v^* is the end of P. We proceed by induction on n. For $n = 1$ the proposition holds. If $n > 1$ let us define
$V_0 = \{v \in V(H) | v_n = 0\}$
$V_1 = \{v \in V(H) | v_n = 1\}$
$M = \{e = \{u, v\} \mid u_i = v_i,\ i = 1, 2, \dots n-1,\ u_n \neq v_n\}$
M is a matching. For $i = 1, 2$, denote by H_i the graph H restricted to the set V_i. For $i = 1, 2$, H_i is an $(n-1)$-dimensional hypercube and thus has an OPPDC $\mathcal{P}_i$ with the required property by induction hypothesis. $E(H) = E(H_0) \cup E(H_1) \cup E(M)$.
We modify $\mathcal{P}_i,\ i = 1, 2$ as follows. For each $P \in \mathcal{P}_i$, if P ends in x and $\{x, y\}$ is an edge of M (which is incident to x), we denote by $P' = P \cup (x, y)$. $\mathcal{P} = \{P' \mid P \in \mathcal{P}_0 \cup \mathcal{P}_1\}$ is an OPPDC of H with the required property. □

4 Concluding remarks

1. Of course, many problems remain. One could conjecture that K_3 and K_5 are the only connected graphs without OPPDC, but perhaps one should first gain more supporting evidence. In the area related to CDC the conuterexamples are rare and the oriented version presented in this paper may be an exception.

2. OCDC may be viewed as a partition of the symmetric orientation of a 2-connected graph into oriented cycles of length > 2. Can one characterize balanced digraphs without 2-cut which can be partitioned into oriented cycles of length > 2 ?

3. Similarly as in 2., OPPDC is a partition of a symmetric orientation of a graph into directed paths. Can one characterize the balanced digraphs which have OPPC (i.e. which can be partitioned into directed paths such that each vertex of G appears just once as a beginning and just once as an end of the paths)?

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