The complexity of H-colouring of bounded degree graphs

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Abstract

We investigate the complexity of the h-colouring problem, and, more generally, of the *H-colouring problem*, restricted to graphs of bounded degree. While the general problems are almost always NP-complete, we present some surprising polynomial algorithms for several of these restricted colouring problems. We also give a number of NP-completeness results, and pose some open problems. One of these may be viewed as the complement of an algorithmic version of the theorem of Brooks.

1 Introduction

Let k be a positive integer; graph G is *k-bounded* if all degrees in G are at most equal to k. Colouring of k-bounded graphs is addressed by the well known theorem of Brooks, which asserts that, for $k \geq 3$, each k-bounded graph, different from K_{k+1}, is k-colourable. This fact leads to a trivial polynomial algorithm for 3-colourability of 3-bounded graphs: a graph is 3-colourable unless it is isomorphic to a K_4 (an easy check), in which case it is not 3-colourable. Since testing 2-colourability is polynomial (for all graphs), the h-colouring problem for 3-bounded graphs is polynomial for any h.

On the other hand, for 4-bounded graphs, while it is still true that Brooks' theorem guarantees 4-colourability (except for K_5), not all h-colourability problems are trivial. In fact, we have the following observation:

Theorem 1.1 *Deciding whether a given* 4*-bounded graph is* 3*-colourable is* NP*-complete.*

Proof. The class of 4-bounded graphs contains all line graphs of 3-regular graphs. Holyer proved that deciding whether a 3-regular graph has a proper 3-edge colouring is NP-complete. □

The complexity of more general notions of colouring has been subject of much recent interest [**?**, **?**, **?**, **?**]. Let H be a fixed graph; an H-colouring of G is a mapping $c : V(G) \to V(H)$ which preserves adjacency, i.e., such that $gg' \in E(G)$ implies $c(g)c(g') \in E(H)$. An H-colouring of G is also called a *homomorphism* of G to H. The H-colouring problem asks whether a given graph G is H-colourable. This is a generalization of the usual h-colouring problem, since a K_h-colouring of G corresponds to an h-colouring of G. However, it turns out that this generalization introduces essentially no new polynomial cases - other than modifications of the usual 2-colouring problem: It is proved in [**?**] that H-colouring is NP-complete unless H is bipartite. On the other hand, the H-colouring problem for *directed* graphs, exhibits interesting duality properties and resulting polynomial algorithms [**?**].

Our interest in studying the complexity of the H-colouring problem was motivated by the following result of Häggkvist and Hell [**?**]:

Theorem 1.2 *For every connected graph* A *there exists a graph* $U[A]$ *with the following property: a* k*-bounded graph* G *is* $U[A]$*-colourable if and only if* A *is not* G*-colourable.*

Since A is fixed, there is a polynomial time algorithm to test whether A is G-colourable: There are only polynomially many (in terms of the size of G) mappings from A to G, and we can quickly test each of them to see if it is a G-colouring of A. (The fastest known algorithm to check for the existence of fixed subgraphs is due to Nešetřil and Poljak [?] and is based on matrix multiplication).

Thus, for each connected graph A, there exists a polynomial time algorithm to check whether a k-bounded graph G is $U[A]$-colourable. Hence, many nontrivial H-colouring problems for k-bounded graphs are polynomial time solvable.

2 Polynomial cases

We now return to considering the graphs $U[A]$. For simplicity we shall consider only cubic graphs in this section. Moreover, we shall take $A = K_3$. Note that saying that K_3 is not G-colourable is equivalent to saying that G is triangle free, since triangles can be mapped only to triangles under a homomorphism. Thus Theorem 1.2 says that a cubic graph G is $U[K_3]$-colourable if and only if G is triangle free.

We now describe a general construction that may be used to define $U[K_3]$. Let X be any set. We define a graph U_X as follows:

- The vertices of U_X are ordered pairs (x, T) where T is a 3-element subset of X and $x \notin T$;
- Two vertices (x, T) and (x', T') are adjacent if and only if $T \cap T' = \emptyset$ and $x \in T', x' \in T$.

(We note this construction is related to the symmetric line graphs of oriented hypergraphs as studied by Ausiello et al [?].)

We now claim that when X has at least 22 elements, then U_X can be chosen to be $U[K_3]$. In other words, every triangle free cubic graph G admits a U_X-colouring. Indeed, suppose that G is triangle free and cubic. Then it is easy to see that the graph G^3 constructed from G by making adjacent all vertices at distance less than or equal to 3 has maximum degree 21. Hence, by Brooks' theorem, it admits an $|X|$-colouring, say c. We assign to each vertex $v \in V$ of colour i, the image $(i, \{j, k, l\})$ where j, k, l are the three (distinct) colours of the neighbours of v in G. We shall show that this mapping preserves adjacency. Let xy be an edge of G and let i, j be the colours of x, y, respectively, in c. Their images are $(i, \{j, k, l\})$ and $(j, \{i, p, r\})$ and they are adjacent in U_X; in fact, there are 4 distinct neighbors of x and y, since G is triangle free, and they all receive different colours in c since their distances are at most 3. Thus, $\{j, k, l\} \cap \{i, p, r\} = \emptyset$. Observe that we have not only found that an U_X-colouring exists, but we have actually constructed it (in polynomial time).

We shall show now that the graphs U_X tend to have high chromatic numbers.

Theorem 2.1 *1. For every k there exists X such that the chromatic number of U_X is at least k.*

2. For every X with at least 15 elements the chromatic number of U_X is at least 4.

Proof. Suppose $X = \{1, \ldots, n\}$.

Part 1. Consider the following graph S_X: the vertices of S_X are all 3-element subsets of X, and two such subsets, say $\{x_1, x_2, x_3\}$ with $x_1 < x_2 < x_3$, and $\{y_1, y_2, y_3\}$ with $y_1 < y_2 < y_3$, are adjacent if $x_2 = y_1$ and $x_3 = y_2$. This is a variant of a general construction of type graphs defined in [**?**]. Note that S_X is a directed graph but we will also call S_X its underlying undirected graph. It follows from the Ramsey theorem for partition of triples that the chromatic number of S_X may be arbitrarily large if n is large.

We now claim that S_X is isomorphic to a subgraph of some $U_{X'}$where X' contains X. Let f be a bijection from the set of all 3-element subsets of X to a set Y disjoint from X and let $X' = X \cup Y$. Now, for $\{x_1, x_2, x_3\} \in V(S_X)$ with $x_1 < x_2 < x_3$, we let $g(x_1, x_2, x_3) = (x_2, \{x_1, x_3, f(x_1, x_2, x_3)\}) \in V(U_{X'})$. It is easy to see that g is an injective homomorphism from $V(S_X)$ to $V(U_{X'})$, i.e. S_X is isomorphic to a subgraph of $U_{X'}$, and hence, the chromatic number of $U_{X'}$ is at least as large as the chromatic number of S_X. (In fact, it is easy to see g is an isomorphism onto an induced subgraph of $U_{X'}$).

Part 2. Suppose that U_X has a proper 3-colouring $c : V(U_X) \to \{1, 2, 3\}$. Consider a 4-element subset M of X, say $M = \{x_1, x_2, x_3, x_4\}$, and the four vertices of U_X corresponding to M, namely $(x_1, \{x_2, x_3, x_4\})$, $(x_2, \{x_1, x_3, x_4\})$,

$(x_3, \{x_1, x_2, x_4\})$ and $(x_4, \{x_1, x_2, x_3\})$. The 3-coloring c assigns the same colour to some two of these vertices. We let $c(M)$ be such a colour.

Next consider the set Z consisting of the ordered pairs (A, M) where M is a 4-element subset of X and A is a 2-element subset of M, say a_1, a_2, such tha $c(a_1, M - a_1) = c(a_2, M - a_2) = c(M)$. Each M admits at least one A such that $(A, M) \in Z$ by the definition of $c(M)$. Thus Z has at least $\binom{n}{4}$ elements.

We now claim that each A can occur in at most $n-3$ elements (A, M) of Z. In fact, consider two elements (A, M) and $)(A, M')$ of Z and assume that $A = \{a_1, a_2\}$, $M = \{a_1, a_2, u, v\}$ and $M' = \{a_1, a_2, x, y\}$. Then $c(a_1, \{a_2, u, v\}) = c(M) = c(a_2, \{a_1, x, y\})$ and hence cannot be adjacent in U_X. Therefore $\{u, v\} \cap \{x, y\} \neq \emptyset$. Thus the set of 2-element subsets B of $X - A$ such that $(A, A \cup B) \in Z$ has the property that any two subsets B intersect, and hence has at most $n - 3$ elements. Hence, Z contains at most $\binom{n}{2}(n - 3)$ elements.

We conclude that $\binom{n}{4} \leq |Z| \leq \binom{n}{2}(n - 3)$ and then $n \leq 14$. □

According to the above theorem, the only known triangle free nonbipartite graphs H, for which we have a polynomial time solvable H-colouring problem for cubic graphs, namely $H = U_X$ and $|X| = 22$, have chromatic number greater than 3. This motivates the conjecture in the last section of the paper.

3 NP-complete cases

In this section, we show that for several families of triangle free nonbipartite graphs H, the H-colouring problem for 3-bounded graphs is NP-complete.

Let us start by considering the C_{2k+1}-colouring problem. We prove that this problem is NP-complete for the class of 3-bounded graphs.

We reduce a graph G into a graph G^* of degree at most three as follows: we replace each vertex x of degree t in G with t odd cycles $C^j = \{v_1^j, \ldots, v_{2k+1}^j\}$ (numbered clockwise), $j = 1, \ldots t$, in G^* such that each edge $e_j \in \delta(x)$ is incident to v_1^j, and moreover $v_2^j v_3^j = v_{2k}^{j+1} v_{2k+1}^{j+1}$, for $j = 1, \ldots, t$. Then:

Theorem 3.1 *G is C_{2k+1}-colourable if and only if G^* is C_{2k+1}-colourable.*

Proof. It suffices to observe that any valid colouring of G^* assigns the same colour to the vertices v_1^j, $j = 1, \ldots, t$. Hence, any valid colouring of G^* leads to a valid colouring of G and viceversa. $\square$

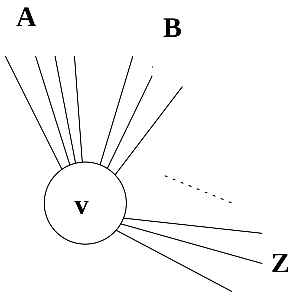
Since the C_{2k+1}-colouring problem is NP-complete for graphs with no degree constraints [?], the NP-completeness of our problem follows.

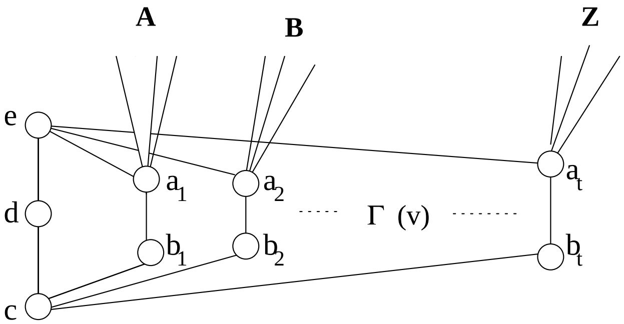
More generally, we can prove NP-complete every H-colouring problem where the graph H has odd girth $2k+1$, every vertex belongs to a C_{2k+1} and no two copies of C_{2k+1} share more than one edge. For simplicity we describe this result in the case $k = 2$:

Theorem 3.2 *Let H be a triangle free graph in which each vertex belongs to a pentagon, and in which no two pentagons share more than one edge. Then the H-colouring problem for 3-bounded graphs is NP-complete.*

Proof. Given a graph $G = G_0$ with n vertices and no degree bound (i.e., n-bounded), we construct in polynomial time a sequence $G_0, G_1, \ldots, G_l$ of graphs such that G_l is 3-bounded and $G \to H$ if and only if $G_l \to H$.

Suppose G_i has already been constructed, and suppose it is k-bounded. Then G_{i+1} is constructed by replacing each vertex v of G_i with a gadget $\Gamma(v)$ as follows: the edges at each v are partitioned into $t = \lceil \sqrt{k} \rceil$ groups of size approximately $\sqrt{k}$ and each group is made incident with a separate vertex of $\Gamma(v)$ as suggested by the figure.





Observe that:

i) G_{i+1} is $2 + \lceil \sqrt{k} \rceil$-bounded;

ii) $|V(G_{i+1})| = |V(G_i)|(3 + 2\lceil \sqrt{k} \rceil)$;

iii) $G_{i+1} \to H$ if and only if $G_i \to H$.

The first two items follow immediately from the construction of G_{i+1}. We will now prove (iii). Suppose first that G_i is H-colourable and let $c(v)$ be the colour of a vertex v of G_i. Since each vertex of H belongs to a pentagon, we have that $c(v)$ belongs to a pentagon, say $\{c(v) = a, b, c, d, e\}$. Then we may colour $\Gamma(v)$ as follows: all vertices a_j, $j = 1, \ldots, \lceil\sqrt{k}\rceil$, receive the colour $c(v) = a$, all vertices b_j receive colour b and so on. It is easy to see that this induces an H-colouring of G_{i+1}.

Let us suppose conversely that G_{i+1} is H-colourable. We claim that all vertices a_j, for any j, have the same colour in any H-colouring. In fact, since H is triangle free, each pentagon of G_{i+1} maps into a pentagon of H and if two a_j's were different then their corresponding pentagons in $\Gamma(v)$ would be mapped into two different pentagons of H sharing two consecutive edges, contradicting the assumption. Hence, any H-colouring of G_{i+1} may be easily transformed into an H-colouring of G_i.

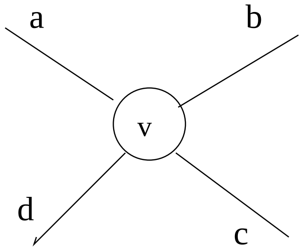
What remains to show is that the construction converges to a 3-bounded graph in a number of steps which is polynomial in n. Consider the recurrence relation:

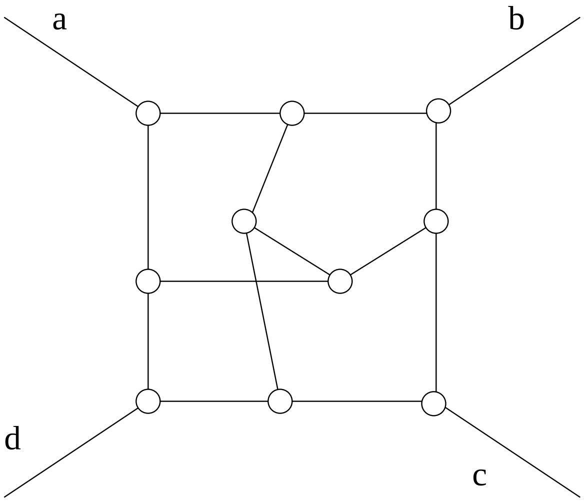
$$q_0 = n$$

$$q_{i+1} = 2 + \lceil\sqrt{q_i}\rceil$$

While q_i is not too small ($q_i > 50$, say) this recurrence is decreasing like $n^{\frac{1}{2^i}}$, i.e., quite fast. In fact, after $O(loglogn)$ steps we have $q_i = O(1)$. When q_i becomes smaller than 6, the influence of the addition of 2 and of the ceiling affects the situation, and in fact $2 + \lceil\sqrt{5}\rceil = 5$. However, for 6-bounded (and hence also 5-bounded) graphs the construction yields a 4-bounded graph G_{l-1}.

Finally, we construct G_l from G_{l-1} as follows:





It is easy to verify that G_l is 3-bounded and that $G_{l-1} \to H$ if and only if $G_l \to H$.

Thus, the above construction provides a reduction from a graph $G = G_0$ to a graph G_l which is 3-bounded and such that $G \to H$ if and only if $G_l \to H$. It can be accomplished in polynomial time (in n) since there are $O(loglogn)$ intermediate graphs and the total number of vertices of G_l is only $O(n^{1+\frac{1}{2}+\frac{1}{4}+\cdots}) = O(n^2)$. Since the H-colouring problem is NP-complete for general graphs unless H is bipartite [**?**], the theorem follows. $\square$

Note that, in the above theorem, H may have arbitrary large chromatic number. As a consequence of the above construction we can prove that, for any integeres k, g, there exists a graph H_k, with $\chi(H_k) = k$, such that H_k-colourability is NP-complete for cubic graphs of girth g.

4 Conclusions

We have investigated the computational complexity of the H-colouring problem for the class of 3-bounded graphs. From our results, some questions arise naturally. In particular, as a consequence of Theorem 2.1, we know that the only triangle free nonbipartite graphs H for which we have a polynomial time solvable H-colouring problem for cubic graphs, namely the graphs $H = U_X$ with $|X| \geq 22$, have chromatic number greater than 3. Since graphs H with chromatic number 3, and which contain triangles, have polynomial time solvable H-colouring problems for cubic graphs (by virtue of Brooks' theorem), we can ask whether all problems of this type, which are not solvable by Brooks' theorem techniques, are in fact NP-complete. As shown in Section 3, several classes of triangle free graphs H for which the corresponding H-colouring problem for 3-bounded graphs is NP-complete exist. Perhaps the simplest unsolved case not covered by Theorem 3.2 is the Petersen graph; we conjecture that this is also an NP-complete case. The results of this paper suggest the following conjecture:

Conjecture *Let H be a triangle free graph* with chromatic number 3. *Then the H-colouring problem for* 3*-bounded graphs is NP-complete.*

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