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# Chooseability and list coloring

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Graph colorings belong to classical graph theoretical problems that are important both for their practical applications and richness of theoretical results. E.g., the Four Color Conjecture has stimulated research in discrete mathematics for more than hundred years, and the recent computer-aided proofs of extensions of the Four Color Theorem by Robertson et al. highlight the development of graph theory in the latest years. Also, graph coloring viewed as graph homomorphism is an important interdisciplinary meeting point of algebra and graph theory.

A real abundance of results in the last 10 years appeared in the area of colorings with local constraints, including list colorings, precoloring extension, and chooseability. The concept of list coloring was introduced in the seventies by Vizing [15] and independently by Erdős, Rubin and Taylor [10], and though it seemed to have been forgotten for more than 10 years, the beginning of nineties suddenly brought this subject into the center of interest. This might have been triggered by the solution of several problems from [10] concerning chooseability of planar graphs. First Alon and Tarsi [4] proved that bipartite planar graphs are 3-choosable, then M. Voigt showed that not all planar graphs are 4-choosable [16], and finally C. Thomassen proved that every planar graph is 5-choosable [14]. Another milestone was Galvin's proof of the Dinitz conjecture on Latin squares, showing more generally that the list chromatic index of a bipartite multigraph equals the maximum degree of the multigraph in question [11]. Many new results have appeared since then, and though maybe not all of them are as earth-shattering as the above mentioned ones, each one sheds light onto some piece of the vividly colorful mosaic of graph coloring problems.

The purpose of these notes is to introduce the concept of list colorings and chooseability to a reader who has reasonable knowledge of graph theory and methods used in discrete mathematics. The text gives all definitions and a few proofs, but mostly it is organized as a tour through the land of list colorings self-guided by excercises and comments. The exercises are marked according to the following scheme:

• – relatively easy statement that the reader might be able to prove himself or herself;

•• – advanced problem, consulting the recommended literature may be necessary;

••• – open problem, solution of which would lead to a publishable paper.

(The rating may be individual. And open problems may turn out solvable when tackled in the appropriate way.)

We consider finite undirected graphs without multiple edges or loops. By a coloring of a graph we mean (if not said otherwise) a coloring of the vertices of the graph, i.e., a mapping from the vertex set into a set of colors (which may be and mostly is a set of positive integers). A coloring is called *proper* if adjacent vertices receive distinct colors. The minimum number of colors in a proper coloring of a graph $G$ is denoted by $\chi(G)$ and called the chromatic number of $G$. The maximum degree in $G$ is denoted by $\Delta(G)$.

# 1 Getting started: List colorings and chooseability of planar graphs

Given a graph $G$ and an assignment $\mathcal{L} = \{L(v) \mid v \in V(G)\}$ of lists of admissible colors to its vertices, we say that $G$ is *$\mathcal{L}$-list colorable* if the vertices of $G$ can be properly colored so that each vertex $v$ is colored with a color from $L(v)$. (If $\mathcal{L}$ is obvious from the context, we simply say that $G$ is *list colorable*.)

If all lists of $\mathcal{L}$ have the same size $k$, $\mathcal{L}$ is called a *$k$-assignment*. The minimum integer $k$ such that $G$ is $\mathcal{L}$-list colorable for every $k$-assignment $\mathcal{L}$, is called the *chooseability* (or the *list chromatic number*) of $G$ and is denoted by $\chi_\ell(G)$. The graph $G$ is called *$k$-choosable* if $\chi_\ell(G) \leq k$.

Note that the *List Coloring* problem, as it asks for *existence* of a coloring is typically not harder (from the computational complexity point of view) than the *Chooseability* problem, which asks if a coloring exists *for every* assignment.

A more general concept of *list set colorings* stems from set colorings of graphs. In this case vertices of a given graph should be assigned sets of colors of prescribed size, so that adjacent vertices are assigned disjoint sets. Formally, let $f, g : V(G) \to \mathbf{N}$ be functions such that $f(v) \geq g(v)$ for every vertex $v$ of $G$. Then $G$ is *$(f, g)$-choosable* if for every list assignment $\mathcal{L}$ with $|L(v)| = f(v)$ for every $v$, there can be chosen subsets $S(v) \subseteq L(v)$ of cardinality $g(v)$ such that $S(u)$ and $S(v)$ are disjoint whenever $u$ and $v$ are adjacent.

Special variant of this general problem is Set-chooseability ($(p, q)$-chooseability), which refers to the case when $f(v) = p$ and $g(v) = q$ are constant functions. A variant of the List coloring problem is Precoloring Extension, which corresponds to the case when some lists are one-element (i.e., their vertices are precolored) while the others are identical. Both these problems are discussed in subsequent sections.

To get better acquinted with the notion of list colorings and chooseability, let us first make a few observations.

**Observation 1** *For every graph $G$, $\chi(G) \leq \chi_\ell(G)$.*

**Proof** Let $k = \chi_\ell(G)$. Since $G$ is $k$-choosable, by definition any $k$-assignment allows a proper list coloring. Hence also the assignment $\mathcal{L}$ which assigns the same list $L(v) = \{1, 2, \ldots, k\}$ of admissible colors to every vertex $v \in V(G)$. An $\mathcal{L}$-list coloring for this particular list assignment $\mathcal{L}$ is a proper coloring of the vertices which uses at most $k$ colors. Hence $G$ is $k$-colorable. ∎

This inequality can be strict:

**Observation 2** $\chi_\ell(K_{2,4}) > 2$.

**Proof** Let the vertices in one part be $a, b$ and in the other one $c, d, e, f$. The assignment

$L(a) = \{1, 2\}, L(b) = \{3, 4\}$,

$L(c) = \{1,3\}, L(d) = \{1,4\}, L(e) = \{2,3\}, L(f) = \{2,4\}$
is not list colorable, showing that $K_{2,4}$ is not 2-choosable. ■

Moreover, the gap between chromatic number and chooseability can be arbitrary large:

**Exercise 1.1** • *For every $k$ there is a non-$k$-choosable bipartite graph.*

It is also well known that $\chi(G) \leq \Delta(G) + 1$ (and in fact, this inequality is strict unless $G$ is a complete graph or an odd cycle). The same inequality holds also for chooseability. (A graph is called *$k$-degenerate* if each of its subgraphs contains a vertex of degree at most $k$.)

**Observation 3** *Let $G$ be a $k$-degenerate graph. Then $\chi_\ell(G) \leq k+1$.*

**Proof** By induction on the number $n$ of vertices of $G$. The statement is clear if $n = 1$. For the inductive step suppose $n > 1$ and let $v$ be a vertex of degree at most $k$ in $G$ ($G$ is a subgraph of itself). Let a $(k+1)$-assignment $\mathcal{L}$ of $G$ be given by the adversory. Then $G' = G - v$ is also $k$-degenerate and has less vertices than $G$, hence by the induction hypothesis, $G'$ can be $\mathcal{L}$-list colored. It remains to color vertex $v$. Since $v$ has at most $k$ neighbors, and these are in the worst case colored by $k$ distinct colors, at least one color from the list $L(v)$ of size $k+1$ can still be used to complete a proper coloring of $G$. ■

As every graph $G$ is $\Delta(G)$-degenerate, we obtain the following corollary:

**Observation 4** *For every graph $G$, $\chi_\ell(G) \leq \Delta(G) + 1$.*

Since by Euler formula every planar graph is 5-degenerate, it follows that every planar graph is 6-choosable. The complete graph on 4 vertices, $K_4$, has chromatic number 4 and as such it is not 3-choosable. Thus *maximum chooseability of planar graphs lies between 4 and 6 (inclusive).* This gap was noted in [10] and it was conjectured that the right number is 5, which was proved more than 10 years later by Thomassen and Voigt. Thomassen's proof of 5-chooseability of planar graphs is a masterpiece of induction, the construction of Voigt strikes with its simplicity.

**Theorem 1** ([14]) *Every planar graph is 5-choosable.*

**Proof** We prove the statement by induction on the number of vertices of $G$. The trick of the tail is to formulate a stronger statement, which would be advantageous in the inductive step, but which at the same time would be modest enough to be still true.

We first observe that (as with other ocloring problems) it suffices to prove the statement for *triangulations* (maximal planar graphs, i.e., all faces in any planar embedding are triangles), because adding edges to a non-5-choosable graph cannot make the graph 5-choosable. However, for the purpose of the induction, we take one step backward and we will be proving the statement for *semi-triangulations*, i.e., planar graphs embedded in the plane so that all faces except the outer one are triangles, and the boundary of the outerface is a cycle in the graph (i.e., the graph is 2-connected).

Secondly, we will strengthen the theorem by shortening the lists assigned to the vertices on the boundary of the outerface. The precise statement we are going to prove is the following:

*Given a planar semi-triangulation $G$ with boundary cycle $C$ of the outerface, and given a list assignment $\mathcal{L}$ such that one pair of consecutive vertices on $C$ are precolored by different colors (i.e., their lists are of size 1 and are different), the lists assigned to other vertices of $C$ are of size 3, and the lists assigned to inner vertices of $G$ are of size 5, then $G$ is $\mathcal{L}$-list colorable.*

The base step of the induction is a triangle with two vertices precolored by different colors, and the third vertex being assigned a list of size 3. This is precisely what is needed to guarantee the existence of a proper list coloring.

For the inductive step, suppose $G$ with $> 3$ vertices and a list assignment satisfying the above specified conditions are given. Let the precolored vertices on $C$ be $a$ and $b$, and let $\alpha, \beta$ be their prescribed colors. We distinguish two cases (cf. also the illustrative Fig. 1):

**1.** $G$ has a diagonal, i.e., some vertices of $C$ are connected by an edge which does not belong to $C$. Let $xy$ be such an edge. The vertices $x$, $y$ form a 2-cut in $G$, and let $G_1$ and $G_2$ be the two subgraphs of $G$ determined by this cut ($x, y$ being included both in $G_1$ and $G_2$). Both precolored vertices $a, b$ must belong to the same subgraph, say $G_1$. Consider $G_1$ first. It has fewer vertices than $G$ and by induction hypothesis, it can be $\mathcal{L}$-list colored. Fix such a proper coloring $\phi$. Now consider $G_2$ with the lists inherited from $\mathcal{L}$, except for vertices $x$ and $y$, they will now be considered procolored by colors $\phi(x)$ and $\phi(y)$, respectively, the colors obtained from the fixed coloring of $G_1$. Since $x$ and $y$ are consecutive vertices on the boundary of $G_2$ and their colors are different, induction hypothesis can be applied to $G_2$ which has less vertices than $G$. Hence there is a proper $\mathcal{L}$-list coloring $\psi$ of $G_2$ which agrees with $\phi$ on $G_1 \cap G_2$. Because of planarity of $G$, no edges connect vertices of $G_1 - \{x, y\}$ to $G_2 - \{x, y\}$, and thus the union of $\phi$ and $\psi$ is a proper $\mathcal{L}$-list coloring of $G$.

**2.** $G$ has no diagonal. Let $x$ be the neighbor of $a$ on $C$ other than $b$. Let $X$ be the set of neighbors of $x$ inside $C$, and let $y$ be the other neighbor of $x$ on $C$. Consider $G' = G - x$. Since $G$ had no diagonals, $G'$ is a semi-triangulation, and as it has fewer vertices than $G$, induction hypothesis can be applied. However, we still have to define the list assignment $\mathcal{L}'$ of $G'$ to which the induction hypothesis will be applied. We define that by altering the lists assigned to vertices of $X$ (the lists of the remaining vertices of $G'$ will be as in $\mathcal{L}$). Let $p, q$ be two distinct colors appearing in $L(x)$ different from $\alpha$, the color of $a$ (since $|L(x)| = 3$ there must be at least two such colors). Delete $p, q$ from the lists $L(t)$ assigned to vertices $t \in X$, and if necessary shorten these lists (to make their size 3). By induction hypothesis, this $\mathcal{L}'$ allows a list coloring of $G'$.

So fix such a proper coloring $\phi$. It remains to color vertex $x$ properly. For this, color $x$ by color $p$ if $\phi(y) = q$, and color it with color $q$ otherwise. To see that this is a proper $\mathcal{L}$-list coloring of $G$, note that both $p, q$ are admissible colors for $x$ in $\mathcal{L}$, and $p, q$ cannot occur at other neighbors of $x$ than $y$ (we chose $p, q$ so that $\phi(a) \neq p, q$ and $p, q \notin L'(t)$ for $t \in X$). ∎

**Theorem 2** ([16]-adjusted) *Not every planar graph is 4-choosable.*

**Proof** Start with the graph $M$ depicted in Figure 1 left. We will use 6 colors, say 1,2,3,4,$\alpha$,$\beta$. Vertices $a, b, c$ and $d$ will be assigned the same lists $L(a) = L(b) = L(c) = L(d) = \{1, 2, 3, 4\}$. The other vertices will have $\alpha$ and $\beta$ in their lists, and in addition two colors from 1,2,3,4. These two colors will be assigned so that to contradict particular choice of colors for $a, b, c, d$.

Suppose for the time being that the adversory decided to color vertex $a$ with color 1 and vertex $d$ with color 2. Then vertices $c$ and $b$ must be colored 3 and 4 or vice versa. If the lists assigned to $e, f, g$ are $L(e) = \{\alpha, \beta, 1, 3\}$, $L(f) = \{\alpha, \beta, 1, 4\}$, $L(g) = \{\alpha, \beta, 3, 4\}$ and the adversory colored $c$ by color 3 and vertex $b$ by color 4, then vertices $e, f, g$ cannot be properly colored, as they form a triangle and only the two colors $\alpha$ and $\beta$ can be used on them. Similarly, lists $L(k) = \{\alpha, \beta, 2, 3\}$, $L(m) = \{\alpha, \beta, 2, 4\}$, $L(h) = \{\alpha, \beta, 3, 4\}$ do not permit a proper list coloring if $c$ is colored 4 and $b$ is colored 3.

So far we have succeeded in constructing an uncolorable list assignment under the assumption that the adversory colors the vertices $a$ and $d$ as we expected. Now we take 12 copies of the graph $M$, identify the vertices $a$ of all of them into one vertex $A$, and also identify all $d$ vertices into one vertex $D$, and make the vertices $A$ and $D$ adjacent. The resulting graph $G$ is planar and has 98 vertices. We claim that it is not 4-choosable. We construct a non-list colorable 4-assignment as follows. For every pair of distinct colors $x, y \in \{1, 2, 3, 4\}$, designate one of the copies of $M$ to this pair, and assign lists to the vertices $e, f, g$ of this copy to prevent a list coloring that would color vertices $A, D, c, b$ with colors $x, y, z, t$ (respectively), and assign lists to vertices $h, k, m$ of the same copy of $M$ to prevent a list coloring that would color vertices $A, D, c, b$ with colors $x, y, t, z$ (respectively) (here $z$ and $t$ denote the complemetary 2 colors to $x$ and $y$ so that $\{x, y, z, t\} = \{1, 2, 3, 4\}$). Such assignments are obtained from the one described in the previous paragraph by substitution $(1, 2, 3, 4) \leftrightarrow (x, y, z, t)$. This copy of $M$ then bans colorings that would color $A$ by color $x$ and $D$ by color $y$. Since $A$ and $D$ have to be colored by different colors, in this way we prevent all possible colorings of $A, D$. Hence $G$ is not 4-choosable. ∎

Other authors have later found other constructions of planar non-4-choosable graphs, the current world record is kept by Mirzakhani with 63 vertices.

**Exercise 1.2** ••• *Can you find a planar non-4-choosable graphs with less than 63 vertices?*

Of course, the ambitious goal would be to find the *minimum number of vertices of a planar non-4-choosable graph*, but neither a nontrivial lower bound is known.

## 2 Chooseability with separation

The idea of requiring that lists assigned to adjacent vertices be farther apart was introduced in [12]. The effect of such distance contraints was studied in [13], which is the universal recommended paper for this section. As the first promotion let us see what happens to List coloring for planar graphs:

**If all lists are the same,** then list size 4 guarantees existence of a list coloring (Four Color Theorem);

**If we let the lists be arbitrary,** then list size 4 is not always enough (but list size 5 is sufficient), cf. the preceding section;

**If we require that adjacent vertices are assigned almost dijsoint lists** (i.e., lists with intersection of size at most 1), then list size 4 is again sufficient to guarantee existence of a list coloring (cf. below).

Further discussion of the effect of larger intersections of lists is among the exercises. But the definitions first:

Let $p, r$ be natural numbers. A $p$-assignment $\mathcal{L}$ of $G$ is called a $(p,r)$-*assignment* if $|L(u) \cup L(v)| \geq p + r$ for all $uv \in E(G)$. The graph $G$ is called *(p,1,r)-choosable* if it admits an $\mathcal{L}$-list coloring for every $(p,r)$-assignment $\mathcal{L}$. (The '1' in the definition of $(p,1,r)$-chooseability may seem obsolete, but it will be clear in Section 3. Note also that $(p,r)$-chooseability mentioned in Section 1 is substantially different from $(p,1,r)$-chooseability.) For $r = 0$, we are getting ordinary $p$-chooseability.

**Exercise 2.1** • *Every planar graph is* $(4,1,3)$*-choosable.* (Hint: Try to adjust Thomassen's proof of 5-chooseability.)

**Exercise 2.2** • *Show that not every planar graph is* $(4,1,1)$*-choosable.*

**Exercise 2.3** ••• *Is every planar graph* $(4,1,2)$*-choosable?*

Given a graph $G$, let $m = \max_{H \subseteq G} \lceil \frac{|E(H)|}{|V(H)|} \rceil$.

**Exercise 2.4** • *The edges of $G$ can be oriented so that every vertex has out-degree at most $m$.* (Hint: Use the Hall matching theorem or network flows.)

**Exercise 2.5** • *If $m$ is as above, then $G$ is* $(cm+1, 1, c(m-1)+1)$*-choosable for every $c$.*

Using the preceding Exercise, show

**Exercise 2.6** • *Planar bipartite graphs are* $(3,1,2)$*-choosable.*

and also

**Exercise 2.7** $^{\bullet}$ *Give a second proof to Exercise 2.1.*

Another open problem is

**Exercise 2.8** $^{\bullet\bullet\bullet}$ *Is every planar graph* $(3,1,2)$*-choosable?*

We will further look at cases when $r$ is large compared to $p$, namely when $p-r=c$ is constant. Note that the requirement $|L(u)\cup L(v)|\geq p+r$ is equivalent to $|L(u)\cap L(v)|\leq p-r$. Somewhat surprisingly, for $c=1$ we can get sharp results (larger $c$ provides open problems), and complete graphs play important role here. Let us first have a look at complete graphs with different lists. In general, the complete graph $K_n$ with $n$ vertices is $n$-choosable, but not $(n-1)$-choosable (e.g., if all lists are the same you need at least $n$ colors). Once we require that lists are pairwise different, chooseability drops even by 2:

**Exercise 2.9** $^{\bullet}$ *For $n\geq 3$, the complete graph $K_n$ is $(k,1,1)$-choosable if and only if $k=1$ or $k\geq n-2$.* (Hint: What is a list coloring of a complete graph?)

Since the rest of the section is devoted to the case of constant $p-r$, it turns handy to introduce the following notation:

For a graph $G$ and a nonnegative integer $c$, we denote by $\zeta(G,c)$ the minimum integer $k$ such that $G$ is $(k,1,k-c)$-choosable. Note that $G$ is then $(k,1,k-c)$-choosable for every $k\geq\zeta(G,c)$.

The expression in the following exercise may seem awful, but it is exactly what comes out of the most natural proof:

**Exercise 2.10** $^{\bullet}$ *Let $n\geq 3$. Then $\zeta(K_n,1)\leq\lfloor\sqrt{n-\frac{11}{4}}+\frac{3}{2}\rfloor$.*

And it turns out that this crazy-looking bound is sharp:

**Exercise 2.11** $^{\bullet}$ *For any prime power $q$, $\zeta(K_{q^2+1},1)\geq q+1$.* (Hint: Finite projective and affine planes.)

If you still doubt the announced sharpness of the previous results, do a bit of calculation:

**Exercise 2.12** $^{\bullet}$ $\lfloor\sqrt{q^2+1-\frac{11}{4}}+\frac{3}{2}\rfloor=q+1$ *for $q\geq 2$.*

With a little bit of number theory, namely using the fact that for any real $\varepsilon$ and any natural number $m$ sufficiently large with respect to $\varepsilon$, there is a prime between $m$ and $(1+\varepsilon)m$, we can summarize:

**Exercise 2.13** $^{\bullet\bullet}$ $lim_{n\to\infty}\frac{\zeta(K_n,1)}{\sqrt{n}}=1$.

In the case of $c>1$ we still have asymptotically tight bounds for $\frac{\zeta(K_n,c)}{\sqrt{n}}$:

**Exercise 2.14** •• *For every integer $c \geq 2$,*

$$\sqrt{\lfloor \frac{c}{2} \rfloor} \leq \liminf_{n\to\infty} \frac{\zeta(K_n, c)}{\sqrt{n}} \leq \limsup_{n\to\infty} \frac{\zeta(K_n, c)}{\sqrt{n}} \leq \sqrt{2\mathrm{e}c}.$$

but the existence of the limit is not known:

**Exercise 2.15** ••• *Does there exist a limit $L = \lim_{n,c\to\infty} \frac{\zeta(K_n,c)}{\sqrt{nc}}$? If yes, determine the limit.*

The upper bound in Exercise 2.14 actually follows from a more general theorem which relates $\zeta(G, c)$ to the maximum degree $\Delta(G)$ of $G$. The proof uses Lovász Local Lemma (first stated and proved in [9], but nowadays you can find it in any textbook on probabilistic methods in graph thoery), and it is interesting that in this case a probabilistic argument gives a result tight upto only a multiplicative constant:

**Exercise 2.16** •• *Prove that $\zeta(G, c) \leq \lceil \sqrt{\mathrm{e}c(2\Delta(G) - 1)} \rceil$ for any graph $G$ and $c \geq 1$, where $\mathrm{e} = 2.718\ldots$ is the Euler constant.*

# 3 Set chooseability and fractional colorings

As mentioned above, the objective of set coloring and set chooseability is to color the vertices with *sets of colors* of prescribed size, so that adjacent vertices receive *disjoint* sets of colors. Recall that $G$ is $(p, q)$-choosable if every $p$-assignment allows a $q$-set list coloring (coloring by sets of size $q$). More generally we say that $G$ is $(p, q, r)$*-choosable* if every $(p, r)$-assignment allows a $q$-set list coloring (cf. $(p, 1, r)$-chooseability in Section 2). We mention $(p, q, r)$-chooseability to show an example where Chooseability is (somewhat surprisingly) easier than List coloring. In the complexity results, $p, q, r$ are fixed parameters of the problems, the input to the problems are a graph (for Chooseability) together with a $(p, r)$-assignment (for List coloring). It is shown in [12] that $(p, q, r)$-List coloring problem is NP-complete for $p \geq \max\{q + 2, r + 1\}$ and polynomially solvable for $p = r \geq q$ and for $q \leq p \leq q + 1$. On the other hand, $(p, q, r)$-Chooseability is solvable in linear time if $8q - 2p > 6r \geq 3p$ or when $4r \leq 2p < 4q - r$. This at first sight slightly suprising situation is due to the fact that in these ranges of parameters most graphs allow non-list-colorable assignments, and so graphs which are $(p, q, r)$-choosable can be characterised. However, the complete discussion of the complexity of $(p, q, r)$-Chooseability is not known:

**Exercise 3.1** ••• *Give a full characterisation of parameter sets $p, q, r$ for which deciding if a given graph is $(p, q, r)$-choosable is polynomial (assuming $P \neq NP$).*

The complexity of the plain $k$-Chooseability is well understood, Gutner and Tarsi showed that it is $\Pi_2^P$-complete for every $k \geq 3$, while 2-choosable graphs can be characterised [10]. The case of $r = 0$ may be tractable:

**Exercise 3.2** ••• *Give a full characterisation of parameter sets $p, q$ for which deciding if a given graph is $(p,q)$-choosable is polynomial (assuming $P \neq NP$).*

A rather innocent looking problem of [10] asks:

**Exercise 3.3** ••• *Is every $(p,q)$-choosable graph also $(pm, qm)$-choosable (for every positive integer $m$)?*

Now it seems that this problem is rather tough and it is still open. An affirmative solution would imply that chooseability is a submultiplicative function on the union of graphs:

**Exercise 3.4** • *Prove that the conjecture of Exercise 3.3 implies that*

$$\chi_\ell(G_1 \cup G_2) \leq \chi_\ell(G_1) \cdot \chi_\ell(G_2)$$

*for any two graphs $G_1, G_2$ on the same vertex set.*

This seems to be just another difficult problem:

**Exercise 3.5** ••• *Is it true that*

$$\chi_\ell(G_1 \cup G_2) \leq \chi_\ell(G_1) \cdot \chi_\ell(G_2)$$

*for any two graphs $G_1, G_2$ on the same vertex set?*

The good news is that the chooseability of the union of graphs is at least bounded by some (though superexponential) function in the choosabilities of the graphs. This follows from the following result of Alon:

**Exercise 3.6** •• [5] *Let $d$ be the average degree in a graph $G$ (i.e., $d = \frac{1}{|V(G)|}\sum_{v \in V(G)} deg\, v$) and let $k$ be an integer such that*

$$d > 4\binom{k^4}{k}\log(2\binom{k^4}{k}).$$

*Then $\chi_\ell(G) > k$.*

**Exercise 3.7** • *There is a function $h$ such that*

$$\chi_\ell(G_1 \cup G_2) \leq h(\chi_\ell(G_1), \chi_\ell(G_2))$$

*for any two graphs $G_1, G_2$ on the same vertex set.* (Hint: Use Exercise 3.6.)

Another related open problem is due to Voigt:

**Exercise 3.8** ••• *Let $G = (V_1 \cup V_2 \cup V_3, E)$ be a graph and $V_1, V_2, V_3$ pairwise disjoint independent sets such that $V_i \cup V_j$ induces a 2-choosable subgraph for every $1 \leq i < j \leq 3$. Is it true that then $G$ is 3-choosable?*

(Voigt proved that such graphs are 4-choosable, and more generally, $(4m, m)$-choosable for every $m$.)

Finally we will relate the set-chooseability to fractional colorings of graphs. After the heavy load of open problems, a candlelight in darkness is that some parameters allow complete characterisation. Namely, we will study the following set

$$Ch(G) = \{\frac{k}{\ell} : G \text{ is } (k, \ell)\text{-choosable}\}$$

and relate it to the *fractional chromatic number*, a graph parameter which was studied before. Denote by $\mathcal{S}$ the set of all independent sets in $G$. The fractional chromatic number of $G$ is

$$\chi^*(G) = \inf \sum_{S \in \mathcal{S}} \phi(S)$$

where the infimum is taken over all functions

$$\phi : \mathcal{S} \to \mathbf{R}^{\geq 0}$$

that satisfy condition

$$\sum_{S: u \in S \in \mathcal{S}} \phi(S) \geq 1$$

for every $u \in V(G)$. As the first example, show the following:

**Exercise 3.9** • $\chi^*(C_{2k+1}) = 2 + \frac{1}{t}$ *for every* $t$.

**Exercise 3.10** • $C_{2k+1}$ *is* $(2t+1, t)$*-choosable for every* $t$.

Now comes the announced theorem, in two bites:

**Exercise 3.11** • *If* $G$ *is* $(k, \ell)$*-choosable then* $\chi^*(G) \leq \frac{k}{\ell}$.

**Exercise 3.12** •• [6] *For every graph* $G$, $\inf Ch(G) = \min Ch(G) = \chi^*(G)$.

# 4 Partial list colorings

If one wants to color a graph $G$ with $t < \chi = \chi(G)$ colors, then choosing $t$ largest color sets in a $\chi$-coloring one can always color at least $\frac{t}{\chi}|V(G)|$ vertices of $G$. It is a natural question to extend this observation to list colorings and to ask how many vertices of a graph can be colored properly from lists with fewer than $\chi_\ell(G)$ elements. Albertson, Grossman and Haas conjecture in [2] that the answer is similar to ordinary colorings.

In this section, let $G$ be a graph with $n$ vertices and let $\chi = \chi(G)$ and $\chi_\ell = \chi_\ell(G)$. For any $t \leq \chi_\ell$, we define $\lambda_t$ to be the *minimum* (taken over all $t$-assignments $\mathcal{L}$ of $G$) of the *maximum possible number of vertices of* $G$ *that can be properly* $\mathcal{L}$*-list colored.*

**Exercise 4.1** ••• ([2]) *Is it true that*

$$\lambda_t(G) \geq \frac{t}{\chi_\ell} n$$

*for any graph $G$ and every $0 \leq t \leq \chi_\ell$?*

Obviously, the conjectured inequality is true for $t \leq 1$ and $t = \chi_\ell(G)$. A first non-trivial result was also given by Albertson, Grossman and Haas:

**Exercise 4.2** •• ([2]) *For any graph $G$ and every $0 \leq t \leq \chi_\ell$,*

$$\lambda_t(G) \geq \left(1 - \left(\frac{\chi - 1}{\chi}\right)^t\right) n.$$

Already this result implies that the conjecture of Exercise 4.1 holds true in some interesting cases:

**Exercise 4.3** • *Show that* $\left(1 - \left(\frac{\chi-1}{\chi}\right)^t\right) \geq \frac{t}{\chi+t-1}$.

**Exercise 4.4** • *Show that the conjecture of Exercise 4.1 holds true for* $0 \leq t \leq \chi_\ell - \chi + 1$.

**Exercise 4.5** • *Show that the conjecture of Exercise 4.1 holds true for $t = 2$ if $\chi(G) < \chi_\ell(G)$.*

**Exercise 4.6** • *Show that the conjecture of Exercise 4.1 holds true for bipartite graphs.*

A breakthrough was achieved by Chappel, showing a linear lower bound for $\lambda_t$:

**Exercise 4.7** •• ([8]) *For any graph $G$ and any $t \leq \chi_\ell$, $\lambda_t(G) \geq \frac{6}{7} \frac{t}{\chi_\ell} n$.*

Albertson et. al. gather few more open problems in [2]. Some questions about planar graphs seem quite interesting. First, they note that $\lambda_2(G) \geq \frac{2n}{5}$ follows from results on acyclic colorings of planar graphs, and strike back with a strenthening of their original conjecture for this particular case:

**Exercise 4.8** ••• ([2]) *For a planar graph $G$, $\lambda_2(G) \geq \frac{n}{2}$.*

and also in the case of planar bipartite graphs (which are known to be 3-choosable)

**Exercise 4.9** ••• ([2]) *For a planar bipartite graph $G$, $\lambda_2(G) \geq \frac{n}{2}$.*

Particular interest deserve also upper bounds on $\frac{\lambda_t}{n}$. Perhaps one of the most attracive is the case $t = 4$ (the best upper bound known so far is $\frac{62}{63}$, rather far even from the conjectured lower bound):

**Exercise 4.10** ••• ([2]) *Find the smallest possible upper bound for $\frac{\lambda_4(G)}{n}$ for planar graphs $G$.*

## 5 Precoloring extension

In this section we consider the situation when some vertices of the given graph already come precolored, and the question is to color properly the remaining vertices, given either a bound on the total number of colors or lists of admissible colors for the vertices to be colored. The former problem is usually refered to as PrExt and the latter one as List coloring extension, but obviously both can be formulated uniformly as List Coloring, with the precolored vertices viewed as vertices assigned lists of admissible colors of size one. Coloring extension is a powerful technique used in many proofs in graph coloring theory.

Here we will focus on the case when the set of precolored vertices satisfies certain distance constraints. Interesting results were recently obtained by Albertson et al. The story begins with a problem posed by C. Thomassen:

*Suppose $G$ is a planar graph and $W \subseteq V(G)$ is such that the distance between any two vertices in $W$ is large, say at least 100. Can any proper 5-coloring of $W$ be extended to a proper 5-coloring of $G$?*

This is a good example that even gods sometimes ask solvable problems. The question of Thomassen was answered in affirmative by Albertson [1], where he notes that Thomassen showed a remarkable prophecy in his conjecture, especially when the conjectured distance constraint is taken in binary:

**Theorem 3** ([1]) *If the distance between any two vertices of $W$ is at least 4 then any proper 5-coloring of $W$ can be extended to a proper 5-coloring of the entire planar graph $G$.*

However, it turns out that planarity of $G$ is far from being essential in this result, only 4-colorability plays role. Prove the following generalization of Theorem 3:

**Exercise 5.1** $^\bullet$ ([1]) *If $G$ is $k$-colorable and the distance between any two vertices of $W$ is at least 4 then any proper $(k+1)$-coloring of $W$ can be extended to a proper $(k+1)$-coloring of the entire graph $G$.*

Under a weaker assumption of distance 3, we still have a positive result but additional colors are necessary:

**Exercise 5.2** $^\bullet$ ([3]) *If $G$ is $k$-colorable and the distance between any two vertices of $W$ is at least 3 then any proper $(k+1)$-coloring of $W$ can be extended to a proper $\lceil \frac{3k+1}{2} \rceil$-coloring of the entire graph $G$. The number of colors $\lceil \frac{3k+1}{2} \rceil$ is best possible.*

However, if we assume $k$-chooseability instead of $k$-colorability then we are again safe, even with List coloring extension:

**Exercise 5.3** $^\bullet$ *If $G$ is $k$-choosable and the distance between any two vertices of $W$ is at least 3 then for any $(k+1)$-assignment $\mathcal{L}$ of $G$, any proper $\mathcal{L}$-coloring of $W$ can be extended to a proper $\mathcal{L}$-coloring of the entire graph $G$.*

One may ask about List coloring extension for $k$-assignments in $k$-choosable graphs. It turns out that even a distance of at least 4 between any two vertices of $W$ is not sufficient to allow always such an extension. Tuza and Voigt constructed a planar graph where any two vertices of $W$ have distance $\geq 4$, the vertices of $W$ have lists of 3 colors, the vertices of $V(G) \setminus W$ have lists of 5 colors, and the graph is not list colorable for this list assignment. On the other hand, Thomassen's proof [14] on planar graph 5-chooseability shows that some precolorings (two adjacent precolored vertices) and short lists (on the same face) allow a list coloring extension. This invokes the following question:

**Exercise 5.4** ••• *Is there a constant $d$ such that for any planar graph $G$, if any two vertices of $W \subseteq V(G)$ are at least $d$ apart then for any 5-assignment $\mathcal{L}$ of $G$, any proper $\mathcal{L}$-coloring of $W$ can be extended to a proper $\mathcal{L}$-coloring of the entire graph $G$?*

# 6 Chooseability on surfaces

Thomassen's theorem on 5-chooseability of planar graphs invokes the question what is the chooseability of graphs embeddable to surfaces of higher genus. As with the ordinary chromatic number, the situation here is surprisingly easier in the sense that the Heawood number is the right answer (however, the proofs get more involved).

Recall that surfaces can be classified according to their genus and orientability. The *orientable surfaces* are the sphere with $g$ handles $\Sigma_g$, where $g \geq 0$. The *non-orientable surfaces* are the surfaces $\Pi_h$ ($h \geq 1$) obtained by taking the sphere with $h$ holes and attaching $h$ Möbius bands along their boundary to the boundaries of the holes. $\Pi_1$ is the projective plane, $\Pi_2$ is the Klein bottle, etc. The *Euler genus* $\varepsilon(\Sigma)$ of the surface $\Sigma = \Sigma_g$ is $2g$, and the *Euler genus* of $\Sigma = \Pi_h$ is $h$. Then $2 - \varepsilon(\Sigma)$ is the *Euler characteristic* of $\Sigma$.

It is well known that a graph embeddable on a surface of Euler genus $\varepsilon = \varepsilon(\Sigma)$ has chromatic number bounded by the Heawood number

$$H(\varepsilon) = \left\lfloor \frac{7 + \sqrt{24\varepsilon + 1}}{2} \right\rfloor .$$

For every surface $\Sigma$ distinct from the Klein bottle, the Heawood number $H(\varepsilon)$ is, in fact, the maximum chromatic number of graphs embeddable on it where the maximum is attained by the complete graph on $H(\varepsilon)$ vertices. Conversely, every graph with chromatic number $H(\varepsilon)$ embedded on $\Sigma$ contains a complete graph on $H(\varepsilon)$ vertices as a subgraph. This result was proved by Dirac for the torus and $\varepsilon \geq 4$ and by Albertson and Hutchinson for $\varepsilon = 1, 3$. (Franklin proved that the coloring problem for the Klein bottle has not the answer $H(2) = 7$ but 6.) In a recent paper, Böhme et. al. show that similar results hold for chooseability as well:

**Theorem 4** ([7]) *Let $\Sigma$ be a surface of Euler genus $\varepsilon$ with $\varepsilon \geq 1$ and $\varepsilon \neq 3$. If $G$ is a graph embedded on $\Sigma$, then $\chi_\ell(G) \leq H(\varepsilon)$ where equality holds if and only if $G$ contains a complete subgraph on $H(\varepsilon)$ vertices.*

The part of this result which concerns non-orientable surfaces is considerably harder, but one may want to see at least that the argument for orientable surfaces is not much different from the ordinary chromatic number:

**Exercise 6.1** •• ([7]) *Let $\Sigma$ be an orientable surface of Euler genus $\varepsilon$. If $G$ is a graph embedded on $\Sigma$, then $\chi_\ell(G) \leq H(\varepsilon)$ and equality holds if and only if $G$ contains a complete subgraph on $H(\varepsilon)$ vertices.*

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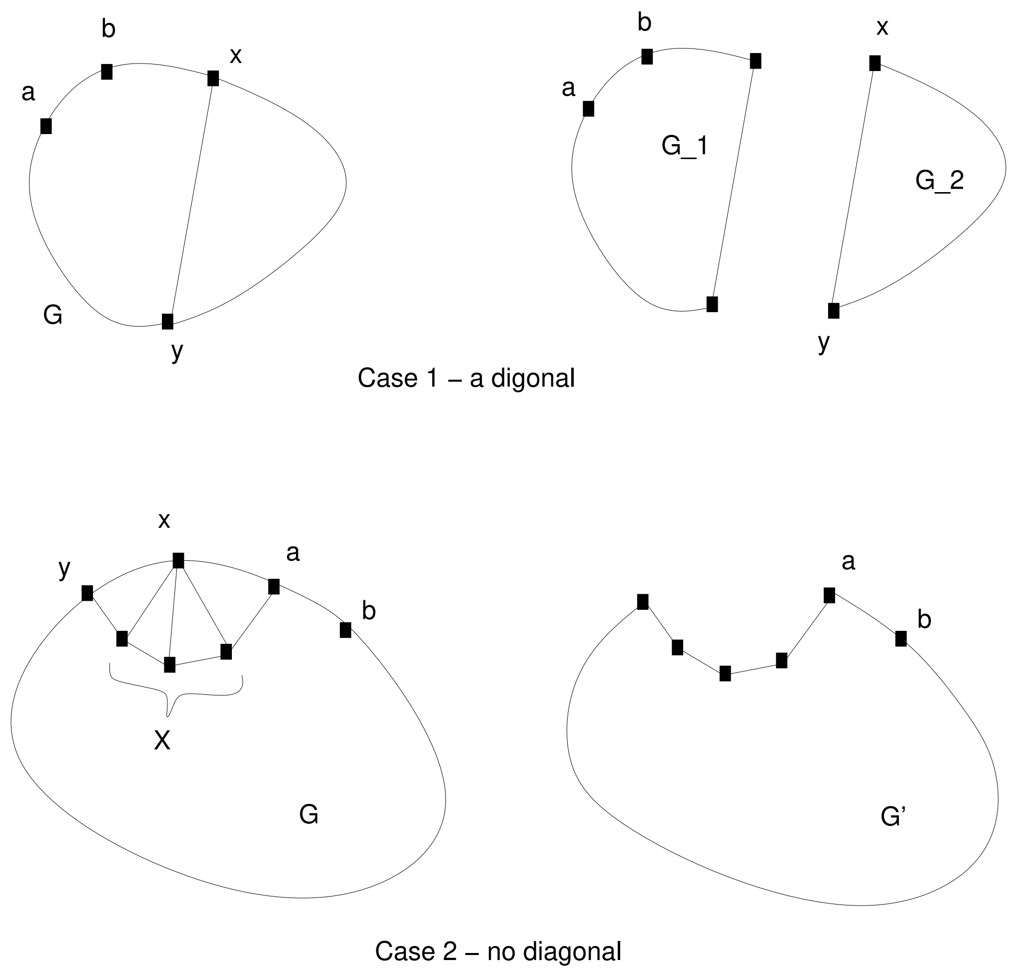


Figure 1: Illustration to the inductive step.

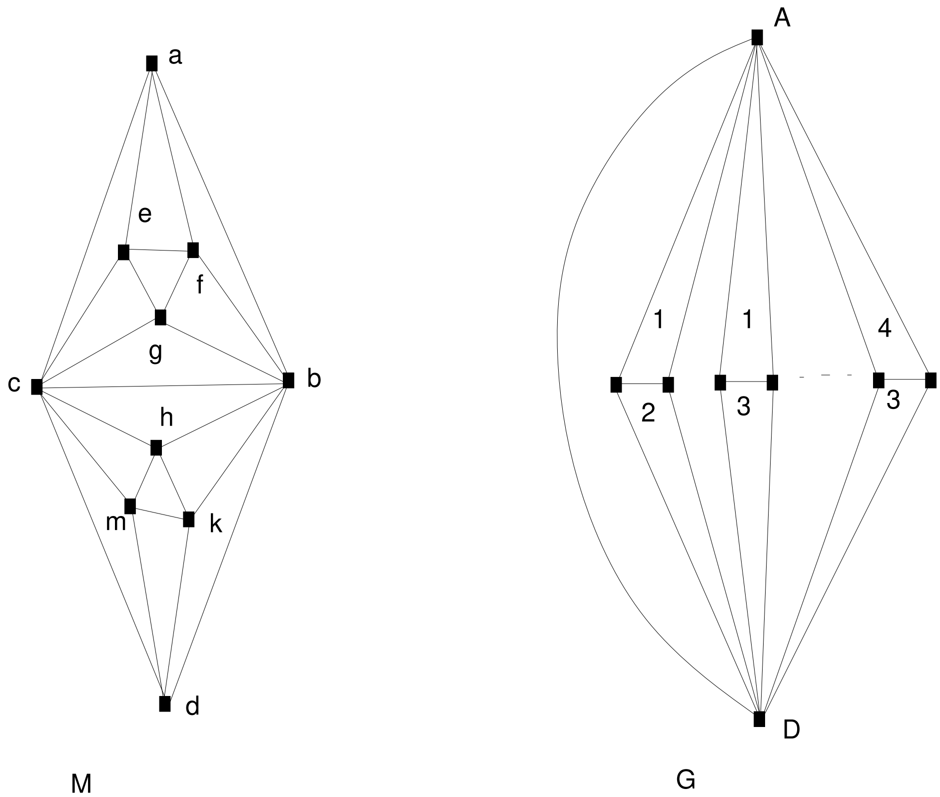


Figure 2: Construction of a non-4-choosable planar graph.