Universal $H-$colorable Graphs Without A Given Configuration

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Abstract

For every pair of finite connected graphs F and H, and every integer k, we construct a universal graph U with the following properties:

1. There is a homomorphism $\pi : U \to H$, but no homomorphism from F to U.
2. For every graph G with maximal degree no more than k having a homomorphism $h : G \to H$, but no homomorphism from F to G, there is a homomorphism $\alpha : G \to U$, such that $h = \pi \circ \alpha$.

Particularly, this solves a problem presented in [1] and [2] regarding the chromatic number of a universal graph.

1 Introduction

For a graph G, let $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. Given two graphs G and H, a map $f : V(G) \to V(H)$ is a *(graph)* *homomorphism* from G to H if and only if $\{f(x), f(y)\} \in E(H)$ for all edges $\{x, y\} \in E(G)$. For simplicity, $f : G \to H$ will denote that f is a homomorphism from G to H, and $G \not\to H$ means that no homomorphism exists from G to H.

Homomorphisms extend the notion of coloring, as $\chi(G) \leq k$ if and only if $G \to K_k$. More generally, we say that G is *H-colorable* if there is a homomorphism $G \to H$. If $F \not\to G$, we say that G is *F-free.* A graph U is *universal* for a class of graphs $\mathcal{K}$ if, for any graph $G \in \mathcal{K}$, there exists a homomorphism $G \to U$. For example, the complete graph on k vertices is universal for the family of k-colorable graphs. Universal graphs were considered earlier in [1] and [2], which established their existence for several classes of graphs, including classes of $F-$free graphs of bounded degree.

When considering $F-$free and $H-$colorable graphs, we can restrict to those graphs F and H that are *cores*, graphs for which every endomorphism is an automorphism. Indeed, a graph is $F-$free if and only if it is free of the (uniquely determined) core of F, and $H-$colorable if and only if it is colorable by the core of H, [3]. We have the following theorem from [5]:

Theorem 1.1 *Let F, H be fixed cores. Then for every universal graph U of the class of all $F-$free, $H-$colorable graphs K_H, there exists a homomorphism $H \to U$.*

In particular, every universal graph for 3-colorable, triangle-free graphs contains a triangle.

This negative result was complemented by Häggkvist and Hell with a positive result for $F-$free graphs of bounded degree [2]. We say that a graph is *$b-$bounded* if $\Delta(G) \leq b$.

Theorem 1.2 *Let F be a core, b a positive integer. Then the class $\mathcal{K}_F^b$ of all $b-$bounded $F-$free graphs has a universal $F-$free graph U_F^b.*

As the chromatic number of a $b-$bounded graph is $\leq b+1$, all of the graphs in the class $\mathcal{K}_F^b$ have chromatic number $\leq b+1$. Häggkvist and Hell [2] have asked the minimal chromatic number of an universal graph for K_F^b. The graphs constructed in [2] all have a large chromatic number. In [1], Galluccio, Hell, and Nešetřil considered the complexity of $H-$coloring a 3-bounded graphs with $\chi(H) \leq 3$, and conjectured that $H-$coloring of 3-bounded graphs is an NP-complete problem for any 3-chromatic triangle-free graph H.

In this paper, we answer both of these questions by proving that the class of all $F-$free 3-bounded graphs has a universal $F-$free graph U with $\chi(U) \leq 3$. More generally, our main result, Theorem 3.1, demonstrates the existence of $F-$free, $H-$colorable universal graphs for the family of $b-$bounded, $F-$free, $H-$colorable graphs.

By a result of Johanson [4], we have the following corollary:

Corollary 1.3 *For every positive integer b, there exists a graph U_b with the following properties:*

1. *U_b is triangle-free, and $\chi(U_b) \leq c \cdot \frac{b}{\log b}$ for a constant c independent of b.*
2. *U_b is universal for the family of $b-$bounded, triangle-free graphs.*

The paper is organized as follows. In Section 2, we give a proof of a theorem covering a particular case of the main theorem (when $F = K_3 = H$). This construction closely follows the construction given in [1] and [2]. In Section 3, we prove the general theorem, giving a new construction for the above theorem. Section 4 contains some remarks and open problems.

We introduce some notation used for the remainder of the paper. As usual, $[n]$ will denote the set of positive integers $\{1, 2, \ldots, n\}$. If x and y are two vertices in a connected graph G, the *distance* $d_G(x, y)$ denotes the number of edges in the shortest path in G joining x and y. For a graph G and a non-negative integer k, G^k is a graph defined by $V(G^k) = V(G)$ and $E(G^k) = \{\{x, y\} : x, y \in V(G), 1 \leq d_G(x, y) \leq k\}$. For a vertex v, $G[v, k]$ denotes the subgraph induced by the vertices $w \in V(G)$ such that $0 \leq d_G(v, w) \leq k$.

If $\Delta(G) = k$, then by induction, for a vertex $v \in G$ and $j \geq 1$, at most $k(k-1)^{j-1}$ points have distance j from v. Putting $X_k(d) = 1 + \sum_{j=1}^{d} k(k-1)^{j-1}$, it follows that $|V(G[v, d])| \leq X_k(d)$.

2 $k-$colorable, triangle-free universal graphs for $k-$bounded, triangle-free graphs

Theorem 2.1 *Given a positive integer k, let $\mathcal{F}$ be the family of $k-$bounded, triangle-free graphs. There exists a universal graph U such that:*

1. *$\chi(U) \leq k$.*

2. *For every $G \in \mathcal{F}$, there exists a homomorphism $\alpha : G \to U$.*

3. *U is triangle-free.*

Proof: Let $G \in \mathcal{F}$. Assume without loss of generality that G is k-regular. Since $\chi(G) \leq k$, we know there exists a proper coloring of G, $c_1 : V(G) \to [k]$. For any vertex in G^3, the degree is at most $X_k(3) - 1$, so there exists a proper coloring of G^3, $c_2 : V(G^3) = V(G) \to [X_k(3)]$. Putting $X = [k] \times [X_k(3)]$, we can define a map $c : V(G) \to X$ by $c(x) = (c_1(x), c_2(x))$. Let $\pi_1 : X \to [k]$ be the projection onto the first coordinate $\pi_1(c_1, c_2) = c_1$.

The universal graph U is defined by:

$$\begin{aligned} V(U) &= X \times X()k \\ E(U) = \{\{(x, A), (x', A')\} :& A \cap A' = \emptyset, \\ & x \in A', x' \in A, \\ & \pi_1(x) \neq \pi_1(x')\}. \end{aligned}$$

1. $\chi(U) \leq k$.

   The map $\pi : V(U) \to [k]$ given by $\pi((x, A)) = \pi_1(x)$ $k-$colors U.

2. For every $G \in \mathcal{F}$, there exists a homomorphism $\alpha : G \to U$.

   Put $\alpha(v) = (c(v), \{c(w) : \{v, w\} \in E(G)\})$. If $\{v, w\} \in E(G)$ with $\alpha(v) = (x, A)$, $\alpha(w) = (x', A')$, then:

   a. $A \cap A' = \emptyset$: Because G is triangle-free, v and w have no common neighbors. No neighbor of v can receive the same $X_k(3)-$color as a neighbor of w, because neighbors of v and w are at most distance three from one another. Thus, any members of A and A' differ in their x_2-coordinate.

   b. $x \in A'$, $x' \in A$: Because $\{v, w\} \in E(G)$, $x' = c(w) \in A$ and $x = c(v) \in A'$.

   c. $\pi_1(x) \neq \pi_1(x')$: $\pi_1(x) = c_1(v)$ and $\pi_1(x') = c_1(w)$, and c_1 properly $k-$colors G, hence $\pi_1(x) \neq \pi_1(x')$.

3. U is triangle-free.

   Suppose that (x, A), (x', A'), and (x'', A'') are three vertices of U that form a triangle. Then $x \in A'$ and $x \in A''$, contradicting $A' \cap A'' = \emptyset$.

Although [1] conjectured that for triangle-free graphs H with chromatic number 3, the $H-$coloring problem for $3-$bounded graphs is NP-complete, setting $k = 3$ in the above theorem proves this conjecture false: All $3-$bounded, triangle-free graphs are $H-$colorable, and all $3-$bounded graphs containing a triangle are not.

3 $F-$free universal graphs for $H-$colorable, $F-$free graphs

The preceding construction mapped triangle-free, $k-$bounded graphs into a universal graph by describing a vertex's neighborhood with the colors used by a coloring of G^3. The condition $A \cap A' = \emptyset$ sufficed to avoid triangles in the universal graph. In order to avoid more complicated graphs, we replace the coloring of G^3 with a labelled copy of the induced subgraph formed by a large neighborhood around the vertex, which will be $F-$free. Edges in the universal graph connect only the neighborhoods that might be had by adjacent vertices in $F-$free graphs. This idea allows us to strengthen the preceding theorem:

Theorem 3.1 *Let F, H be finite connected graphs, b a positive integer. Then there exists a universal graph U with the following properties:*

1. *U is $H-$colorable.*
2. *For every $b-$bounded, $F-$free, $H-$colorable graph G, there exists a homomorphism $G \to U$ that preserves the $H-$coloring of G.*
3. *U is $F-$free.*

This result can be expressed using a commutative diagram:

Proof: Throughout this proof, we assume that given a K_n, its vertices are labelled by $[n]$. Let $m = |V(F)|$, and let $\mathcal{S}$ be the set of all connected subgraphs S of $K_{X_k(m)}$ such that $F \not\to S$.

We will now describe the universal graph U.

$$\begin{tikzcd}
& F \arrow[dl, "/" marking] \arrow[dr, "/" marking] & \\
G \arrow[rr, dashed] \arrow[dr] & & U \arrow[dl] \\
& H &
\end{tikzcd}$$

$$\begin{aligned}
V(U) &= \mathcal{S} \times V(K_{X_k(m)}) \times V(H) \\
E(U) &= \{\{(S,v,n),(S',v',n')\} : \{n,n'\} \in E(H), \\
&\qquad \{v,v'\} \in E(S) \cap E(S'), \\
&\qquad S[v,m-1] = S'[v,m-1], \\
&\qquad S[v',m-1] = S'[v',m-1]\}
\end{aligned}$$

Here, graph equalities mean not only that the two graphs are isomorphic, but that the labels on the vertices and edges in each graph are identical.

1. U is $H-$colorable.

   The map $\pi : U \to H$ defined by $\pi((S,v,n)) = n$ is a homomorphism that preserves the $H-$coloring of G.

2. For every $b-$bounded, $F-$free, $H-$colorable graph G, there exists a homomorphism $G \to U$ that preserves the $H-$coloring of G.

   Let G be a graph that satisfies the conditions above. Because G^m is $X_k(m)-$colorable, we can fix a homomorphism $c : G \to K_{X_k(m)}$. If $v \in V(G)$ and $r \leq m$, then c is injective on $G[v,r]$, so $c(G[v,r])$ is isomorphic to $G[v,r]$. In particular, $F \not\to c(G[v,r])$, since $c(G[v,r])$ is isomorphic to a subgraph of G.

   Because $F \not\to c(G[v,m])$, we can define $\alpha : V(G) \to V(U)$ by $\alpha(x) = (c(G[v,m]), c(x), h(x))$. The map α preserves the $H-$coloring of G, namely $h = \pi \circ \alpha$. We claim that α is a homomorphism from G to U. Suppose $\{x,x'\} \in E(G)$. Because h is a homomorphism, $\{h(x), h(x')\} \in$

$E(H)$. Since $G[x,m]$ and $G[x',m]$ are induced subgraphs of G, both containing x and x', and c is a homomorphism, $\{c(x),c(x')\} \in E(c(G[x,m])) \cap E(c(G[x',m]))$. From $G[x,m-1] \subseteq G[x',m]$ and $G[x',m-1] \subseteq G[x,m]$, we obtain:

$$\begin{aligned} c(G[x,m])[c(x),m-1] &= c(G[x,m-1]) \\ &= c((G[x',m])[x,m-1]) \\ &= c(G[x',m])[c(x),m-1]. \end{aligned}$$

Similarly, $c(G[x',m])[c(x'),m-1] = c(G[x,m])[c(x'),m-1]$. Hence, $\{\alpha(x),\alpha(x')\} \in E(U)$.

3. U is $F-$free.

   Let $V(F) = \{x_1,\dots,x_m\}$. Suppose that a homomorphism $\beta : F \to U$ exists, and $\beta(x_i) = (S_i, v_i, n_i)$. We claim that the map $\beta^* : F \to S_1$ defined by $\beta^*(x_i) = v_i$ is a homomorphism, contradicting the condition on $V(U)$ that $F \not\to S$ for all $S \in \mathcal{S}$.

   If $\{x_i,x_j\} \in E(F)$, $d_F(x_1,x_i) \leq m-1$ because $|V(F)| = m$. Since $\{\beta(x_i),\beta(x_j)\} \in E(U)$, $\{v_i,v_j\} \in E(S_i) \cap E(S_j)$.

   If $\{x_{k_1} = x_1, x_{k_2}, \dots, x_{k_p} = x_i\}$ is the shortest path in F from x_1 to x_i, then $\{v_i,v_j\} \in E(S_{k_p}[v_{k_p},1])$. Since $\{v_{k_{p-1}}, v_{k_p}\} \in E(S_{k_p}[v_{k_p},1])$,

   $$\{v_i,v_j\} \in E(S_{k_p}[v_{k_{p-1}},2]) = E(S_{k_{p-1}}[v_{k_{p-1}},2]).$$

   By induction, we can show that for any q, $1 \leq q \leq p < m$, that

   $$\{v_i,v_j\} \in E(S_{k_{p-(q-1)}}[v_{k_{p-(q-1)}},q]).$$

   Taking $q = p$,

   $$\{v_i,v_j\} \in E(S_{k_1}[v_{k_1},p]) = E(S_1[v_1,p]) \subseteq E(S_1).$$

   Thus, β^* is a homomorphism from F to S_1, producing a contradiction and completing the proof.

4 Remarks and Open Problems

The theorems above suggest that one needs some restrictive condition to guarantee the existence of an universal graph. In the theorems above, the $b-$boundedness condition in the graph family is essential. The following problems appear to be open.

- Does there exist a triangle-free graph U which is universal for every triangle-free planar graph?
- Does there exist a triangle-free planar graph U which is universal for all 3-bounded planar graphs?

We can observe that there is no triangle-free planar graph U which is universal for all triangle-free planar graphs. To see this, consider "sunflower graphs" Sun_n. To construct Sun_n, begin with a $K_{1,n}$, where the lone vertex is labelled x and the n vertices are labelled by $[n]$. Vertices i and $i+1$ are joined by edge-disjoint paths of length 3, as well as vertices n and 1. Then, the neighbors of each vertex in $[n]$ (not including x) are joined by edge-disjoint paths of length three. The sunflower graph Sun_n has $5n+1$ vertices and $7n$ edges. The graphs of Sun_n are shown below for $n = 1, 2, 3$, and 6.

One can check that there is a homomorphism $Sun_n \to Sun_{n'}$ if and only if $n'|n$. One can also check that any planar homomorphic image of Sun_n must have a vertex of degree n', where $n'|n$ and $n' > 1$. If n is prime and exceeds $\Delta(U)$, then $Sun_n \not\to U$.

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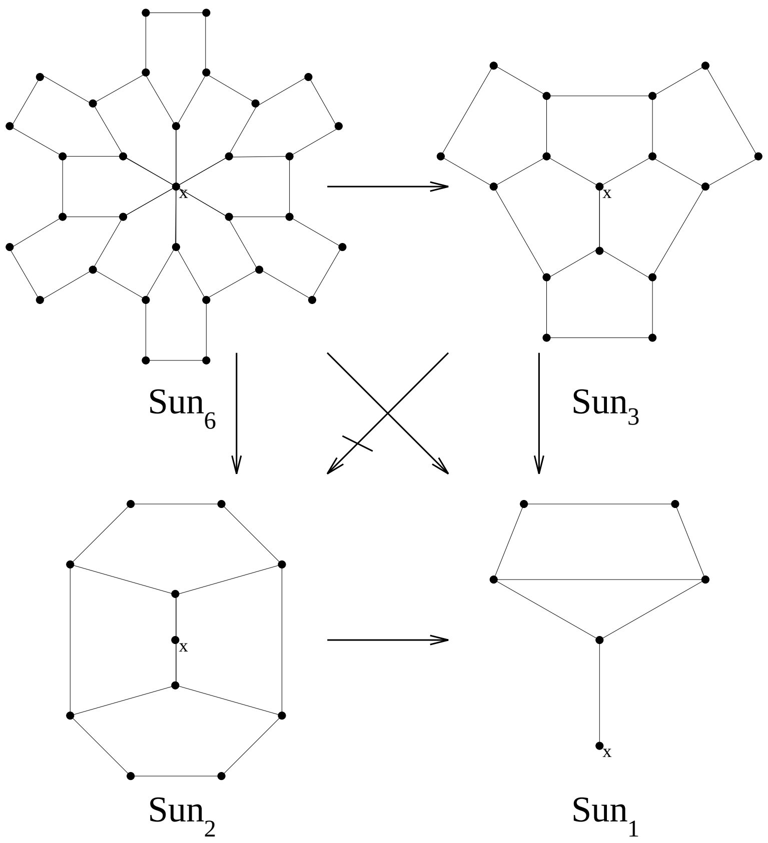

Figure 1: Sunflower graphs for $n = 1, 2, 3$, and 6. Note that although Sun_1 can be constructed, it is not triangle-free. Arrows denote that homomorphisms exist from one graph to the other; slashed arrows indicate that no homomorphism exists.

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