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Duality,
Nowhere-Zero Flows,
Colorings and Cycle Covers [*]

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Abstract

Nowhere-Zero Flows Problems, Coloring problems, Cycle cover problems lie in the core of graph theory. The strong relationship between them is the duality. Several important (and beautiful) conjectures are still open and this a very active field of study. We presente this subject with the enlightening notion of duality.

1 Introduction

This paper was written for the traditional Spring School of the Combinatorial Seminar at Charles University which was held in April 1999 in Borová Lada and in Finsterau. In 1999 this school was organized jointly with Humboldt Universitaet Berlin, Universitaet Bonn and Université Bordeaux I. Teachers from these schools took part in the meeting. The text tries to provide a study text for (undergraduate) students and it should serve as a background for the discussions and lectures at Spring School.

Some additional material and some complementary information can be found in the following articles:

[19] F. Jaeger. *Flows and Generalized Coloring Theorems in Graphs.* J.Combin. Theory Ser. B 26 (1979),pp. 205-216.

[22] F. Jaeger. *Nowhere zero-flow Problems.* Selected topics in Graph Theory 3 Academic Press, London 1988, 71–95.

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[39] J. Nešetřil, A. Raspaud. *Antisymmetric Flows and Strong Colourings of Oriented Graphs.* to appear in Annales de l'Institut Fourier.

[50] P.D. Seymour. *Nowhere-zero 6-flows.* J. Combinatorial Theory (B) **30** 1981, 130-135.

[51] P.D. Seymour. *Nowhere-zero flows.* Handbook of Combinatorics, edited by R. Graham, M. Grötschel and L. Lovász. (1995) 289–299.

[65] C.Q. Zhang. *Integer flows and cycle covers of graphs* Pure and Applied Mathematics, Dekker 1997.

For general terminology and introduction we refer to

[6] B. Bollobás: *Modern Graph Theory*, Graduate Texts in Mathematics, vol. 184, Springer (1998).

[33] L. Lovász: *Combinatorial problems and exercises*, North-Holland, 2nd Edition, 1993.

[34] J. Matoušek and J. Nešetřil: *Invitation to Discrete Mathematics*, Oxford University Press, 1998.

This paper is organized as follows:

Contents

The problems stated as questions in the text (there are 36 of them) are mostly (with some exceptions) routine problems which serve to improve understanding. We tried to make the paper selfcontained (for the benefit of students which are coming with different background) so the first part reviews the material from the (almost) very beginning.

2 Duality

Mathematicians love dualities. We postpone any effort to try to explain and to describe this phenomenon generally and instead we give a few examples. These examples are of course combinatorially biased (to fit our Spring School). But even in this restricted framework the examples are abundant. We shall list some without any special preference and order:

A. *Geometric Duality* for lines and points in a projective plane.

$$Lines \longleftrightarrow Points$$

This duality stems from the symmetry of the defining axioms of a projective plane and thus in "all statements and definitions" we can exchange lines for points and vice versa. An usefull example: We say that a *set of points* is in a *general position* if no three of them lie on a common line; likewise we say that a *set of lines* is in a *general position* if no three of them meet in a common point.

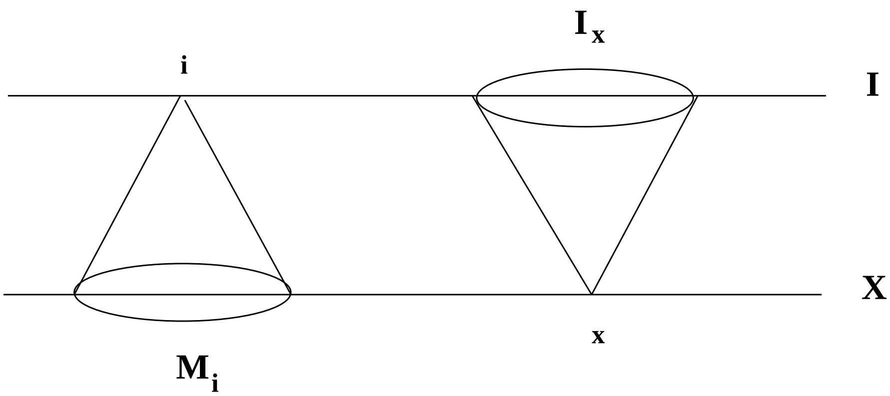
Geometric Duality transcendents projective planes and some geometric dualities fall in the examples which will be stated later (e.g. planar graph duality extending the duality of polyhedra).

In a combinatorial setting the *Geometric Duality* is related to more general:

B. *Set Systems Duality.*

Given a set system $(X, \mathcal{M}), \mathcal{M} = (M_i; i \in I), M_i \subset X$, we define *dual set system* by $(I, \mathcal{I}), \mathcal{I} = (I_x; x \in X)$ where $I_x = \{i; x \in M_i\}$.
The dual set system is denoted by $(X, \mathcal{M})^*$. Then we can also consider dual $((X, \mathcal{M})^*)^*$ of the dual $(X, \mathcal{M})^*$ of set system $(X, \mathcal{M})$, or shortly $(X, \mathcal{M})^{**}$, and as expected we can prove $((X, \mathcal{M})^*)^* \cong (X, \mathcal{M})$. Note that we have to deal with systems of sets, several sets may be repeated, and they may be empty too.

The principle of Set System Duality may be explained by the following picture:



Question 1
i. Prove $((X,\mathcal{M})^*)^* \sim (X,\mathcal{M})$.
ii. A set system $(X,\mathcal{M})$ is *simple* (or *linear*) if $(X,\mathcal{M})$ contains no pair of sets $M_i, M_j, i \neq j$, such that $|M_i \cap M_j| \geq 2$. Prove that $(X,\mathcal{M})$ is linear iff $(X,\mathcal{M})^*$ is linear.
iii. Define a *cycle* of length l in a set system and let the *girth* $g(X,\mathcal{M})$ of $(X,\mathcal{M})$ denotes the shortest length of a cycle in set system $(X,\mathcal{M})$. Decide whether $g(X,\mathcal{M})^* = g(X,\mathcal{M})$.
iv. Prove (or remember) that dual of a projective plane is itself a projective plane. What is the dual of Fano plane?

It is surprising how much structure is on this elementary level. We shall return to set system duals later on in a different context.

C. *Linear Programming Duality*

You all know this result (which goes back to beginning of linear programming and it is attributed to J. von Neumann). Let us review briefly this familiar situation:

Suppose we are given a real $m \times n$ matrix $\mathbf{A} = (a_{ij})$, a real vectors $\mathbf{b} = (b_1, b_2, ..., b_m)$ and $\mathbf{c} = (c_1, c_2, ..., c_n)$.

Primal problem of LP asks to find

$$\text{Primal Problem} \begin{cases} max\ \mathbf{c}^T x = max\ c_1x_1 + c_2x_2 + ... + c_nx_n. \\ \mathbf{Ax} \leq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{cases}$$

This is also called *primal LP problem.* The *dual problem* to this primal LP problem is then the following:

$$\text{Dual Problem} \begin{cases} min\ \mathbf{y}^T\mathbf{b} = min\ y_1b_1 + y_2b_2 + ... + y_mb_m. \\ \mathbf{y}^T\mathbf{A} \geq \mathbf{c}^T \\ \mathbf{y} \geq \mathbf{0} \end{cases}$$

Then the following holds:

Theorem 1 *If either of primal or dual LP has a solution, then both problems have a solution and the two solutions are equal.*

We stated the result in a simplified form and in a full generality this is another example of geometric duality. But we want to stress very briefly only one feature. (There is vast literature here, to name just a few see [9, 15, 49]).

In a combinatorial setting the matrix $\mathbf{A}$ often consists from *incidence matrix* of a set system $(X, \mathcal{M})$, $\mathcal{M} = (M_i; i \in I)$. This is matrix $(a_{ij}$ of order $|I| \times |X|$ (so that its rows may be indexed by the set I and its columns by set X) such that $a_{ix} = 1$ whenever $x \in M_i$, otherwise $a_{ix} = 0$. If $\mathcal{S}$ is a set system then we denote by $\mathbf{I}_{\mathcal{S}}$ its incidence matrix.

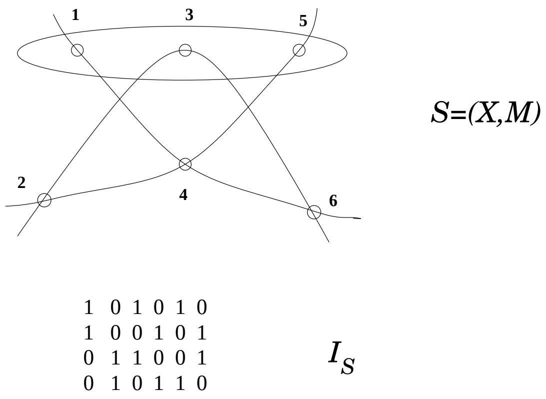


Figure 1: Example

Question 2
i. Prove that the transpose of $\mathbf{I}_{\mathcal{S}}$ is $\mathbf{I}_{\mathcal{S}^*}$.
ii. Prove (and remember) that the incidence matrix of an undirected graph is *totally unimodular* (i.e. all its square submatrices have determinant ± 1 or 0 iff the graph G is bipartite.

Thus when the primal problem is related to a set system $\mathcal{S}$ then we may consider the dual problem as related to its dual $\mathcal{S}^*$. Sometimes this analogy is perfect. For example if $G = (V, E)$ is a graph with $|V| = n$ then the primal problem (for $\mathbf{c} = \mathbf{1}, \mathbf{b} = \mathbf{1}, \mathbf{A} =$ the matrix of incidence of the graph G) can be written as:

$$(1) \begin{cases} max\ \mathbf{1^T x} \\ \mathbf{Ax} \leq \mathbf{1} \\ \mathbf{x} \geq \mathbf{0} \end{cases}$$

or more explicitely formula as:

$$\begin{cases} max\ x_1 + x_2 + ... + x_n \\ a_{i1}x_1 + a_{i2}x_2 + ... + a_{ij}x_j + ... + a_{in}x_n \leq 1 \quad \text{for every } i = 1, ..., n \end{cases}$$

The dual problem then has the form:

$$\begin{cases} min\ \mathbf{y^T 1} \\ \mathbf{y^T A} \geq \mathbf{1} \\ \mathbf{y} \geq \mathbf{0} \end{cases}$$

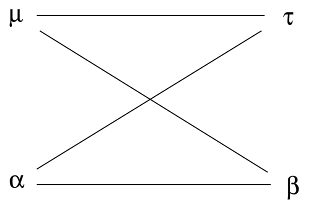
Now if G is bipartite graph then the basic solution $\mathbf{x}$ of primal and the basic solution $\mathbf{y}$ of dual problem are $0-1$ vectors and thus they are incidence vectors of a subset A of V and a subset C of E. If we now translate the conditional inequalities of primal and dual problems then we get the following properties of sets A and C:
No two vertices of A are joined by an edge (and thus A is an *independent set in* G, the maximal size of an independent set in G is denoted by $\alpha(G)$); every vertex of G is covered by an edge of C (and thus C is an *edge cover of* G, the minimal size of an edge cover of G is denoted by $\beta(G)$).

Thus we claim that $\alpha(G) = \beta(G)$ for every bipartite graph and inspired by this we may consider independent set as a *dual notion* of edge cover (and vice versa). We extend this from bipartite graphs (where the equality $\alpha(G) = \beta(G)$ holds) to general graphs and even set systems (where the equality clearly does not hold).
In attempts to compute (as you know hard to determine) parameters $\alpha(G)$ and the like we are trying to estimate , or approximate its value and thus we sometimes substitute $\alpha(G)$ by value $\alpha^*(G)$ which does not denotes here a dual value but a *fractional value*, by which we mean the (generally non-integer) solution of the above primal linear program (1).

But this is not the end of the story in this fertile area of combinatorial optimization : we may also consider the LP primal problem (1) for the matrix $\mathbf{A^T}$, i.e. for the incidence matrix of the dual system $\mathcal{S}^*$.

This of course has the same integral properties and thus the primal solution finds for any bipartite graph a subset M of E and dual solution finds a subset T of V with the following properties: no two edges of M meet in a vertex (and thus M is a *matching in* G, the maximal size of matching in G is denoted by $\mu(G)$); every edge of E is hitted at least once by a vertex of T (and thus T is a *vertex cover* or *hitting set* of graph G, the minimal size of a hitting set of G is denoted by $\tau(G)$).

Thus also matching and vertex cover are dual notions (in sense of the LP duality). But also matching (in a set system generally) is a notion dual to independent set and vertex cover is dual to edge cover (both by virtue of set-system duality). So picture gets more complicated (and we get Klein group of dualities):



And this is not superficial only, this also mirrores the following

Question 3
Prove Gallai equality: $\alpha(G)+\tau(G) = \beta(G)+\mu(G)$ for any graph G. What is this common value?

Question 4
Formulate an LP program to compute $\chi(G)$ and give a combinatorial interpretation to its dual LP.

Let us stop this short (very short) introduction to LP duality here, an interested reader can find further information in in [9, 15, 31, 49]. Our next example will be shorter because it is very general.

D. *Oriented Graph and Poset Duality*

Given an oriented graph $G = (V, E)$ or a relation R on a set X we can form a new graph $G' = (V, E')$ by reversing all the arcs of G; formally $E' = \{(y, x); (x, y) \in E\}$. In the case of relation R we form simply the inverse relation R^{-1}. Iterating once more this will clearly bring us back to the original graph.

This simple idea leads to abundance of dual notion when we interchange directions ("in" and "out") and in the case of relation when we exchange "$\leq$ " for "$\geq$ ". Thus we get pairs of dual notions:

in-degree	out-degree
sink	source
minimal	maximal
minimum	maximum
supremum	infimum
subset	superset

Of course theory of poset found some non-trivial dualities on their own. (Let us mention at least Boolean algebra duality. Also Galois connection between two posets can be interpreted as a duality.)

E. *Category Theory Duality*

The preceeding example was too easy (or superficially treated). So let us make it a bit more complicated.

Consider categories. Categories are just directed graphs (multigraphs with labeled arcs) where in addition we define composition operations between consecutive arcs.

For example all finite sets and all mappings between them form a category. Another category we get when we consider all (or some) finite graphs and all (or some) mappings between them. These maps can be for example isomorphisms, or only automorphisms, or relation "to be a minor" or homomorphisms which we define later in this article.

Categories in concrete examples presents a "world" for mathematicians, that is where we are working and living (without much thinking whether our world is or is not a category - as long we have TV and peanuts). But our world is here (although we think that it is only "there") and now imagine that we reverse all the arrows, all the arcs. Simple as that, as before for oriented graphs and posets. We reverse all the arcs and define composition as before. Call such a system *dual category.* See Fig 2 bellow:

But the actual meaning of the reversal will be of course profound. The arrows have names and "meaning" and they form our world, right? (Well, the first question of everybody: shall we reverse time too?)

Leaving fairy tales aside we can try to interpret those notions which were captured, or defined by arrows only (i.e. "categorical" notions). This then leads to the following pairs of dual concepts:

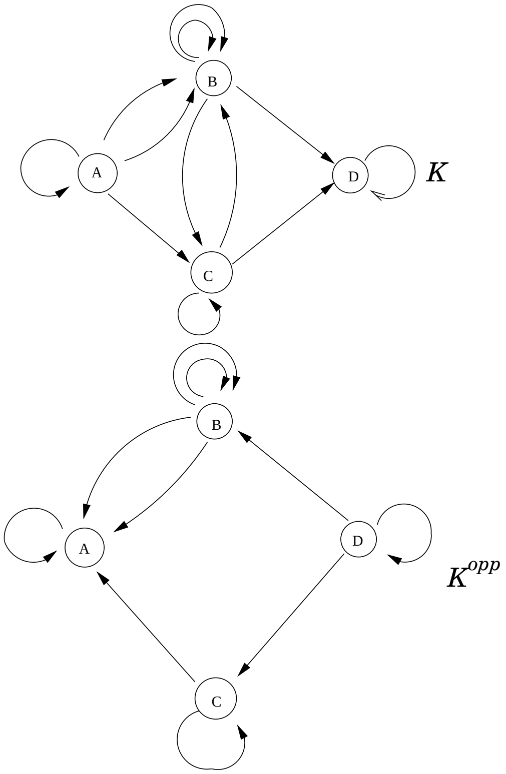


Figure 2: Category

injective mappings	surjective mappings
monomorphisms	epimorphisms
products	sums
limits	colimits
complete	cocomplete
intersection	union

and many others.

However easy to state (especially when we do not define all these notions) and arguably too general, yet this may be usefull. If only to organize otherwise complicated situation. Let us give an example from Ramsey theory, [37].

Let us state *Finite Ramsey Theorem* which you all know only too well:

Theorem 2 *Let p, k, n be positive integers. Then there exists N with the following properties: If X is any set of size at least N and if $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_k$ is any partition of the set $\binom{X}{p}$ of all p-element subsets of X then there exists a subset $Y \subset X$, Y of size at least n such that all p-element subsets of Y are in one of the classes of the partition.*

There are various ways how to simplify this (admittedly complicated) statement. One of them is to write $N \to (n)^p_k$ which does not explain much to a non-specialist.

Another way is to identify subsets with monotone injections. Let us be more precise: Let now natural numbers denote corresponding initial segments natural numbers: $k = \{0, 1, .., k-1\}$. Then $\binom{n}{p}$ will have both meaning of the combination number and also of the set of all p-subsets of the set $n = \{0, 1, .., n-1\}$. As these sets are linearly ordered each p-subset of n corresponds uniquelly to a monotone injection $p \to n$. The sets of all monotone injections $p \to n$ will be denoted by $< p, n >$. The Finite Ramsey Theorem then claims the following:

If N is sufficiently large then for every partition of the set $<p, N>$ of all monotone injections $p \to N$ into k-parts there exists a monotone injection $f : n \to N$ such that all monotone injections $f \circ g$ where $g \in < p, n >$ are in one class of the partition.

The following schematic diagram may be then usefull:

$$\begin{tikzcd}
\mathbf{n} \arrow[r] & \mathbf{N} \\
\mathbf{p} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

(i.e. $\longrightarrow$ denotes here a monotone injection.)

Let us now dualize this diagram, in the sense that we formally reverse all arrows. We obtain then:

$$\begin{tikzcd}
\mathbf{n} \arrow[d] & \mathbf{N} \arrow[l] \arrow[dl] \\
\mathbf{p} &
\end{tikzcd}$$

What are these arrows, what is their meaning? The dual of injectivity are surjections and as we considered monotone injections we shall have now surjections with the right ordering of preimages. Let us call them (for this moment) ordered surjections: A surjective mapping $f : n \to p$ is said to be a *monotone surjection* if for every $i, j, 0 \leq i < j < p$ holds $minf^{-1}(i) < minf^{-1}(j)$.

We can now translate this *Dual Ramsey Theorem* in the language of sets. It is not hard to see that we get the following (a *p-partition is a partition into p parts*):

Theorem 3 *Let p, k, n be positive integers. Then there exists N with the following properties: if X is any set of size at least N and if $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_k$ is any coloring of the set of all p-partitions of the set X by k colors then there exists a partition π of X into n classes such that all p-partitions of X which are coarser then π are in one of the class of the partition.*

This approach helped to isolate this theorem. Dual Ramsey Theorem is a true and powerfull statement (see [41, 7]).

Question 5
Can you give yet different interpretation for Dual Ramsey Theorem for p=2?

F. *Matroid duality*

A *matroid* M is a finite set S and a collection $\mathcal{F}$ of subsets of S (called *independent* sets) such that the following conditions are satisfied.
1. $\emptyset \in \mathcal{F}$
2. If $X \in \mathcal{F}$ and $Y \subset X$ then $Y \in \mathcal{F}$
3. If U, V are members of $\mathcal{F}$ with $|U| = |V| + 1$ then there exists $x \in U \setminus V$ such that $V \cup \{x\} \in \mathcal{F}$.

A subset of S not belonging to $\mathcal{F}$ is called *dependent.*

A *base* of M is a maximal independent subset of S, the collection of bases is denoted by $\mathcal{B}$. Let G be a graph, let S be its set of edges $E(G)$ and let $X \in \mathcal{F}$ if and only if X does not contain a cycle of G. Then $\mathcal{F}$ is the collection of independent sets of a matroid on S, called the *cycle matroid* of the graph G and denoted by $M(G)$.

Question 6
i. What are the bases of $M(G)$?
ii. Prove that all bases have the same size
(in $M(G)$ and in every matroid M)

A *circuit* of a matroid M is a minimal dependent set. The collection of its circuits is denoted by $\mathcal{C}$. The cycles of a graph G are the circuits of the cycle matroid $M(G)$ on the set of the edge set $E(G)$.
If $\{B_i : i \in I\}$ is a set of bases of a matroid M the $\{S \setminus B_i : i \in I\}$ is the set of bases of a matroid M^* on S. M^* is called the *dual matroid* of M. A subset is *spanning* in M if and only if it contains a base.

Question 7
i. Prove that a subset $X \subseteq S$ is independent in M^* if and only if $S \setminus X$ is spanning in M.
ii. Prove that dual matroid is indeed a matroid.
iii. Give interpretation of dual matroid of $M(G)$, what are bases of $M^*(G)$.
iv. What are cocircuits of G?

G. *On Dualities in General*

What is then a duality in general?

It is difficult to grasp this concept by an exhausting definition which would suit to all particular instances. Somehow each area of mathematics has its own dualities. Dualities are usually nice theorems which turn (in many instances) to definitions. Their importance is then perhaps more in concept formation and as a basic tool of understanding. They usually mirror some deeper laws and regularities in formation of particular theory and thus belong to broader class of phenomena like symmetry in general ([63]).

And even this setting is not exhausting all examples . There is also *Načeradský Duality* depicted on the following Figure ([38]).

This duality led to the 1990 meeting of J. Načeradský and the first author and to the whole project *Antropogeometry* on which they are working since, see e.g. [38].

Subject of this text is yet another duality between coloring and flows in graphs.

3 Flows

3.1 Definitions

Let $G = (V, E)$ be an oriented graph, if $S \subset V$ we denote by $\omega^+(S)$ the set of the edges which begin in S and terminate not in S. We write $\omega^-(S) = \omega^+(V \setminus S)$. An *integer flow* is a mapping ϕ from E to integers such that:

$$\forall v \in V : \sum_{e \in \omega^+(\{v\})} \phi(e) - \sum_{e \in \omega^-(\{v\})} \phi(e) = 0. \tag{1}$$

We will also say that the graph G has a an *integer flow.*

Every graph G has an integer flow. In fact, we can put integral capacities on the edges and then we know that there exists a *maximal flow* with all its values integral. This is a standard topic of theory of *network flows* and this integral property (which in turn is a consequence of total unimodularity of the incidence matrix of every directed graph; you know that) is the basis of all combinatorial applications of network flows. However this is of little relevance to us as we want to go in other direction. The following is the key definition:

Definition 1 *A* nowhere zero (integer) flow *in a (an oriented) graph G is an integer flow ϕ which associates to every arc of G a non-zero value. We could also define the support $\sigma(\phi)$ of a flow ϕ in a digraph $G = (V, E)$ as the set of arcs e such that $\phi(e) \neq 0$. Then a nowhere-zero flow is a flow ϕ satisfying $\sigma(\phi) = E$.*
To shorten this lengthy expression we shall also say that ϕ is an integer-NZF.

For an undirected graph we say that the graph has an integer flow if it has a integer flow for a some orientation of its edges. It is easy to see (really?) that if a graph G has a integer-NZF for some orientation then it has a integer-NZF for any orientation?

The existence of integer-NZF is a more restrictive property then the existence of a integer flow. In fact, for a given graph G, an integer-NZF need not exist. Can you find an example? Or, positively, can you find integer-NZF for some classes of graphs?

Question 8
i. Prove that any complete graph, has integer-NZF.
ii. Find integer-NZF for the Petersen graph.
iii. Prove that if $G_i = (V_i, E_i), i = 1, 2$ are oriented graphs (not necessarily disjoint!) and if $\phi_i, i = 1, 2$ is positive integer-NZF in the graph G_i then $\phi_1 + \phi_2$ is integer-NZF for the graph $G_1 \cup G_2$.

In fact, not every graph has an integer-NZF and the following theorem characterizes such graphs:

Theorem 4 *For an undirected graph G the following two statements are equivalent:*
i. G has integer-NZF (for an orientation, or equivalently for every its orientation);

ii. G does not contain a bridge. (A bridge *in a graph G is an edge e for which the graph* $G - e = (V(G), E(G) - \{e\})$ *has more components than the graph G. According to this definition a bridgeless graph need not be connected.)*

The condition *ii.* is obviously a necessary one, see Fig.3.

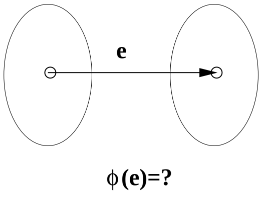


Figure 3: Bridge

The reverse direction is slightly more interesting.

Question 9
Try to prove this Theorem. You can see some results bellow (which of course generalize this simple result) but give different (and easy) proof!

Why is the notion of an integer-NZF important? Such a question is always difficult to answer but let us try:

3.2 4CC

The Four Color Conjecture (shortly 4CC; people like it so much that they call it a conjecture even long after it became theorem!) is one of the problems which shaped graph theory as we know it today. And the definition of integer-NZF allows us to put coloring of planar graphs into a new perspective.

In order to exhibit the (important) relationship of integer-NZF to colorings of planar graphs we have to introduce some further definitions:

We start with a planar graphs and we recall the definition of the *dual* of a plane graph:

Let $G = (V, E)$ be an (undirected) plane graph i.e. a planar graph with a fixed non crossing drawing in the plane. It is now important that we allow loops and multiplicities of both loops and edges. Thus in fact E is list of edges and loops (or a "set with repetitions").
Another way of looking at this is to say that the edges are given by a mapping (of *incidence*) $\epsilon : E \to \binom{V}{2} \cup V$.
This allows us to describe homomorphisms of multigraphs as follows:
A pair of mappings $f : V \to V'$, $g : E \to E'$ is a homomorphism of $G = (V, E, \epsilon)$ into $G' = (V', E', \epsilon')$ if the following diagram commutes:

$$\begin{tikzcd}
f : V \arrow[r] & V' \\
g : E \arrow[r] \arrow[d, "\epsilon"'] & E' \arrow[d, "\epsilon'"'] \\
f^{\#} : \binom{V}{2} \cup V \arrow[r] & \binom{V'}{2} \cup V'
\end{tikzcd}$$

Here $f^{\#} : \binom{V}{2} \cup V \to \binom{V'}{2} \cup V$ is the mapping defined by $f^{\#}(x, y) = \{f(x), f(y)\}$.

Denote $F(G)$ the set of *faces* of (the given drawing) of G and let $f : F(G) \to \{1, ..., k\}$ be an arbitrary mapping. This is the subject of 4CC: we want to find such a mapping f where the neighboring faces get different value (i.e. a *color*) while the value k is as small as possible. You know that this is the same thing as to ask for chromatic number of the graph G^* dual to the graph G. However here we shall need an oriented version of this concept:

Given an orientation G_o of a plane graph G we define the *oriented dual graph* $G_o^* = (V^*, E^*)$ as follows: $V^* = F(G)$, i.e. we assign to each face f of G a vertex f^* of the dual graph G_o^*, and corresponding to each arc e of G_o there is an arc e^* of G_o^*. Two vertices f^* and g^* are joined by the arc e^* if and only if the corresponding faces are separated by the arc e in G and the arc e^* is oriented so that if we go from the initial vertex of e to the terminal vertex then we cross the arc e^* so that its orientation is obtained by a clockwise rotation from the arc e. If $G = (V, E, \epsilon)$ then this rule defines $G^* = (V^*, E^*, \epsilon^*)$. Note that there is a one to one correspondence between E and E^* (and we could even put $E = E^*$).
This is in perhaps too many words describing a simple construction. See Fig.4

One can easily see that oriented dual graph G_o^* is again planar and if we use its drawing resulting from its definition the also $(G_o^*)^*$ is isomorphic to G_{-o} (the oriented graph obtained by reversing the orientation of all the arcs of G_o). So this is *oriented planar graph duality.* But this is not all and this duality extends to flows and potentials, or tensions as we shall now

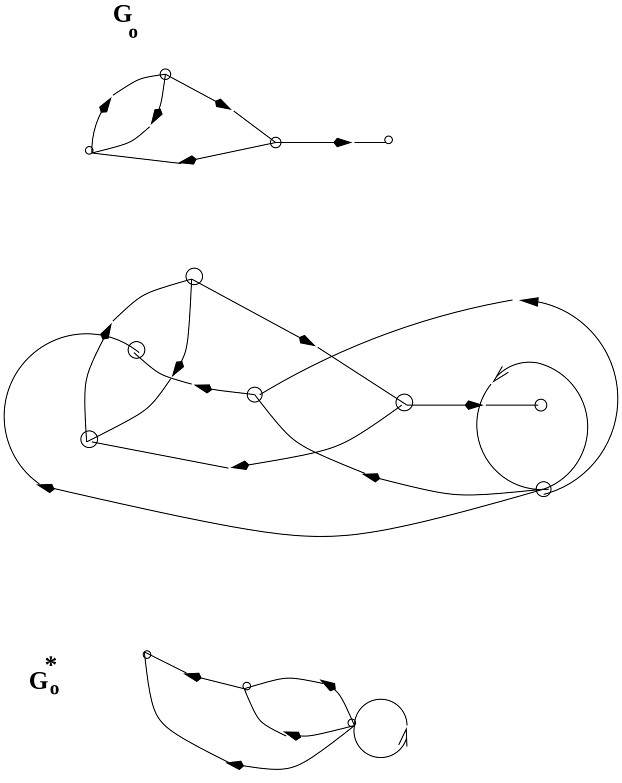


Figure 4: Dual graph

demonstrate.
Let again $\phi : F(G) \to \{1,...,k\}$ be an arbitrary mapping. Define a mapping $\partial\phi : E_o^* \to \{0, \pm 1, ..., \pm k-1\}$ by $\partial\phi(x,y) = \phi(y) - \phi(x)$, for $(x,y) \in E_o^*$. By duality the mapping $\partial\phi$ may be considered as a function defined in E_o (or formally $\partial\phi(e) = \partial\phi(e^*)$). In other words: as the edges E_o^* and E_o are in one to one correspondence we can define by the same formula mapping $\partial\phi : E_o \to \{0, \pm 1, ..., \pm k-1\}$.

We have then the following:

Theorem 5 *For any plane graph G the following statements are equivalent:*

i. The faces of the plane graph G are k-colorable;

ii. G^* is k-colorable;

iii. There exits an integer-NZF of G which uses values $\pm 1, \pm 2, ..., \pm(k-1)$

Proof. The equivalence of *i.* and *ii.* follows from the definition. Assume *ii.* Let G_o^* and G_o be coherent orientations of the plane graph and its dual. Let ϕ be a coloring of G_o^*. It means that the colors are numbers in $\{1,...,k\}$. To prove *iii.* it suffices to consider an arbitrary vertex x of G and to prove that $\sum_{x \in e \in E} \partial(\phi(e)) = 0$. However $\sum_{x \in e \in E} \partial(\phi(e)) = \sum_{x \in (x,y) \in E}(\phi(y) - \phi(x)) + \sum_{x \in (y,x) \in E}(\phi(x) - \phi(y))$.

Let $f_1, ..., f_t$ be all the faces which have x on their boundary listed in the clockwise order. By duality the previous sum is $\sum_{(f_i, f_{i+1}) \in E(G_o^*)}(\phi(f_{i+1}) - \phi(f_i)) + \sum_{(f_{i+1}, f_i) \in E(G_o^*)}(\phi(f_i) - \phi(f_{i+1})) = 0$. Thus ϕ is a k-NZF.

Conversely let $\phi : E \to \mathbb{Z}$ be a NZF. Then fix a face f_o and define $\theta(f_o) = 0$. Extend the mapping θ to all faces f as follows: if $\theta(f)$ is not yet defined, $\theta(f')$ is defined and $(f, f') \in E^*$ then we put $\theta(f) = \theta(f') - \phi(e)$ where e is the edge of G corresponding to (f, f'); similarily if $\theta(f)$ is not yet defined, $\theta(f')$ is defined and $(f', f) \in E^*$ then we put $\theta(f) = \theta(f') + \phi(e)$ where e is the edge of G corresponding to (f', f). This definition is correct (of course this is where we use that ϕ is a flow) and as ϕ was NZF we get a coloring of G^*. Of course some of the colors may be negative numbers. This may be corrected by starting not with $\theta(f_o) = 0$ but to choose a sufficiently large positive integer. □

Question 10

Why is the mapping θ correctly defined?

This theorem throws a new light on the definition of integer-NZF. From now on it will matter which values we shall use for NZF.
Thus define *k-nowhere zero flow*, shortly k-NZF, as an integer NZF which uses only values $\pm 1, \pm 2, ..., \pm(k-1)$.

Using this we have then the following

Corollary 1 *Every planar graph is 4 colorable iff every planar graph has a* 4*-NZF.*

This is part of duality between coloring and flows where we have 1-1 correspondence. But this is not restricting us (in the same way as in the case of duality of Linear Programming) to use this concepts and problems *for graphs in general.* And this leads (as we want to demonstrate) to a very fruitfull area of contemporary combinatorics.

One more question: Why do we care about another formulations of 4CC at the time when the problem is solved?

One reason may be in the fact that the known proofs of 4CC are still not very satisfactory, see [2, 46, 54, 28]. But another reason (and we think that more important) is that there is such a vast relevance of this problem that simply any new (let us say essential new) reformulation is welcome and studied intensively. New equivalences are a rare article, see [25] and [10] for recent spectacular additions to the list.

3.3 Complexity

This is a more philosophical aspect (or psychological aspect; believe it or not this plays in active mathematics and science in general an important rôle). By complexity we mean here more like another way to express difficulty or hardness, more then (computational) complexity.

Coloring problems are mostly complicated questions and as decisions problems (and even approximate problems) one has mostly only negative results.

On the other side flows problems are well understood and they are mostly tractable. Many problems related to flows (it seems indeed that most of such problems) have polynomial solution. Not only that the theory of *network flows* is particularly elegant, it is one of the very few combinarorial instances of a *mathematical theory* which you know from other more abstract fields. (Remark: This is a bit curious situation. For directed graphs everything seems to be more complicated than for ordinary undirected graphs. One of the few exemptions are network flows.)

So if we reduce something to flows we hope (perhaps naively as we may discover later) that the problem will be more tractable. Well we shall see...

So these two aspects ($4CC$ and Complexity) increased the curiosity of researchers and the problem of the existence NZF took life of its own. One started to study dual problems without its relevance to the primal problem, as indeed for most instances of the dual problem the primal problem does not exist.

But then suddenly we discover curious facts:

Question 11

i. Every complete graphs of odd order has 2-NZF.
ii. And then every eulerian graph has 2-NZF.
Can you prove it?

This seems to be contradicting the chromatic number connection of flows which we described above for planar graphs. So we start to search and a beginning of theory emerges. This will be explained in the next section.

4 Group valued Flows

We shall proceede now more abstractly and we define flows with values in an abelian group. In this way we put some of the above results in a new context. As we shall see later this abstraction is a usefull one.

4.1 Definitions

Let $G = (V, E)$ be a digraph, if $S \subset V$, we recall that $\omega^+(S)$ is the set of the edges which begin in S and terminate not in S and that $\omega^-(S) = \omega^+(V \setminus S)$. Let M be an abelian group (with additive notation). An *M-flow* is a mapping ϕ from E to M which satisfies:

$$\sum_{e \in \omega^+(S)} \phi(e) - \sum_{e \in \omega^-(S)} \phi(e) = 0, \tag{2}$$

for every subset $S \subset V$.

Question 12
Prove that it is sufficient to verify the equation 2 on each vertex of G.

We will say that the graph G has an *M-flow*.
Now if $B \subset M$, an *B-flow* of a graph G is a flow of G where the values are taken in B [22].
Thus a M-NZF is defined as a B-flow for $B = M \setminus \{0\}$.

When $M = \mathbb{Z}$ and $B \subseteq [1-k, -1] \cup [1, k-1]$ ($k \geq 2$), we will speak about k-NZF. This coincide with our definition given in the section 3.1.
The *support* $\sigma(\phi)$ of a flow ϕ in a digraph $G = (V, E)$ is the set of arcs e such that $\phi(e) \neq 0$.
For an undirected graph we say that the graph has a M-flow if it has a M-flow for a some orientation of its edges. It is easy to see that if a graph G has a B-flow for some orientation then it has a B-flow for any orientation. Thus again the existence of NZF is the property of underlying undirected graphs G.

4.2 Tensions

Let μ be a circuit in a digraph G, we denote by μ^+ the set of arcs with one direction and μ^- the set of arcs with the opposite direction (we can think that we walk around the circuit, then μ^+ are the arcs that we traverse from the initial vertex to the terminal vertex and μ^- are the arcs that we traverse from the terminal vertex to the initial vertex). With the same context as in the paragraph 3.2, a *M-tension* is a mapping θ from E to M such that :

$$\text{for every circuit } \mu : \sum_{e \in \mu^+} \theta(e) - \sum_{e \in \mu^-} \theta(e) = 0. \tag{3}$$

A tension will be called *nowhere-zero* if it never takes the value zero.

4.3 Tensions and Colorings

A *M-potential* of G is a mapping π from V to M. The mapping $\partial\pi$ (*potential difference*) of $E \to M$ defined by: for every edge $e = (v, v') \in E$, $\partial\pi(e) = \pi(v') - \pi(v)$ is a M-tension of G, said derived from π. This is related to Theorem 5.

Question 13
Prove that any M-tension of G is derived from a potential of G.

We identify good colorings of the vertices of G, where the colors are taken in M with the *M-potentials* of G. It follows that the corresponding M-tension is nowhere-zero. This is a basic observation linking flows and colorings and this leads to:

Proposition 1
Let G be a graph and let M be an Abelian group of order $k \geq 2$. $\chi(G) \leq k$ if and only it admits a nowhere-zero M-tension.

Question 14
Prove it.

4.4 Flows and face colorings of planar graphs

Consider a plane representation of an oriented planar graph $G = (V, E)$ and $F(G)$ its faces. Let M be an abelian group. A *M-potential* on the faces of G is a mapping π fom $F(G)$ to M. Let $\partial\pi$ be the mapping from E to M such that for every $e \in E$: $\partial\pi(e) = \pi(f_r) - \pi(f_l)$, where f_r (resp. f_l) is the face to the right (resp. left) of the arc e.

Question 15
Prove that $\partial\pi$ is a M-flow of G (said to be derived from π).

If we apply these concepts to plane graphs and their duals we get:

Proposition 2
By duality we have a bijective mapping between:
i. The M-potentials of the faces of G and the M-potentials of G^*.
ii. The M-flows of G and the M tensions of G^*.

Question 16

Prove it.

Proposition 3
Let be a planar representation of a planar graph G and M be an Abelian group of order $k \geq 2$. The faces can be good colored with k colors if and only if G has a M-NZF.

Question 17
Prove it.

We have to notice that a necessary condition in order that G satisfies the two equivalent properties of the Proposition 1 (resp. Proposition 3) is that it does not have loops (resp. bridge).

These two propositions are in a paper of Tutte [58] where the author defines for the first time the dichromatic polynomial of a graph and shows that it allows to compute in the same time the number of nowhere-zero M-tensions and the number of nowhere-zero M-flows of the graph (for any Abelian group). In both case the number does not depend on the structure of the group M, but only on its order. This is the subject of the next section.

4.5 Equivalence Theorem

Here we prove a surprising and (as we shall later see usefull) theorem that the existence of M-NZF for a graph G does not depend on the structure of the group but only on its order $|M|$.

Theorem 6 *Let G be a digraph. For every $k \geq 2$, the following conditions are equivalent:*

1. *There exists a $\mathbb{Z}_k$-NZF in G.*
2. *For any Abelian group M of order k, there exits a M-NZF in G.*
3. *There exists a k-NZF in G.*

Combining this with Proposition 3 we get:

Corollary 2 *A planar graph is face-k-colourable if and only if it has a nowhere-zero* $\mathbb{Z}_k$*-flow.*

The proof of Theorem 6 will be given at the end of this section. Its proof is based on the construction of a polynomial, called *flow polynomial* associated to every graph.
Do you remember what is the chromatic polynomial of a graph? (If you don't then see the lecture notes by Nešetřil and Sopena on coloring [42]; flow polynomial is just another of the relatives, they all live under the same roof called *Tutte-Grothendieck polynomial.*)
Let G be a graph and let M be an abelian group of order k. For a given orientation D of G, let $F_M(G;k)$ be the number of M-NZF of G under the orientaton D. As we have seen previously the number of M-NZF of G does not depend of the orientation of G. The numbers $F_M(G;k)$ have the following important properties which we state as [58]:

Proposition 4 *(deletion - contraction procedure)*
i. $F_M(G;k) = F_M(G/e;k) - F_M(G \setminus e;k)$ *if* $e \in E$ *is not a loop;*
ii. $F_M(G;k) = (k-1)F_M(G \setminus e;k)$ *if* $e \in E$ *is a loop;*
iii. $F_M(G;k) = 0$ *if* $G = K_2$*;*
iv. $F_M(G;k) = (k-1)$ *if* G *is a loop.*

That is easy to state especially when we didn't define all the symbols. However we defined them in one of the duality examples in the introduction: G/e is the result of contraction of the edge e in a graph G (sometime this is denoted also by $G.e$) and $G \setminus e$ is just G - e. Because we are considering multigraphs and loops so perhaps we should give a formal definition for this operation:

Let $G = (V,E)$ be a graph specified by incidence mapping $\epsilon : E \to \binom{V}{2} \cup V$. Then the graph $G/e = (W,F)$, $\epsilon(e) = \{x,y\}$, is defined as follows: $W = (V \setminus \{x,y\}) \cup \{z\}$, where $z \notin V$, $E = F \setminus \{e\}$ and the incidence mapping ϵ' is defined by:

i. $\epsilon'(e') = \epsilon(e')$ for every $e \neq e' \in E$ such that $\epsilon(e) \cap \epsilon(e') = \emptyset$.

ii. If $e \neq e'$ and $\epsilon(e') = \{x,t\}$ (resp. $\{y,t\}$) then $\epsilon'(e') = \{x,z\}$ (resp. $\{y,z\}$).

iii. If $e \neq e'$ and $\epsilon(e') = \epsilon(e)$ then $\epsilon(e') = \{z\}$.

See schematic Fig.5.

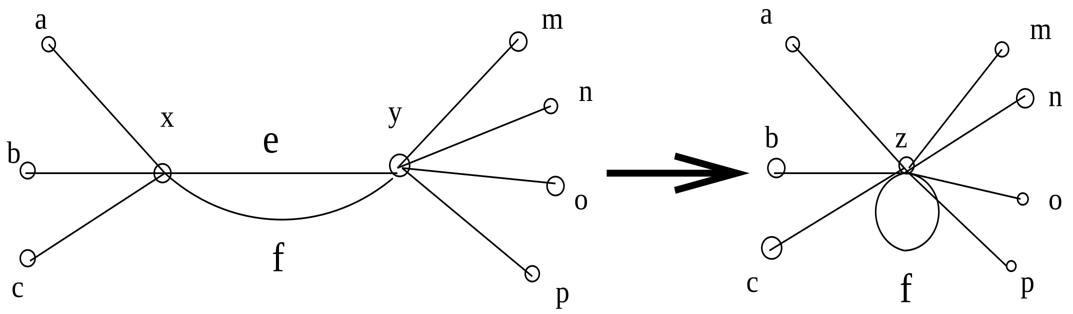


Figure 5: Contraction

Question 18

i. What is $F_M(K_3; k)$?
ii. What is $F_M(G, k)$ if G is a disconnected graph?

Proof of Deletion-Contraction Proposition: If $e \in E$ is not a loop, then $F_M(G/e; k)$ is the number of flows G nowhere-zero, except for e where it can be zero. $F_M(G \setminus e; k)$ is the number of fows of G which are zero only at e. Check the last three properties by yourself. $\square$

A typical consequence of edge deletion-contraction procedure is the following which justifies the name given to the numbers $F_M(G; k)$: they are all evaluation of one polynomial, called *flow polynomial.*

Theorem 7 *For every graph G there exists a real polynomial $F(G; x)$ with integer coefficients such that the number of M-NZF is equal to $F(G; k)$, for any abelian group $M, |M| = k$.*

Proof. We proceed by double induction on number of vertices of G and for a fixed number of vertices on the number of edges. The complementary informations in *ii.*, *iii.* and *iv.* serves to cover the boundary cases: One vertex graph (with arbitrary many loops) and K_2. $\square$

Question 19
i. Prove the previous Theorem in a greater detail.
ii. Determine the flow polynomial of a tree

We give now a Proposition which will be used to have an explicite expansion of the flow polynomial:

Proposition 5 *Let $G = (V, E)$ be a connected oriented graph and $T = (V, E')$ be a spanning tree of G. Let M be an additive group and f be any mapping $E \setminus E' \to M$. Then there exists exactly one M-flow $\phi : E \to M$ such that $\phi(e) = f(e)$ for every edge $e \in E \setminus E'$.*

The proof is a variant of the matroid argument about rank of the cycle space of a graph:
To each $e \in E \setminus E'$ there exits a cicuit C_e such that $(E(C_e) \setminus \{e\}) \subseteq E'$ and $e \in E(C_e)$. Let ϕ_e be a M-flow with support $E(C_e)$ such that $\phi_e(e) = f(e)$. Define the flow ϕ on E as the sum of the flows ϕ_e. The flow satisfies the requirements of the proposition and it is the unique flow satisfying $\phi(e) = f(e)$ for every edge $e \in E \setminus E'$.

$\square$

The explicite expansion of the flow polynomial is:

Theorem 8 *Let M be a finite additive group of order k, and let $G = (V, E)$ be a directed graph. The number of M-NZF of G is :*

$$F_M(G; k) = \sum_{F \subseteq E} (-1)^{|E-F|} k^{|F|-r(F)}.$$

If $F \subseteq E$, we denote by $r(F)$ the maximum number of edges in a forest of G contained in F.

Proof. It follows from the Proposotion 5 that $k^{|F|-r(F)}$ is the number of the M-flows of the subgraph (V, F) of G. We are interested in nowhere-zero flows, i.e. in flows which have support $E(G)$. As $k^{|F|-r(F)}$ is the number of M-flows with support contained in the set F, we get the formula by the inclusion-exclusion principle. □

Now we are in position to prove Theorem 6.

Proof. We know that conditions 1. and 2. are equivalent as both these statements claim that $F_M(G;k) \neq 0$. The implication $3 \longrightarrow 1$ is easy and we can leave it as an exercise. Let us prove the converse, this is less easy.

Suppose that ϕ is a $\mathbb{Z}_k$-NZF for G. Consider the set of all valuations $f : E \to \mathbb{Z}$ which satisfy:

i. $f(e) \in [-k-1, -1] \cup [1, k]$ for every $e \in E$;

ii.. $\bar{f}(e) = \phi(e)$ ($\bar{x}$ means here the modulo k residue of x).

Properties *i.* and *ii.* are called *modulo conditions.*

It is clear that for every valuation f with the modulo conditions holds:

$$\forall S \subset V : \sum_{e\in\omega^+(S)} f(e) - \sum_{e\in\omega^-(S)} f(e) = 0 \bmod k. \tag{4}$$

Define $D(f) = \sum_{v\in V} | \sum_{e\in\omega^+(v)} f(e) - \sum_{e\in\omega^-(v)} f(e)|$, and choose one f satisfying the modulo condition and with $D(f)$ minimum. If $D(f) = 0$ then we are done, f is k-NZF.

Assume that $D(f) \neq 0$. If we reverse the direction of an arc e and replace $f(e)$ by $-f(e)$, then the new function f satisfies the modulo condition for the new flow ϕ obtained by the same way, but working in $\mathbb{Z}_k$. So we can assume that all the values of f are positive. Let $S = \{v \in V : \sum_{e\in\omega^+(v)} f(e) > \sum_{e\in\omega^-(v)} f(e)\}$ and $T = \{v \in V : \sum_{e\in\omega^+(v)} f(e) < \sum_{e\in\omega^-(v)} f(e)\}$.

We claim that there is no direct path from S to T. If such path P exists then take the new valuation f' such that $f'(e) = f(e) - k$ if $e \in P$ and $f'(e) = f(e)$ if e is not belonging to P. f' satisfies the modulo condition and we have: $D(f') = D(f) - 2k$, which contradicts then minimality hypothesis. Now let a partition $\{A, B\}$ of V such that $S \subseteq A$ and $T \subseteq B$ with $\omega(A)^+ = \emptyset$. Such partition exists because of the previous claim. We denote by $\omega^0(A)$ the set of edges with both edges in A. Then we have :

$\sum_{v\in A}(\sum_{e\in\omega^+(v)} f(e) - \sum_{e\in\omega^-(v)} f(e)) = \sum_{e\in\omega^0(A)} f(e) + \sum_{e\in\omega^+(A)} f(e) - \sum_{e\in\omega^0(A)} f(e) - \sum_{e\in\omega^-(A)} f(e)$. Summing up we get $-\sum_{e\in\omega^-(A)} f(e) \leq 0$. Since $\sum_{e\in\omega^+(v)} f(e) - \sum_{e\in\omega^-(v)} f(e) \geq 0$ for all $v \in A$, it follows that $S = \emptyset$. The same holds for T. Hence f is a k-NZF. □

4.6 Elementary Results

The *Theory of Nowhere Zero Flows* which we are beginning to build is very elegant. The informed student of mathematics may have seen something similar in a course of Homological Algebra (or some related lecture which was covering elements of this theory). This is not an accident as flows on graphs, potentials, tensions are beginning of that theory (a graph is 1-dimensional complex). However in this combinatorial context the theory of NZF has little to do with very general approaches of algebraic topology. But these connections certainly influenced research on both sides (for example in the theory of invariants of knots [26]; compare also [27, 30]).

However what makes the theory of flows and tensions also interesting is that this theory unified and explained connections and gave easy proofs of several results which were regarded as isolated and peculiar. Let us give some examples:

Proposition 6 *If* Φ *is a* $\mathbb{Z}_2$*-flow of a graph* $G = (V, E)$ *then the support of* Φ*, is an edge-disjoint union of cycles.*

Proof. By 2, the number of edges with non zero edge-value incident to a given vertex is even, then the support of the $\mathbb{Z}_2$-flow defines an Eulerian graph. This completes the proof. $\square$

Observe that this has a corollary:

Proposition 7 *The faces of a planar representation of a planar graph can be good colored in two colors if and only if all its vertex-degrees are even.*

Question 20

Prove the **Proposition 7**.

Proposition 8
A $G = (V, E)$ *cubic graph has a* $\mathbb{Z}_3$*-NZF if and only if it is a bipartite graph.*

Proof. Let V_1 and V_2 a good vertex-coloring of G. Orient all edges of G from V_1 to V_2 and assign value 1 to all of them. Observe that this is a $\mathbb{Z}_3$-NZF. Conversely if f is a $\mathbb{Z}_3$-NZF then find such an orientation G_o of G so that the value on any arc of G_o is 1. Then any vertex of G is either source or sink and thus this is a desired 2 coloring of the vertices of G. $\square$

When translated to planar graphs this give:

Proposition 9 *A planar triangulated graph is* 3*-chromatic if and only if its faces can be good colored with two colors.*

Question 21

Prove the **Proposition 9**.

Thus we find examples of graphs with 3-NZF. What about a 4-NZF? Let us try to give some examples. You probably heard of this one:

Proposition 10 *A cubic simple planar graph has a* 3*-edge coloring if and only if its faces can be good colored with* 4 *colors.*

(A graph G is said k-edge colorable if we can color the edges of G with k colors such that two incident edges have different colors.)

This is historical Tait's theorem (from 1880)[53] which was isolated in order to give one of the first proofs of 4CC. It also led to study of *hamiltonian graphs* and to the *Petersen graph.* This text wouldn't be complete without its picture.

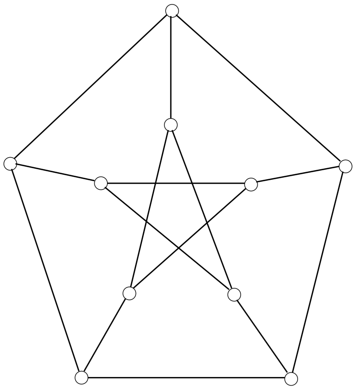


Figure 6: Petersen Graph

The same argument proving proposition 10 can be generalized to get :

Proposition 11
A cubic graph G has a 4-NZF if and only if it is 3-edge colorable.

Proof. By Theorem 6 a graph has a 4-NZf if and only if it has $\mathbb{Z}_2 \times \mathbb{Z}_2$-NZF. Let the non-zero elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ be a, b, c. We have $a + b + c = 0$ and $a + a = b + b = c + c = 0$. From this it is easy to see that a mapping $f : E(G) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-NZF iff it is a 3-edge coloring using the colors a, b, c. □
The Petersen graph has no 4-NZF.

Let us end this section by the following characterisation of graphs with 4-NZF:

Proposition 12 *A graph $G = (V, E)$ has a 4-NZF if and only if $E = E_1 \cup E_2$ and each of the graphs (V, E_i), $i \in [1, 2]$, has all vertices of even degree.*

Proof. By Theorem 6 has a 4-NZF if and only if it has $\mathbb{Z}_2 \times \mathbb{Z}_2$-NZF ϕ. Write $\phi = (\phi_1, \phi_2)$ and observe that the ϕ_i's are $\mathbb{Z}_2$-flows which are $\mathbb{Z}_2$-NZF on the support E_i of ϕ_i. However according to Proposition 6 this happens if and only if E_i has all its degrees even. Moreover ϕ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-NZF if and only if $E = E_1 \cup E_2$. □

5 Art of Conjecturing and Proving

5.1 Tutte's conjectures

With all the information which we so far gathered the situation looks inverted (to chromatic number situation): In the dual situation to colorings we are facing problem of large flows. Are the large flows necessary at all? We have seen that graphs with the 4-NZF exists. Can we get 5-NZF graph?

One has to be carefull here for the following holds:

Proposition 13
i. If G has $\mathbb{Z}_k$-NZF then it has also $\mathbb{Z}'_k$-NZF for any $k' \geq k$.
ii. If G has k-NZF then it has also k'-NZF for any $k' \geq k$.

Proof. *ii.* follows from definition, *i.* follows from *ii.* if we use the Equivalence Theorem 6. □

Thus can we get a graph G without 4-NZF? We know the answer already: The Petersen graph. Petersen graph has a 5-NZF and this is indicated on Fig. 7

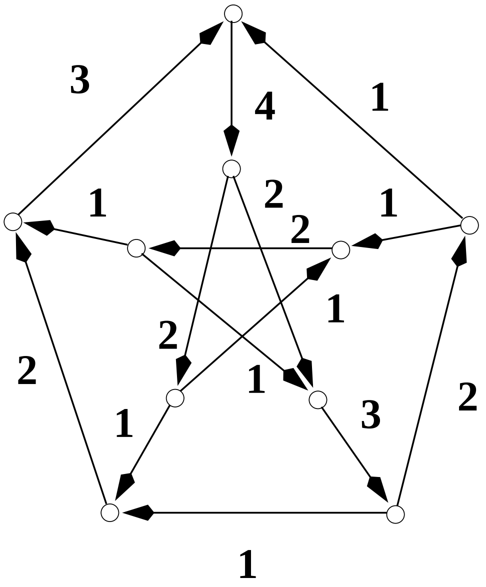


Figure 7: 5-NZF

Now we can try to go further. Any more examples? Tutte did this in the fifties and made (in several papers [57, 58, 59, 60]) the following beautiful conjectures:

Conjecture 1 *Every bridgeless graph has a* 5*-NZF.*

We can notice that the smallest counterexample for this conjecture is a 3-edge connected graph. Assume the contrary, let G be a graph with a 2-cut $C = \{e_1, e_2\}$, by contracting the edge e_1 the obtained graph G' has a 5-NZF f for some orientation, we have $f(e_2) \neq 0$. We extend the 5-NZF f of G' to a 5-NZF f' of G such that $f'(e) = f(e)$ if e is an edge of G' and $f'(e_1) = \pm f(e_2)$ following the direction we choose for e_1.

Conjecture 2 *Every bridgeless graph without* 3*-cut has a* 3*-NZF.*

This last conjecture is true for planar graphs, it is a consequence of what you proved in Question 17 and of a famous theorem of Grötzsch [16] asserting that every loopless planar graph without triangles has a chromatic number equal to 3.
An other fascinating conjecture is the following:

Conjecture 3 *Every bridgeless graph containing no Petersen minor admits a* 4*-NZF.*

A graph H is a minor of the graph G if H can be obtained from a subgraph of G by contracting some edges of the subgraph and deleting the possible resulting loops.
The last year Robertson, Seymour, Thomas presented the proof of Conjecture 3 for cubic graphs. This is certainly the main achievement related to Conjecture 3 so far, see references [47, 48]. One could remark that this implies the $4CC$.

All these conjectures (known as *Tutte 3-,4-,5- flow conjectures*) are presently open despite all the efforts which were so far made. However a significant progress has been recently made and we hope to review this in this text.

One should stress that these were and still are very brave conjectures. Tutte experience came from $4CC$. He was the first to construct the first 3-connected cubic planar graph without a hamiltonian cycle thus finally after 70 years destroying Tait's approach to $4CC$. He also defined Tutte polynomial (both as early as 1947). But still his conjectures were very brave . At the time when he conjectured the existence of 5-flow for every graph it was not even known whether any graph has a flow bounded by some universal constant (compare our question 9). This had to wait another 15 years before Jaeger proved this (Kilpatrick proves it too independently [29]). We shall see this in the next section.

5.2 Jaeger's theorems

In this section we will prove the following theorems of Jaeger. The first one is close to the last Tutte's conjecture given in the previous paragraph, the second one is close to the first:

Theorem 9 *Every* 4*-edge connected graph has a* 4*-NZF*

Theorem 10 *Every bridgeless graph has a* 8*-NZF*

(So Tutte was right, it is bounded!).
In order to prove Theorems 9 and 10, we have to deal with seemingly unrelated questions, edge-disjoint trees in graphs. We start with the following theorem of Tutte and Nash-Williams [61, 36], see also Welsh [62]:

Theorem 11 *A graph G has k pairwise edge-disjoint spanning trees if and only if the number of edges (counting parallel edges) in any contraction graph G_p of G satisfies the following condition:*

$$|E(G_p)| \geq k(|V(G_p)| - 1).$$

The original proof of the Theorem was a complicate ad hoc argument. By now the best approach is to prove a much more general Theorem on matroid bases.

The *rank function* of a matroid M defined on a set S is a function $r : 2^S \to \mathbb{Z}$ defined by :

$$r(A) = max(|X| : X \subseteq A, X \in \mathcal{F}).$$

The *rank of the matroid* M is the rank of the set S.

Question 22
Prove that if M^* is the dual matroid of a matroid M on S then the rank of M^* is $|S| - r(M)$.

The rank function r of M and the rank function r^* of M^* are linked by the following relation:

Question 23
Prove the following equation:
for $A \subseteq S$:

$$r^*(S \setminus A) = |S| - r(S) - |A| + r(A).$$

Edmonds [11] proved this theorem:

Theorem 12 *A matroid M on S has k disjoint bases if and only if the following inequality holds for each subset A of S.*

$$kr(A) + |S \setminus A| \geq kr(S). \tag{5}$$

We shall not prove Theorem 12 here, see for proofs [31, 49, 62].
Let $V(G)$, $E(G)$ denote respectively the vertex and the edges sets of a graph G. For a partition P of the vertex set $V(G)$ let $E(P; G)$ denote the number of edges of G which join two distinct classes of the partition P.
We will prove in fact the following form of the theorem of Nash-Williams :

Theorem 13 *A simple connected graph G has k edge disjoint spanning trees if and only if for any partition P of $V(G)$*

$$E(P; G) \geq k(|P| - 1) \tag{6}$$

Question 24
Prove that Theorem 11 and 13 are equivalent.

Proof. First we prove the necessity of 6. Suppose there exist edge-disjoint spanning trees $T_1, \dots T_k$. Then if $E(P; T_i)$ denotes the number of edges of T_i which join distinct classes of P

$$E(P; T_i) \geq |P| - 1.$$

Hence

$$E(P; G) \geq \sum_{i=1}^{k} E(P; T_i) \geq k(|P| - 1).$$

Conversely suppose that (6) holds, then it is clearly sufficient to show that the cycle matroid $M(G)$ has k pairwise disjoint bases. By (5) this so if and only if for all $A \subseteq E = E(G)$

$$kr(A) + |E \setminus A| \geq kr(E) \tag{7}$$

where r is the rank function of $M(G)$.
Suppose that it exists $A \subseteq E(G)$ with

$$|E \setminus A| < k(r(E) - r(A)).$$

Let $k(A)$ be the number of connected components in the subgraph $G|A$ ($G|A$ denotes the subgraph generated by A). Let $D_1, D_2, \ldots, D_{k(A)}$ denote the vertex sets of these connected components.
If there are t vertices $v_1, v_2, \ldots, v_t$ which are not incident with any edge of A take D to be the partition

$$D = (D_1, D_2, \ldots, D_{k(A)}, \{v_1\}, \ldots \{v_t\})$$

so that

$$|D| = k(A) + t$$

$$E(D; G) = |E \setminus A|.$$

Then since $r(E) = |V(G)| - 1$, $r(A) = |V(G)| - t - k(A)$,
we have

$$E(D; G) < k(|V(G|) - 1 - |V(G)| + t + k(A))$$

$$E(D; G) < k(|D| - 1)$$

which contradicts the hypothesis. This completes the proof. $\square$

The proofs of theorems 9 and 10 uses the following result [32]:

Theorem 14 *For any $k \geq 1$ every $2k$-edge connected graph contains k edge-disjoint spanning trees.*

Proof. Assume that t is the maximum number of edge-disjoint spanning trees in a $2k$-edge connected graph G. Then by the Theorem 11 it means that there exists a contraction G_p such that it has m vertices and at most $(t+1)(m-1)-1$ edges. Hence there is a vertex of G_p with a degree less than $2(t+1)$. The graph G_p is also $2k$-edge connected, it implies that the minimum degree $\delta(G_p)$ is at least $2k$. We have : $2k \leq \delta(G_p) < 2(t+1)$, which inplies that $t \geq k$. $\square$

The result which we state now is more general that needed for the proof of Theorem 9:

Theorem 15 *If a graph $G = (V, E)$ contains k ($k \geq 2$) edge-disjoint spanning trees then it has a $\mathbb{Z}_2^k$-NZF.*

Proof. Let $T_1, T_2, \ldots, T_k$ be the k edge-disjoint spanning trees and $\bar{T_i} = G \setminus T_i$, $i \in 1, 2, ..., k]$. For each T_i ($i \in 1, 2, .., k$), by proposition 2, we know that G has Z_2-flow ϕ_i sucht that $\bar{T_i} \subset \sigma(\phi_i)$. We have $\bigcup_{i=1}^{i=k} \bar{T_i} = E$. For every $e \in E$ we define the flow Φ such that $\Phi(e) = (\phi_1(e), \phi_2(e), \ldots, \phi_p(e))$. It is easy to see that Φ is a $\mathbb{Z}_2^k$-NZF of G. $\square$

Proof of theorem 9. By Theorem 14 the graph G contains two edge-disjoint spanning trees. Then by Theorem 15 we have the result. □

Proof of theorem 10. We proceed by induction on $|E(G)|$. By using the remark following the 5-NZF conjecture it is enough to prove it for 3-edge connected graphs . Then by replacing each edge of G by two parallel edges, we obtain a new graph G' that is 6-edge connected. Now let T_1, T_2, T_3 be the three edge-disjoint spanning trees of G' given by Theorem 14. These spanning trees of G' may be considered as spanning trees of G. In G they are no more edge-disjoint, but $E(T_1) \cap E(T_2) \cap E(T_3) = \emptyset$. Let $\bar{T_i} = G \setminus T_i$. For each T_i ($i \in 1, 2, 3$), we know that G has Z_2-flow ϕ_i sucht that $\bar{T_i} \subset \sigma(\phi_i)$. $\bigcup_{i=1}^{i=3} \bar{T_i} = E$, then (ϕ_1, ϕ_2, ϕ_3) is a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-flow. By theorem 6 we have the result. □

Jaeger's theorem gives already a small number, considering that we cannot go under 5. But remember what an effort was necessary to get in coloring of planar graphs from 5 to 4! Still there is an improvement here:

5.3 Seymour's theorem

In 1981 Seymour prove the best answer concerning the 5-flow's conjecture of Tutte:

Theorem 16 *Every bridgeless graph has a* 6*-NZF*

To prove the the theorem, first, we construct a family of undirected graphs $\mathcal{C}$ obtained recursively from an isolated vertex by applying a finite number of constructions of the form $R1$, $R2$ and $R3$:

$R1$: add an isolated vertex to G;
$R2$: add an edge within one connected component of G;
$R3$: add two edges joining two distinct components of G.

Now we have:

Theorem 17 *Let $G = (V, E)$ be a graph of $\mathcal{C}$, considered with an arbitrary orientation. For each mapping c from E to $\mathbb{Z}_3$, there exists a $\mathbb{Z}_3$-flow ϕ of G such that $\phi(e) \neq c(e)$ for each $e \in E$.*

Proof. We proceed by an induction on the number of edges. If $|E| = 0$, there is nothing to prove. Suppose that $G' = (V', E')$ is constructed from G using construction $R2$ or $R3$, and let c' be a mapping from E' to $\mathbb{Z}_3$. Let μ be a $\mathbb{Z}_3$-flow of G' whose support is a cycle containing $E' \setminus E$. Since $|E' \setminus E| \leq 2$, we may use μ to obtain a $\mathbb{Z}_3$-flow μ' of G' such that $\mu'(e) \neq c'(e)$ for each e in $E' \setminus E$ (μ' is equal to μ, $-\mu$ or the zero flow). By the induction hypothesis, there exist a $\mathbb{Z}_3$-flow ϕ of G such that $\phi(e) \neq c'(e) - \mu'(e)$, for each e in E. Then $\phi' = \phi + \mu'$ is a $\mathbb{Z}_3$-flow of G' such that $\phi'(e) \neq c'(e)$, for each e in E'. □

We can remark that every graph of $\mathcal{C}$ has a $\mathbb{Z}_3$-NZF.

Proof of Theorem 16 (sketch):
First one has to realize that the theorem is enough to prove for simple 3-connected graph G. By the remark after Tutte's 5-flow conjecture, we can assume that the graph G is 3-edge connected. Moreover we can also assume that G is cubic. For if some vertex x has degree greater than 3 then there are edges $\{y_1, x\} = e_1$ and $\{y_2, x\} = e_2$ such that $(G - \{e_1, e_2\}) \cup \{y_1, y_2\} = G'$ is a graph with less edges and it is also bridgeless. So we could proceed by induction on $|E(G)|$.

Next we show that in G there are subgraphs $C_1, ..., C_r$ such that if we contract each of the C_i in one point we get a graph belonging to $\mathcal{C}$. Here each of C_i is either a cycle or a vertex. To see that let $C_1, ..., C_r$ be such a set of C_i such that the subgraph which we get on the set $\cup V(C_i)$ satisfies the property (belongs to $\mathcal{C}$) and has largest number of vertices.

If $W = \cup V(C_i) \neq V(G)$. Consider the block decomposition of $G - W$ and let B be an end block of $G - W$, by connectivity argument, then either there exists a vertex x which is joined by two edges to W or B is a cycle which contains two vertices joined to W. In any case we get a larger subgraph with the above property, which contradicts the maximality hypothesis of $\cup V(C_i)$.

Thus let H be the graph which we get by contracting $C_1, ..., C_r$. By the previous theorem H has a $\mathbb{Z}_3$-NZF, which can be extended to a $\mathbb{Z}_3$-flow ϕ_3 of G with $E \setminus \cup_{i=1}^{r} C_i \subseteq \sigma(\phi_3)$. Consider now a $\mathbb{Z}_2$-flow ϕ_2 with $\sigma(\phi_2) = \cup_{i=1}^{r} C_i$. Then (ϕ_2, ϕ_3) defines a $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF. □

5.4 Petersen Flow

We want to review here the following little known conjecture of Jaeger.
Let $C_1, \cdots, C_6$ be a basis of $\mathcal{C}(P)$, the cycle space of the Petersen graph $P = (V, E)$ (as $6 = 15 - (10 - 1)$, you surely recognize that). For $i \in 1, 2, ..., 6$, we denote by Φ_i the $\mathbb{Z}_2$-flows of P such that $\sigma(\Phi_i) = C_i$. It is clear that $\Phi = (\Phi_1, \cdots, \Phi_6)$ is a nowhere-zero $\mathbb{Z}_2^6$-flow of P. The set of the values of Φ is denoted by B. As for any two distinct edges there exists a cycle in the basis which contains exactly one of them (or as any basis in the Petersen graph is edge-distinguishing; compare Lemma 3 in the last section), we clearly have $|B| = 15$.
We shall say that a graph has a *Petersen flow* if it has a B-flow. It is equivalent to say that G has nowhere-zero $\mathbb{Z}_2^6$-flow taking its values in B.
We do not take care of the orientation here because if we consider flows over $\mathbb{Z}_2^k$ then obviously a $\mathbb{Z}_2^k$-flow for an orientation is a $\mathbb{Z}_2^k$-flow for any orientation of G.
It is easy to see that the property to have a Petersen flow is independent of the chosen basis. Indeed assume a graph has a Petersen flow constructed on a cycle basis then for any new basis each cycle of this new basis is a linear combination of the cycles of the previous basis. Hence if we take the same linear combination of the supports of the $\mathbb{Z}_2$-flows defined by the cycles of the first basis then we have a Petersen flow constructed on the second basis.
In [22] Jaeger conjectured the following property:

Conjecture 4 *Every bridgeless graph has a Petersen flow.*

The Petersen flow conjecture can be restricted to cubic graphs.

Question 25
Prove that a cubic graph G satisfies the Petersen flow conjecture if and only if we can color ist edges, using the edges of the Petersen graph P as colors, in such way that every triple of mutually incident edges of G is colored as a similar triple of P.

We give now an other formulation of the conjecture. Let $G_1 = (E_1, V_1)$ and $G_2 = (V_2, E_2)$ be two graphs. We say that $G_1 \leq G_2$ if and only if there exits a subdivision $G_1' = (V_1', E_1')$ of G_1 and a bijective mapping β from E_2 to E_1' such that, for each $\mathbb{Z}_2$-flow ϕ of G_1', $\phi \circ \beta$ is a $\mathbb{Z}_2$-flow of G_2. We write $G_1 \simeq G_2$ if $G_1 \leq G_2$ and $G_2 \leq G_1$.
Let K_2^3 be the graph formed by two vertices and three edges linking these two vertices.

Question 26
Let G be a 3 edge-colorable cubic graph, prove that $K_2^3 \leq G$.

It is shown in [22] that the relation $\leq$ is a quasi-order and that for each graph G of a class of graphs $\mathcal{C}$ there exist a G_o such that $G_o \leq G$ and minimal for the quasi-order.

Proposition 14 *If G_1 and G_2 are two graphs with $G_1 \leq G_2$ and if G_1 has a B-flow ($B \subseteq \mathbb{Z}_2^k - \{0\}$), then G_2 has also a B-flow.*

Proof. Let G_1' a subdivision of G_1, and let β be a bijective mapping β from E_2 to E_1' such that for each $\mathbb{Z}_2$-flow ϕ of G_1', $\phi \circ \beta$ is a $\mathbb{Z}_2$-flow of G_2. If G_1 has a B-flow $(\phi_1, \ldots, \phi_k)$, then G_1' has also a B-flow $(\phi_1, \ldots, \phi_k)$. Then $(\phi_1 \circ \beta, \ldots, \phi_k \circ \beta)$ is a B-flow of G_2. $\square$
We denote by K_1^1, the graph concisting of one vertex and one loop at this vertex an by P the Petersen graph.
The Petersen flow conjecture is equivalent to the following:

Conjecture 5 *For every bridgeles graph G we have:*
$K_1^1 \leq G$ *or* $K_2^3 \leq G$ *or* $P \leq G$

Question 27
Prove that:
$i.$ If $K_1^1 \leq G$ then G has a $\mathbb{Z}_2$-NZF;
$ii.$ If $K_2^3 \leq G$ then G has a $\mathbb{Z}_2 \times \mathbb{Z}_2$-NZF.

It is easy to see that in the two cases the graph has a Petersen flow.

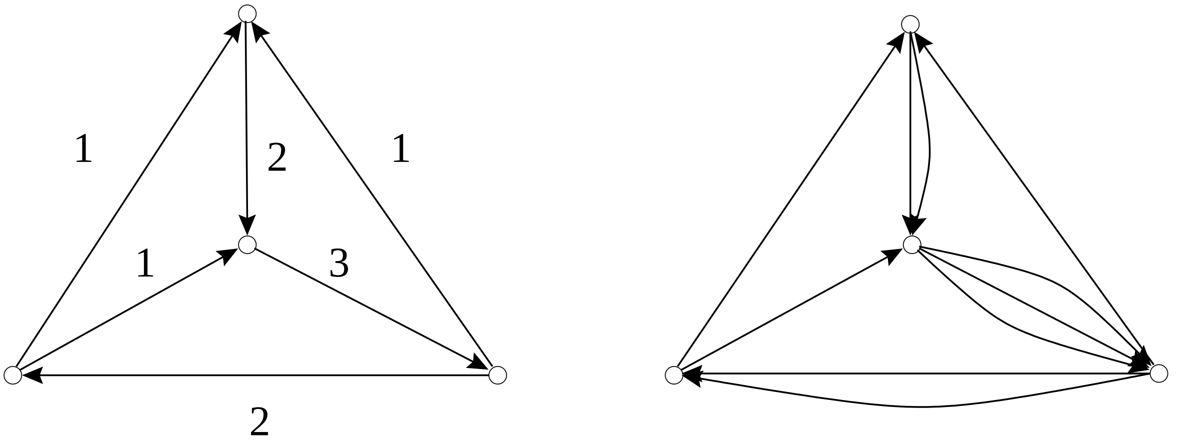
In what follows you will see some consequences of the existence of the Petersen flows in a graph.

6 Cycle covers

Flows in networks and also nowhere-zero flows have a common roots and they are related to cycle covers.

6.1 Shortest Cycle Cover Problem

This can be formulated as follows: If ϕ is a NZF in an oriented graph G which has only positive values (that as we know can be achieved by changing the orientation) then we can alternatively view the flow ϕ as a multigraph G' which we get by multiplying each arc e of G $\phi(e)$ times. The multigraph G' is a balanced multigraph (meaning that each in-degree is equal to out-degree for any vertex of G) and thus it is a disjoint union of cycles. Here we adopt a ususal terminology in this area: a *circuit* is undirected cycle in a graph while *cycle* is any edge-disjoint union of circuits. This terminology is usual in this area but unusual in general and we always remaind the reader when using the word cycle in this more general sense.
The problem of k-NZF is then the following problem: Given an undirected graph G find an orientation G_o of G such that there exists a cycle G'_o which we get by multiplying each edge of G_o by at most $k-1$ times. That sounds easy, isn't it? See fig.8.



4-NZF

Cycle Cover

Figure 8: Integer flow and cycle cover

We can also say that G'_o or its symmetrization G' provides a cycle cover of G. If we put NZF in this framework then it is only natural to consider also other measures of the "effectivity of cycle covers". For example we can ask for the minimal number of arcs in a cycle cover. This is so called SCCP problem.

Let us describe now this context more formaly:
A *circuit* is a connected 2-regular subgraph and a *cycle* is a union of edge-disjoint circuits. A *cycle cover* of a graph G is a family of cycles of G such that each edge of G is contained in at least one cycle of the family; a cycle cover is called a *k-cycle cover* if it is formed by at most k cycles. The *length* of a cycle cover $\mathcal{C}$, denoted by $l(\mathcal{C})$, is the sum of the lengths (number of edges) of all the cycles in $\mathcal{C}$.
Given a graph G and an integer k, does G have a cycle cover of total length at most k? This is the *Shortest Cycle Cover Problem (SCCP).*
Thomassen proved that the SCCP is a NP-complete problem [56]. Bounds are proposed by the the following conjectures. The first one is from Jaeger [20].

Conjecture 6 *Every bridgeless graph $G = (V, E)$ has a cycle cover $\mathcal{C}$ such that:*

$$l(\mathcal{C}) \leq \frac{7}{5}|E|.$$

We have the following Theorem [44]:

Theorem 18 *Petersen flow Conjecture implies Conjecture 6.*

To prove this we have first to prove a lemma. Let P be the Petersen graph and $w : E(P) \longrightarrow \mathbb{N}$, a weight function. $w(P)$ is the sum of the weights of its edges, the weight of a cycle C ($w(C)$) will be the sum of the weights of its edges and the weight of a cycle cover $\mathcal{C}$ will be the sum of the weights of its cycles.

Lemma 1 *There are four cycles $\{C_1, C_2, C_3, C_4\}$ of a weighted Petersen graph such that:*
i. $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$ is a cycle cover of P;
ii. $w(\mathcal{C}) \leq \frac{7}{5}w(P)$.

Proof. Let $\mathcal{C}_1 = \{C_1, C_2, C_3, C_4\}$ and $\mathcal{C}_2 = \{C_1', C_2', C_3', C_4'\}$ be two cycle covers of P depicted in Fig.9 and Fig.10. $\mathcal{C}_1$ and $\mathcal{C}_2$ are each formed by four cycles.

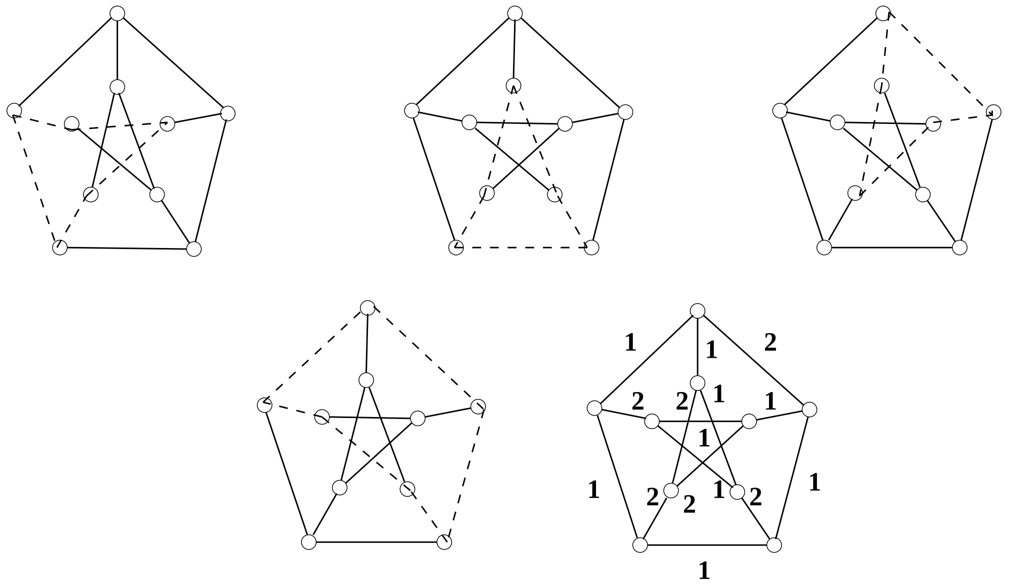


Figure 9: $\mathcal{C}_1$

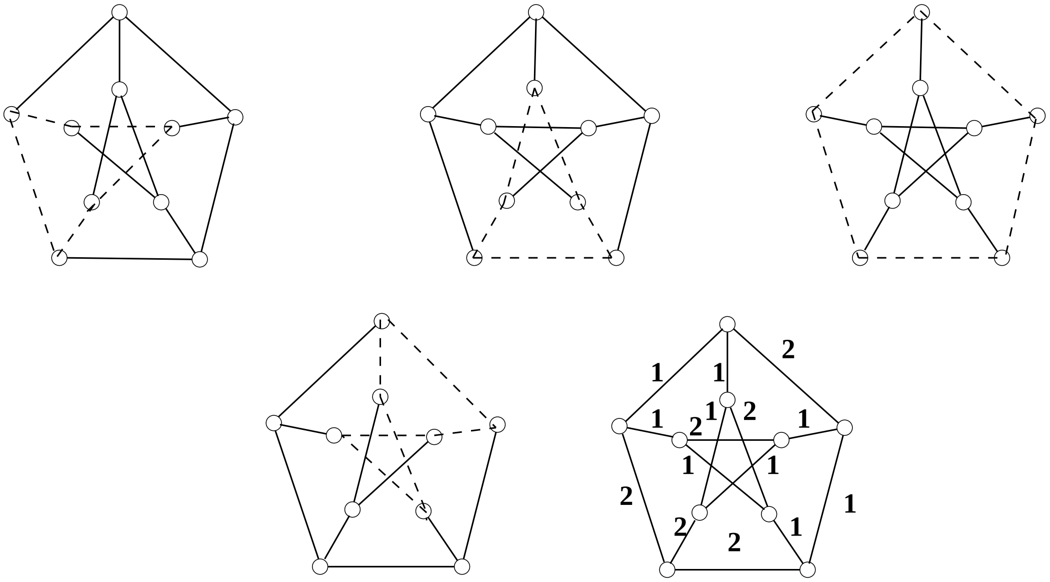


Figure 10: $\mathcal{C}_2$

By making a clockwise rotation of angle $\frac{2\Pi}{5}$ of center 0 (the center of the circonscribed circle to the pentagon of the Petersen graph, see Fig. 11), we obtain from the cycle cover $\mathcal{C}_1$ an other cycle cover of P. By repeating this rotation 4 times, we obtain in total five different cycle covers $\mathcal{C}_1$, $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_5$, $\mathcal{C}_6$.
Those cycle covers use each edge of the pentagon six times, each edge of the star seven times and each edge of the perfect matching eight times. By doing the same with the cycle cover $\mathcal{C}_2$ we also obtain five different cycle covers $\mathcal{C}_2$, $\mathcal{C}_7$, $\mathcal{C}_8$, $\mathcal{C}_9$, $\mathcal{C}_{10}$. Those cycle covers use each edge of the pentagon eight times, each edge of the star seven times and each edge of the perfect matching six times.

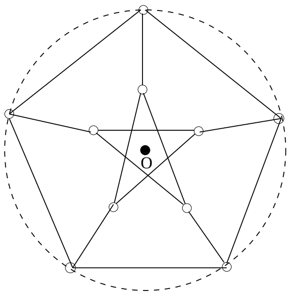


Figure 11: Rotation

Then

$$\sum_{i=1}^{10} w(\mathcal{C}_i) = 14.w(P).$$

Hence there is $j \in \{1, \ldots, 10\}$ such that:

$$w(\mathcal{C}_j) \leq \frac{7}{5} w(P).$$

$\square$

We prove now the Theorem 18.
Let $\Phi = (\Phi_1, \cdots, \Phi_6)$ be a Petersen flow of a graph G, defined from a cycle basis of he Petersen graph. Let $w(a) = |\Phi^{-1}(a)|$, $a \in B$; we identify $a \in B$ and the corresponding valued edge of the Petersen graph. We have : $w(P) = |E(G)|$. Consider by the previous lemma the cycle cover of the Petersen graph such that $w(\mathcal{C}) \leq \frac{7}{5} w(P)$. The cycles of this cycle cover are linear combinations of the cycles of the considered basis. Take as flows of G the flows with supports the cycles obtained by the corresponding linear combinations which we use to obtain the cycles of the cycles cover $\mathcal{C}$. Then the union of those supports of the corresponding $\mathbb{Z}_2$-flows in G form a cycle cover $\mathcal{C}'$ of G and we have : $l(\mathcal{C}') = w(\mathcal{C})$. Since $w(\mathcal{C}_j) \leq \frac{7}{5} w(P)$ and $w(P) = |E(G)|$, we have:

$$l(\mathcal{C}') \leq \frac{7}{5} |E(G)|.$$

$\square$

An other fascinating conjecture was due to Itai et Rodeh [18] which was very recently solved by Fan [13, 14] using deeply the structure of the 6-NZF:

Theorem 19 *Every bridgeless graph $G = (V, E)$ has cycle cover $\mathcal{C}$ such that:*

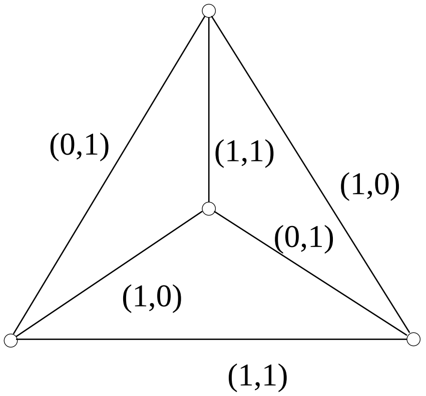
$$l(\mathcal{C}) \leq |E| + |V| - 1.$$

The three following results show the relationship between nowhere-zero flows and shortest cycle covers.

Proposition 15 *If a graph G has a 4-NZF then it has a $\mathcal{C}$ such that:*

$$l(\mathcal{C}) \leq \frac{4}{3} |E|.$$

Proof. By Theorem 6, we assume that G has a $\mathbb{Z}_2 \times \mathbb{Z}_2$-NZF denoted by ϕ. Then the mappings ϕ_1, ϕ_2 from E to $\mathbb{Z}_2$ defined for every $e \in E$ by $\Phi(e) = (\phi_1(e), \phi_2(e))$ are $\mathbb{Z}_2$-flows of G. It is easy to see that $E = \sigma(\phi_1) \cup \sigma(\phi_2)$. For each $i \in \{1, 2\}$, $\sigma(\phi_i)$ is a cycle C_i. Denote by $C_3 = C_1 \Delta C_2$ the cycle obtained by taking the edges belonging to the symmetric difference of $E(C_1)$ and $E(C_2)$. $\{C_1, C_2, C_3\}$ is a double cycle cover of the graph (each edge of G belongs to exactly two different cycles of the cycle cover), any two cyles of them is a cycle cover of the graph. We have $|C_1| + |C_2| + |C_3| = 2|E|$, see Fig.12. Hence there is a $j \in \{1, 2, 3\}$ such that $|C_j| \geq \frac{2}{3} |E|$. Assume without loss of generality that $j = 1$. Then the cycle cover $\mathcal{C} = \{C_2, C_3\}$ satisfies: $l(\mathcal{C}) \leq \frac{4}{3} |E|$. This completes the proof. $\square$



C_1	C_2	C_3
0	1	1
1	0	1
1	1	0

$$|C_1| + |C_2| + |C_3| = 2|E|$$

Figure 12: $\mathbb{Z}_2 \times \mathbb{Z}_2$-NZF and Cycle cover

In [12] Fan proposed the following conjecture:

Conjecture 7 *If a graph $G = (V, E)$ has a 3-NZF, then it has a cycle cover $\mathcal{C}$ such that*

$$l(\mathcal{C}) \leq |E| + |V| - 3.$$

Zhang proved this conjecture [64] and in [45] it is even proved that:

Theorem 20 *If a graph G has a 4-NZF and is different from K_4, then it has a cycle cover $\mathcal{C}$ such that:*

$$l(\mathcal{C}) \leq |E| + |V| - 3.$$

We can notice that the previous theorem implies the result of Zhang, since if a graph has a 3-NZF then it has a 4-NZF. In [23] we set the following theorem in the general framework of regular matroids.

Theorem 21 *If a graph G has a 5-NZF then it has a $\mathcal{C}$ such that:*

$$l(\mathcal{C}) \leq \frac{8}{5}|E|.$$

This result is optimal for the regular matroid when considering the dual matroid of the cycle matroid of K_5.

By using the existence of a 8-NZF in a bridgeless graph Bermond, Jackson, Jaeger proved the following theorem [4] which is the best bound up to now:

Theorem 22 *Every bridgeless graph $G = (V, E)$ has a cycle cover $\mathcal{C}$ such that:*

$$l(\mathcal{C}) \leq \frac{5}{3}|E|.$$

Alon and Tarsi proved the same result by using the existence of a 6-NZF [1].

Bermond, Jackson and Jaeger [4] considered the problem of covering the vertices of a graph by cycles, they conjectured that the vertices of a 2-connected graph G can be covered by cycles of total length at most $2(|V| - 1)$. Fan prove this conjecture by proving [14]:

Theorem 23 *The vertices of a 2-edge connected graph $G = (V, E)$ can be covered by cycles of total length at most $2(|V| - 1)$, where $|V| \geq 2$.*

Question 28
Prove that Theorem 23 implies Theorem 19.

6.2 Cycle Double cover

A *cycle double cover* **CDC** of a graph $G = (V, E)$ is a family of cycles of G such that every edge appears in exactly two cycles of this family. Cycles here can be a union of edge-didjoint cycles.
The following double cover conjecture was proposed by several authors.

Conjecture 8 *Every bridgeless graph has a cycle double cover.*

This conjecture may be viewed as another measure of covering of graphs by balanced subgraphs: NZF is one of them, SCCP is another, yet another presents the following:

Question 29
Do you remember Chinese Postman Problem?

chinese postman problem

We thank to Dr. Wu-Zeng Chou (Academia Sinica) for his writing.
However cycle covering is just one side of the Conjecture 8. The conjecture seems to be originally formulated in a geometrical context which we now include for completeness (see [21]). An embedding of a graphs G on a surface is called *strong embedding* if each face boundary is a cycle.
You may remember from your classes the following (not entirely easy) result [35]:

Theorem 24 *Every plane embedding of a planar 2-connected graph is strong.*

The following then comes as a bit of surprise:

Conjecture 9 *Every 2-connected graph has a strong embedding on some surface.*

One can show that Conjecture 9 implies CDC conjecture (assuming Jordan Theorem for 2-dimensional surfaces, see [55]).

Question 30
Prove that it suffices to prove CDC conjecture for any (vertex) 2-connected graph.

We shall not pursue here this geometrical motivation further. Our concern here is the combinatorial background of CDC Conjecture and its relationship to NZF and Cycle Covers.
There is a numerous evidence for this. For example it has been proved in [24] that if the Conjecture 6 concerning the existence of a cycle cover $\mathcal{C}$ of length at most $\frac{7}{5}|E|$ is true then the Cycle double cover conjecture is true.
A cycle double cover is said to be *k-colorable* ($k \geq 2$) if we can color its cycles with k-colors in such way that each edge appears in two cycles of different colors.
Alternatively we can say that there are k cycles $E_1, E_2, ..., E_k$ in G (cycles now in the sense of union of edge-disjoint circuits) such that every edge belongs to two of these cycles.
We give now an easy proposition:

Proposition 16 *If a graph G has a k-colorable cycle double cover ($k \geq 2$) then it has a cycle cover $\mathcal{C}$ such that:*

$$l(\mathcal{C}) \leq \frac{2(k-1)}{k}|E|.$$

Question 31
Prove the proposition 16.

In [8, 43] a strongest conjecture is proposed:

Conjecture 10 *Every bridgeless graph has a 5-colorable cycle double cover.*

Question 32 Prove that if a graph G has a Petersen flow then it has 5-colorable cycle double cover.

However at this time it is not clear whether there exists some (universal) k_0 such that every graph which has CDC has also k_0-colorable CDC.

We can notice that if the previous conjecture is true, then every graph $G = (V, E)$ has a cycle cover $\mathcal{C}$ such that $l(\mathcal{C}) \leq \frac{8}{5}|E|$. This is a conjecture proposed in [4].

In the proof of proposition 15 we have seen that:

Proposition 17 *If a graph G has a 4-NZF then it has a cycle couble cover.*

The last strong result concerning the cycle double cover conjecture is the following proved by Alspach, Goddyn and Zhang [3]:

Theorem 25 *If a graph G does not contain a subdivision of the Petersen graph then it has cycle double cover.*

Another link of CDC to NZF is the following:

Let $k \geq 1$ be fixed. Let D_k be the subset of Z_2^k consisting of those elements containing exactly two 1s. Let ϕ be a D_k- flow of a graph G (see section 4.1 where we defined a B-flow). Then $\phi = (\phi_1, ..., \phi_k)$ and by Proposition 6 each set $E_i = \sigma(\phi_i)$ is a cycle (again in the sense of union of edge-disjoint cycles). As ϕ is and D_k-flow these cycles induce a cycle double cover of G. Conversely given a cycle double cover $C_1, ...C_t$ which is k-colorable (e.g. for $k = t$), we can induce a D_k-flow on G by reversing the above procedure.
Thus the existence of k-colorable CDC is equivalent to D_k-flow on G.
In the next section we shall see another influence of restricting flow values. For yet another example of this phenomenon see [5].
For these two last sections we advise to consult the book of Zhang [65].

7 Antisymmetric flows

The theory of NZF is the theory of undirected graphs: if a graphs G posseses a k-NZF for an orientation of G then every orientation of G has k-NZF. In this section we discuss the possibility to extend the theory to oriented graphs. This is not an easy task as for oriented graphs the right definition of a "minor" is still discussed and tested. In minor based theories the difference between undirected and directed graphs is large. So we may view this part as a contribution to this area. Our text is based on authors article [39] where a complementary information can be found.

7.1 Definitions

Let $G = (V, E)$ an oriented graph. If M is an abelian group (with additive notation), then a M-flow is a mapping ϕ from E to M. If $B \subset M$, is such that $B \cap -B = \emptyset$, then a flow with values taken in B will be called an antisymmetric-flow. In this case we will say that G has a M-antisymmetric-flow (shortly M-ASF). One could also say that an antisymmetric-flow is a nowhere-inverse flow.

Question 33
i. Prove that no graph has $(\mathbb{Z}_2)^k$-ASF;
ii. Prove that any Eulerian graph with an Eulerian orientation (i.e. balanced : $\forall x \in V$, $d^+(x) = d^-(x)$) has a $\mathbb{Z}_3$-ASF.

If $M = \mathbb{Z}$ and all the values of the antisymmetric-flow are belonging to $[1-k, k-1]$ ($k \geq 2$) then we will say that G has a k-ASF.

If G is an undirected graph we will say that G has a M-ASF, if it has a M-ASF for some orientation.
It is a non trivial fact that 3-edge connected graphs have ASF (this will be proved below).

We review our approach somewhat in the reverse order. The notion of antisymmetric flow was inspired by the notion of oriented chromatic numer about which one can get basic information in the lecture notes by Nešetřil and Sopena on coloring [42]. We shall not elaborate on this. Let us briefly say that one can define the notion of strong-oriented coloring and then to prove that for every planar graph G, G has $\mathbb{Z}_k$-ASF iff the dual graph G^* has $\mathbb{Z}_k$-strong oriented coloring. However due to lack of space (and time) we ommit this, see [39].

7.2 Elementary properties

At this stage of reading of this survey, perhaps the following is an easy task:

Question 34
ii. Prove that:
Let G be an undirected graph, M an abelian group. Assume that G has a M-flow ϕ such that $\phi(e) \neq -\phi(e)$ for any $e \in E$, then it has a M-ASF for some orientation.
ii. Prove the following:
If an undirected graph G has a M-NZF, M being a group of odd order then there is an orientation of G with an M-ASF.
iii. Prove the following:
if a graph has a k-ASF, then it has a k'-ASF for any $k' \geq k$. Moreover if a graph has a $\mathbb{Z}_k$-ASF then it has $\mathbb{Z}_{k'}$-ASF for any $k' \geq k$ (k and k' odd).

Following the general schema for NZF we first investigate to which extend the analogy of Equivalence Theorem 6 (see 5.5) holds. In the following we will use the following result of Tutte [58] which is a corollary of Theorem 6 (see 4.5):

Theorem 26 *If $G = (V, E)$ has a $\mathbb{Z}_k$-NZF f, then it has a k-NZF g such that: $f(e) \equiv g(e) \bmod k$ for any $e \in E$ and the flows f and g are defined under the same orientation.*

Using the previous theorem we have the following:

Proposition 18 *For any odd k an undirected graph G has k-ASF if and only if it has $\mathbb{Z}_k$-ASF.*

Proof. Suppose first that an undirected graph G has a k-NZF (k odd) then it has a $\mathbb{Z}_k$-ASF. Then it admits a $\mathbb{Z}_k$-NZF under the same orientation. By reversing the direction of some arcs, we find an orientation of G for which the NZF uses only the values $\{\bar{1}, \ldots, \overline{\frac{k-1}{2}}\}$. It is obvious that this is a $\mathbb{Z}_k$-ASF.
Conversely we choose an orientation for which the $\mathbb{Z}_k$-ASF f is such that : $f(E) = \{\bar{1}, \ldots, \overline{\frac{k-1}{2}}\}$
. By the Theorem 26 there exits a k-NZF g of G under the same orientation, such that :
$f(e) \equiv g(e) \bmod k$. If $f(e) = i$ then g(e) can be i or $i - k$. We have then :

$g(E) \subseteq \{1, 1-k, 2, 2-k, \ldots, \frac{k-1}{2}, \frac{k-1}{2} - k\}$. It is easy to see that $g(E)$ contains no opposite values. □

It may happen that for k even that for a graph G there is an k-ASF while no $\mathbb{Z}_k$-ASF exists. This is due to the fact that an k-ASF f needs not satisfy the condition $f(e) + f(e') \neq 0 \bmod k$ for any pair of edges e,e' of G. A k-ASF satisfying this additional property (which is obviously necessary) implies the existence of $\mathbb{Z}_k$-ASF. In this case we have the equivalence between the existence of a k-ASF and a $\mathbb{Z}_k$-ASF for any integer $k > 2$.
In this context the following is of an interest:

Theorem 27 *Let k be an odd number, G an undirected graph. If for an orientation G has a $\mathbb{Z}_{2k}$-ASF then for some other orientation it has a $\mathbb{Z}_k$-ASF. On the other hand every $\mathbb{Z}_k$-ASF may be thought as $\mathbb{Z}_{2k}$-ASF and thus the existence of orientations with $\mathbb{Z}_{2k}$-ASF and $\mathbb{Z}_k$-ASF are equivalent.*

Proof.
First, suppose that G has a $\mathbb{Z}_{2k} - ASF$, denoted by f. Clearly we may assume that the values of the antisymmetric-flow f are taken in $\{\bar{1}, \ldots, \overline{k-1}\}$. By the theorem of Tutte there exists a k-NZF g under the same orientation, such that :
$f(e) \equiv g(e) \bmod 2k$. If $f(e) = i$ then g(e) can be i or $i - 2k$. We have then: $g(E) \subseteq \{1, 1-2k, 2, 2-2k, \ldots, k-1, k-1-2k\}$. Let $g'(e)$ be the residue of $g(e)$ modulo k. Since $g(e) \neq 0, \pm k$, it follows that $g'(e) \in \mathbb{Z}_k$ is non-zero. Hence G has a $\mathbb{Z}_k$-NZF. And by reversing the orientation of certain arcs, we will have a $\mathbb{Z}_k$-ASF in G.
The reverse implication follows by Question 34.*iii*: if G has a $\mathbb{Z}_k$-ASF then it has a $\mathbb{Z}_{2k}$-ASF. □

Let us remark that for k even we can only prove that if a graph has a $\mathbb{Z}_{2k}$-ASF then it has a $\mathbb{Z}_k$-NZF.

This remarks leads us to the following two invariants:
Let G be a graph and D an orientation of G, we will denote by $asf(G, D)$ the smallest order k of a group M s.t. G has a M-ASF, under the orientation D. The *upper antisymmetric-flow number* of G is by definition : $asf_U(G) = max_D asf(G, D)$, the maximum is taken over all the possible orientations of G. We also call $asf_L(G) = min_D asf(G, D)$, the *lower antisymmetric-flow number* of graph G.
Of course at this moment we do not know whether asf_U is a correct definition (and we do not want to put there infinity as we shall see shortly that this is not necessary).
We shall now study asf_L.

7.3 Tutte's analogies

Theorem 28 *Every bridgeless graph G has an orientation for wich G has a $\mathbb{Z}_7$-ASF (i.e. $asf_L(G) \leq 7$ for every bridgeless graph G).*

Proof. By Seymour's theorem [50], any bridgeless graph has a $\mathbb{Z}_2 \times \mathbb{Z}_3$-NZF. By the theorem of Tutte it has a 6-NZF wich is also a 7-NZF, by observation 1 it admits a $\mathbb{Z}_7$-ASF. □

Also the Jaeger's theorem [19] giving the existence of a 8-NZF for any bridgless graph, assures that any bridgless graph has a $\mathbb{Z}_9$-ASF.

Conjecture 11 *$asf_L(G) \leq 5$ for every bridgeless graph G.*

Observe that the Petersen graph has a $\mathbb{Z}_5$-ASF. See Figure 13.

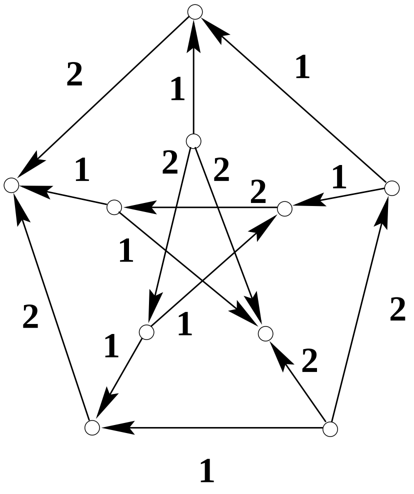


Figure 13: $\mathbb{Z}_5$-ASF

We have also:

Corollary 3 *If G has a $\mathbb{Z}_k$-ASF for $k \leq 4$ then it is an Eulerian graph.*

> **Question 35**
> Prove the Corollary 3:
> i. By using the same argument as in theorem 27 prove that G has a $\mathbb{Z}_2$-NZF.
> ii. Prove that G is an Eulerian graph.

For a given orientation of a graph G, the existence of a $\mathbb{Z}_k$-ASF is stronger that the existence of a k-ASF. This may be seen as follows:
Consider a cubic bipartite graph $G = (V, E)$ with the bipartition $V = X \cup Y$ ($|X| = |Y|$). The edges of a perfect matching are oriented from X to Y and the remaining edges from Y to X. Let f be the valuation of the edges of G such that $f(e) = 2$ if e is an arc belonging to the matching and $f(e) = 1$ for the arcs not belonging to the matching. It is easy to see that f is a 3-ASF, but neither does G admit a $\mathbb{Z}_3$-ASF nor a $\mathbb{Z}_4$-ASF under this orientation. It admits a $\mathbb{Z}_5$-ASF under the same orientation. Thus by theorem 27 and Corollary 3 the oriented versions of Tutte conjectures (from section 4.7) collapse to the only one conjecture 11.
However this is not the end of the story and situation is more complicated at the other side of the spectrum and we turn to the numbers asf_U in the next section.

7.4 Antisymmetric flows for a given orientation

Let G be an undirected graph. If G contains an isthmus or a 2-cut then G does not admit a nowhere-zero flow. If G contains an edge-cut made of two edges e_1,e_2 between components

K_1 and K_2 and if we take an orientation of the graph such that the edges e_1 and e_2 are both oriented from K_1 to K_2 then clearly G with this orientation has no antisymmetric-flow. A 2-edge-cut will be also called a 2-cut.
We prove that every graph without a 2-cut has an antisymmetric-flow. We shall use the following concept and an easy lemma:

Lemma 2 *A connected bridgeless graph G has no 2-cut if and only if for every pair $\{e, e'\}$ of distinct edges of G there exits a separating circuit C (i.e. a circuit which contains exactly one of the edges e, e').*

Proof. Suppose that $G \setminus \{e, e'\}$ is connected, let $e = \{x, y\}$. Then a path which joins x to y in $G \setminus \{e, e'\}$ together with the edge e forms a cycle which does not contain e' in G.
In the reverse direction, suppose that for any pair of distinct edges there exist a separating cycles. Thus given a pair $\{e, e'\}$ of distinct edges find separating cycles with $e \in C_e$, $e' \notin C_e$. The end vertices of e belong to the same component thus $G \setminus \{e, e'\}$ is connected. □

Let G be a graph. A family $C_1, \dots, C_t$ of circuits of G is said to be *edge-distinguishing* if any pair $\{e, e'\}$ of distinct edges of G there exists C_i which is containing exactly the edge e and not containing the edge e'. The above lemma can be formulated as follows : every 3-edge connected graph G has a family of distinguishing cycles.
Note that for 3-edge connected planar graphs the set of facial cycles is distinguishing.

More generally we have:

Lemma 3 *For a 3-edge connected graph any cycle basis is edge distinguishing.*

Proof. Assume the contrary: let $\mathcal{C}$ be a cycle basis and let e,e' be distinct edges such that any cycle $C \in \mathcal{C}$ contains either both edges e,e' or none of them. As $\mathcal{C}$ is a basis there is no circuit which contains exactly one of the edges. This contradics lemma 2. □

Theorem 29 *Let G be a graph, $C_1, \dots, C_t$ a family of edge-distinguishing cycles. Then then for any orientation G has a $\mathbb{Z}_3^t$-ASF.*

Proof. Let $G = (V, E)$ with a fixed orientation. Let C_i denote also the set of arcs. For each C_i by doing a walk on C_i we define the flow $f_i : C_i \to \mathbb{Z}_3$:
$f_i(xy) = +1$ for forwarding arcs of C_i,
$f_i(xy) = 2$ for the backwarding arcs of C_i,
$f_i(xy) = 0$ for the arcs not belonging to C_i.
Put $f(xy) = (f_i(xy); i = 1, \dots, t)$. We prove that f is an antisymmetric-flow.
Obviously $f(x, y) \neq (0, \dots, 0)$. Let a and b two arcs of G. Let C_i be a cycle containing exactly one of the arcs a,b; let say $a \in C_i$ and $b \notin C_i$. Then $f_i(a) \neq 0$ and $f_i(b) = 0$. Thus $f(a) \neq -f(b)$ (in $\mathbb{Z}_3^t$) (note that we must use $\mathbb{Z}_3^t$ as in $\mathbb{Z}_2^t$ for every $x = -x$). □

Denote by $\mathcal{C}(G)$ the cycle space of G.

Corollary 4 *(* **Existence of ASF** *) Any orientation of a 3-edge connected graph G has a $\mathbb{Z}_3^{dim\mathcal{C}(G)}$-ASF.*

Remark: Another, although less explicite proof of the fact that any orientation of any 3-edge connected graph has an antisymmetric-flow may be sketched as follows:

Existence of ASF (second proof).
Let $G = (V, E)$ be an orientation of a 3-edge connected graph with m edges and n vertices. The set of all IR-flows on G (IR-flows are usually called circulations) is the set of all solutions $(x_e; e \in E)$ defined by equations :

$$\forall u \in V : \sum_{v \in V} x_{uv} - \sum_{v \in V} x_{vu} = 0 \tag{8}$$

the variable x_{uv} is defined if and only if $(uv) \in E(G)$.
The set of all solutions form a $m-n$-dimensional sub-space $\mathcal{V}$ of $\mathbb{R}^E$. The preimage of $\mathcal{V}$ on any coordinate (i.e. edge e of G) is 1-dimensional as otherwise e would be a cut. Similarly the projection of $\mathcal{V}$ to any pair of coodinates e, e' has dimension 2 (for otherwise there would be no circuit which would distinguish e and e' and thus $\{e, e'\}$ would be a 2-cut). Thus there exists a solution of the system 8 which misses any of the hyperplanes $x_e = 0$, $x_{uv} + x_{u'v'} = 0$. And there exists also an integer solution. □

Let us remark that this algebraical argument implies a slight improvement of the existence of ASF:

Corollary 5 *Let G be a 2-edge connected oriented graph without an oriented 2-cut. Then G has an ASF.*

We shall end this survey with the following question which brings us back to fifties (or before 1954): before it was not known whether any graph has a bounded NZF. The history repeats itself, only this time we consider oriented graphs. We can formulate it as follows:

Final Problem

Decide whether there exists a (universal) constant $k(0)$ such that any oriented graph without an oriented 2-cut has $k(0)$-ASF

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