A note on reconstruction of a space from the open set lattice

A. Pultr* and A. Tozzi

It is a well-known fact that a sober space X can be reconstructed from the lattice $L = \mathfrak{O}(X)$ of its open sets as the system of all complete filters of L endowed by the natural topology. Another known fact of this nature is the Thron's theorem ([**10**]) stating that if spaces X and Y are T_D and if the lattices $\mathfrak{O}(X)$ and $\mathfrak{O}(Y)$ are isomorphic then X and Y are homeomorphic. The question naturally arises as to whether, or, rather, how, a space can be reconstructed from $L = \mathfrak{O}(X)$ under T_D (which does not imply – nor is implied by – sobriety). One does not have to reconstruct a space as the system of *all* the complete filters; it suffices to have a formula in lattice terms which tells the "good" complete filters from the bad ones (in a given class of spaces). In this paper we present such a formula for the class of T_D-spaces, and a simpler one for T_1-spaces (also not including nor included in the sober ones). The formulas also yield reconstructing continuous maps from (special) homomorphisms. On the other hand, it is shown that for the class of all T_0-spaces no such formula in lattice terms exists.

1. Preliminaries

1.1. A *frame* is a complete lattice L satisfying the distributivity law

$$a \wedge \bigvee S = \bigvee \{a \wedge b \mid b \in S\}$$

for any $a \in L$ and any $S \subseteq L$. A *frame homomorphism* is a mapping preserving all suprema (including the bottom 0) and all finite infima (including the top 1).

The lattice $\mathfrak{O}(X)$ of all open sets of a topological space X is a frame and if $f : X \to Y$ is a continuus map we have the frame homomorphism $\mathfrak{O}(f) : \mathfrak{O}(Y) \to \mathfrak{O}(X)$ defined by $\mathfrak{O}(f)(U) = f^{-1}(U)$.

For more information about frames see [**7**] or [**11**]

1.2. Recall that a filter F on a frame (more generally, on a distributive lattice) is said to be *prime* if

$$a \vee b \in F \quad \Rightarrow \quad a \in F \text{ or } b \in F;$$

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it is said to be *completely prime* (briefly, *complete*) if

$$\bigvee_{i\in J} a_i \in F \quad \Rightarrow \quad \exists i,\ a_i \in F$$

for any system $\{a_i \mid i \in J\}$.

1.3. An element $a \in L$, $a \neq 1$, is said to be *meet irreducible* if $a = b \wedge c$ implies that either $a = b$ or $a = c$. For any space X, the open sets $U \smallsetminus \overline{\{x\}}$ are meet irreducible in $\mathfrak{O}(X)$. If the converse holds, that is, if each meet irreducible is of the form $U \smallsetminus \overline{\{x\}}$ for some $x \in X$, we say that X is *weakly sober*. In a *sober* space, moreover, the point x is uniquely determined (thus, sober $\equiv$ weakly sober & T_0).

1.4. Another condition on a space we will discuss is the T_D introduced by Aull and Thron in [**1**]:

for each $x \in X$ there is an open U in X such that $x \in U$ and $U \smallsetminus \{x\}$ is open.

(this T_D is a separation condition situated strictly between T_0 and T_1). Also, we will consider the T_D and T_1 relaxed by "removing T_0": Set $\widetilde{x} = \bigcap\{U \mid U \text{ open},\ x \in U\}$ (in other words, $\widetilde{x}$ is the class of x in the equivalence relation $x \sim y \ \equiv\ \overline{\{x\}} = \overline{\{y\}}$). A space is

T_{D-0} : if for each $x \in X$ there is an open U in X such that $x \in U$ and $U \smallsetminus \widetilde{x}$ is open,

and

T_{1-0} : if $\overline{\{x\}} = \overline{\{y\}}$ implies $\widetilde{x} = \widetilde{y}$.

(Note that T_{1-0} is the symmetry condition from [**4**]; for a discussion of the properties T_D and T_{D-0} see also [**9**].)

2. Locally minimal prime filters

2.1. Recall that each complete filter F on a frame L can be expressed as

$$F = \{u \mid u \nleq a\}$$

where $a \in L$ is a meet irreducible element (see [**7**]; it suffices to take $a = \bigvee\{v \mid v \notin F\}$).

2.2. The prime filters F and prime ideals on a frame L are in a one-one correspondence given by

$$F \mapsto J = L \smallsetminus F, \quad J \mapsto F = L \smallsetminus J.$$

Since maximal ideals are exactly those J for which one has the implication

$$\forall u \in J,\ u \vee v \neq 1 \quad \Rightarrow \quad v \in J,$$

and since each maximal ideal is prime, we have the *minimal* prime filters characterized by the implication

$$\forall u \notin F,\ u \vee v \neq 1 \quad \Rightarrow \quad v \notin F. \tag{2.2.1}$$

Now let the minimal prime filter be complete and let a be the associated meet-irreducible element. Then, as a is largest in J, the implication (2.2.1) reduces to

$$a \vee v \neq 1 \Rightarrow v \leq a. \tag{2.2.2}$$

2.3. A prime filter F on L is said to be *locally minimal* if there is a $b \in F$ such that $F_b = F \cap \downarrow b$ is a minimal prime filter on $\downarrow b = \{x \mid x \leq b\}$.

Lemma. *Let F be a complete filter which is locally minimal prime. Let a be the element from 2.1 and b that from the definition of local minimality. Set $a_1 = a \wedge b$. Then we have the implication*

$$v \leq b \ \ \&\ \ a_1 \vee b \neq b \quad \Rightarrow \quad v \leq a_1.$$

Proof: As obviously $F_b = \{u \mid u \leq b,\ u \not\leq a_1\}$, the implication follows from (2.2.2). $\square$

2.4. Let X be a topological space. For $x \in X$ denote by $\lambda(x)$ the complete filter $\{U \mid U \text{ open},\ U \ni x\}$.

Proposition. *Let X be a topological space. Then each complete filter F on X which is locally minimal prime is $\lambda(x)$ for some $x \in X$.*

Proof: We have open A and B such that $B \not\subseteq A$, $F = \{U \text{ open} \mid U \not\subseteq A\}$ and for all open $V \subseteq B$ and $A_1 = A \cap B$,

$$A_1 \cup V \neq B \quad \Rightarrow \quad V \subseteq A_1. \tag{2.4.1}$$

Claim. *For each $x \in B \smallsetminus A_1$, $B = A_1 \cup (\overline{\{x\}} \cap B)$.*

(suppose there are $x, y \in B \smallsetminus A_1$ with $y \notin \overline{\{x\}}$. Then there is an open W such that $x \notin W \ni y$. Set $V = W \cap B$. Then $A_1 \cup V \neq B$ but $V \not\subseteq A_1$, contradicting (2.4.1).)

Now take any $x \in B \smallsetminus A_1$. Let $U \in F$. Then $U \cap B \not\subseteq A_1$, hence $U \cap B \cap \overline{\{x\}} \neq \emptyset$ and hence $x \in U \cap N \subseteq U$. On the other hand, if $x \in U$ then $x \in U \cap B$ and hence $U \cap B \not\subseteq A_1$, that is, $U \cap B \in F_A \subseteq F$ and hence $U \in F$. Thus, $F = \lambda(x)$. $\square$

2.5. Proposition. *A space X is T_{D-0} iff the complete filters F which are locally minimal primes are exactly the $\lambda(x)$, $x \in X$.*

Proof: Let X be T_{D-0} and $x \in X$. Choose an open B such that $x \in B$ and $B \smallsetminus \widetilde{x}$ is open. Let an open $V \subseteq B$ be such that $V \cup U \neq B$ for all U open such that $x \in U \subseteq B$. Then $V \cup (B \smallsetminus \widetilde{x}) \neq B$ and hence $x \notin V$. Thus, $\lambda(x)$ is minimal.

On the other hand, let the second statement hold. Take an $x \in X$ and the B witnessing the local minimality of $\lambda(x)$. As $B \smallsetminus \overline{\{x\}}$ is the largest open $U \subseteq B$ such that $x \notin U$, the minimality condition yields the implication

$$(B \smallsetminus \overline{\{x\}}) \cup V \neq B \quad \Rightarrow \quad x \notin V.$$

Thus, if $x \in V$, V contains any $y \in B$, $y \in \overline{\{x\}}$; in other words, $x \in \overline{\{y\}}$ for any $y \in B$ such that $y \in \overline{\{x\}}$ and hence $B \smallsetminus \overline{\{x\}} = B \smallsetminus \widetilde{x}$. $\square$

2.6. Proposition. *A space X is T_{1-0} iff the complete filters F which are minimal primes are exactly the $\lambda(x)$, $x \in X$.*

Proof: The first part is obtained similarly as in 2.5 setting $B = X$. The second part is obvious: If X is not T_{1-0}, there are x and y with $x \in \overline{\{y\}}$ and $y \notin \overline{\{x\}}$. Then $\lambda(x) \subsetneq \lambda(y)$ and hence $\lambda(y)$ is not minimal. $\square$

2.7. To summarize: We have the inclusions

locally minimal prime complete filters $\subseteq$
$\subseteq$ centred filters ("real points") $\subseteq$
$\subseteq$ complete filters ("spectral points").

The second inclusion is an equality iff the space in question is weakly sober (which is well known, see also [**9**]), and the first one is an equality iff it is T_{D-0}.

3. Reconstruction

3.1. Recall that a *spectrum* of a frame L is the topological space

$$\Sigma L = (\{F \mid F \text{ complete filter on } L\}, \{\Sigma_a \mid a \in L\})$$

where $\Sigma_a = \{f \mid a \in F\}$ and that for a space X we have the natural continuous map

$$\lambda : X \to \Sigma\mathfrak{O}(X)$$

defined by $\lambda(x) = \{U \mid x \in U\}$.

3.2. If X is not T_0 it obviously cannot be reconstructed from the lattice $\mathfrak{O}(X)$: no two points x, y such that $\widetilde{x} = \widetilde{y}$ can be told apart by the open sets of X. If X is

T_0, however, the mapping λ from 3.1 is an embedding of a subspace. Thus, if there is a formula Φ such that

$$\text{(3.2.1)} \qquad \text{a complete filter } F \text{ is } \lambda(x) \text{ for an } x \in X \text{ iff } \Phi(F)$$

(where $\Phi(F)$ stands for "Φ holds for F"),

then X can be reconstructed as the subspace of $\Sigma\mathfrak{O}(L)$ constituted by the F with $\Phi(F)$.

Let $\mathcal{C}$ be a class of topoplogical spaces. We say that Φ *is a reconstruction formula for $\mathcal{C}$* if (3.2.1) holds for any $X \in \mathcal{C}$.

3.3. The following is a standard fact

PROPOSITION. *The void formula is the reconstruction formula for the class of all sober spaces.*

(See 2.1: as each meet irreducible is of the form $X \smallsetminus \overline{\{x\}}$, $F = \{U \mid U \nsubseteq X \smallsetminus \overline{\{x\}}\} = \{U \mid x \in U\} = \lambda(x)$.)

3.4. From 2.5 we immediately obtain

PROPOSITION. *The formula*

$$\Phi(F) \quad \equiv \quad F \textit{ is a locally minimal prime filter}$$

is a reconstruction formula for the class of all T_D spaces.

3.5. By 2.6 we have a simpler reconstruction formula for the class of all T_1 spaces, namely

$$\Phi(F) \quad \equiv \quad F \text{ is a minimal prime filter.}$$

3.6. A reconstruction formula allows for reconstructing continuous maps as well as spaces. We have

PROPOSITION. *Let Φ be a reconstruction formula for a class $\mathcal{C}$. Then, for $X, Y \in \mathcal{C}$, a frame homomorphism $h : \mathfrak{O}(Y) \to \mathfrak{O}(X)$ is equal to $\mathfrak{O}(f)$ for a continuous $f : X \to Y$ iff $\Phi(F)$ implies $\Phi(h^{-1}(F))$ for any $F \in \Sigma\mathfrak{O}(X)$; if this holds, the f is uniquely determined.*

PROOF: If $h = \mathfrak{O}(f)$ and $\Phi(F)$ then $F = \lambda(x)$ and we have

$$h^{-1}(F) = \{U \mid x \in f^{-1}(U)\} = \{U \mid f(x) \in U\} = \lambda(f(x)).$$

On the other hand, let the condition hold. Then we have

$$h^{-1}(\lambda(x)) = \{U \mid h(U) \in \lambda(x)\} = \{U \mid x \in h(U)\} = \lambda(y)$$

for some $y \in Y$. Set $y = f(x)$. We have

$$x \in f^{-1}(U) \text{ iff } f(x) \in U \text{ iff } U \in \lambda(y) \text{ iff } x \in h(U).$$

Thus, $f^{-1}(U) = h(U)$ and hence f is continuous and $h = \mathfrak{O}(f)$. The unicity follows from T_0. □

3.7. As a reconstruction formula is assumed to be in terms of the lattice language, it has to be preserved by isomorphisms. Thus, we immediately obtain the following generalization of Thron's theorem (see [**10**]);

COROLLARY. *Let a class of spaces $\mathcal{C}$ have a reconstruction formula. Then any two $X, Y \in \mathcal{C}$ such that $\mathfrak{O}(X) \cong \mathfrak{O}(Y)$ are homeomorphic.*

3.8. Note : (Recall also 2.7.) For sober spaces and for T_D spaces we have more than what is said in 3.6. Namely: If X is *arbitrary* and Y sober then $f \mapsto \mathfrak{O}(f)$ is a one-one correspondence between the continuous maps $f : X \to Y$ and frame homomorphisms $h : \mathfrak{O}(Y) \to \mathfrak{O}(X)$. On the other hand, from 2.5 we immediately see that if X is T_D and Y *arbitrary* T_0 then $f \mapsto \mathfrak{O}(f)$ is a one-one correspondence between the continuous maps $f : X \to Y$ and frame homomorphisms $h : \mathfrak{O}(Y) \to \mathfrak{O}(X)$ such that $h^{-1}(F)$ is locally minimal prime for any locally minimal prime complete F.

If we drop the T_0 we have (without the unicity that if X is arbitrary (resp. T_{D-0}) and Y weakly sober (resp. arbitrary) than the frame homomorphisms frame homomorphisms $h : \mathfrak{O}(Y) \to \mathfrak{O}(X)$ (resp. frame homomorphisms $h : \mathfrak{O}(Y) \to \mathfrak{O}(X)$ such that $h^{-1}(F)$ is locally minimal prime for any locally minimal prime complete F) are exactly the $\mathfrak{O}(f)$ with $f : X \to Y$ continuous maps.

3.9. PROPOSITION. *Let $\mathcal{C}$ be a class of spaces containing all T_1-spaces and all the sober ones. Then $\mathcal{C}$ has no reconstruction formula.*

PROOF: Define a space X on the ordinal $\omega + 1$ by taking for the topology the system consisting of $\emptyset$ and all complements of finite sets $K \subseteq \omega$. Thus, $\overline{\{\omega\}} = X$. Evidently, each $X \smallsetminus K$ with $|K| \geq 2$ is meet reducible and hence besides the $X \smallsetminus \{x\}$, $x \in \omega$, we have only one more meet irreducible, namely $\emptyset = X \smallsetminus \overline{\{\omega\}}$. Hence X is sober.

The subspace Y of X carried by ω is T_1. As we have $\mathfrak{O}(X) \cong \mathfrak{O}(Y)$, there cannot be, by 3.7, a reconstruction formula even for the class $\{X, Y\}$. □

4. More notes on minimality and local minimality

4.1. The fact of 2.5 accounts for a T_1-space being naturally embedded into its Wallman compactification (which is the set of all minimal prime filters with the natural topology – see, e.g., [**3**], the immediate translation of the classical construction yields, of course, the set of all maximal ideals first), and also for at least T_{1-0} being necessary for having such an embedding.

4.2. As there are T_1 spaces which are not sober, the embedding into the Wallman compactification concerns all true points of X but not necesarily all the spectrum ones. More generally, the question naturally arises as to for which spaces all the spectrum is contained in the set of all locally minimal prime filters.

LEMMA. *Let L be a frame. Let each complete F on L be locally minimal. Then ΣL is T_D.*

PROOF: For a complete F consider an $a \in F$ such that F_a is minimal on $\downarrow a$. Thus we have the implication $G \in \Sigma_a \;\Rightarrow\; F \subseteq G$ and hence if $a \in G$ and $G \in \overline{\{F\}}$ then $G = F$. Thus, $\Sigma_a \smallsetminus \{F\} = \Sigma_a \smallsetminus \overline{\{F\}}$. □

PROPOSITION. *For a T_0 space X, each complete filter F on $\mathfrak{O}(X)$ is locally minimal prime iff X is sober and T_D.*

PROOF: If X is sober and T_D then the first statement holds by 2.5. On the other hand, if each complete filter is locally minimal prime then by Lemma $\Sigma\mathfrak{O}(X)$ is T_D. $\Sigma\mathfrak{O}(X)$ is the soberification of X and being T_D it cannot be a soberification of a proper subspace (see II.1.7 in [**7**]). Thus, $X \cong \Sigma\mathfrak{O}(X)$. □

COROLLARY. *A spatial frame L is isomorphic to $\mathfrak{O}(X)$ for a sober T_D space X iff ΣL is T_D.*

4.3. Recall from [**6**] the notions of *subfit* and *fit*:

(subfit) $$a \not\leq b \quad \Rightarrow \quad \exists c,\ a \vee c = 1 \neq b \vee c,$$

and

(fit) $$a \not\leq b \quad \Rightarrow \quad \exists c,\ a \vee c = 1 \text{ and } c \to b \neq b$$

(where $c \to b$ is the Heyting operation in L; this second formula is a translation of the characterization theorem 2.2 in [**6**]).

Further, following [**2**] let us say that a frame is *weakly Hausdorff* if it is subfit and, moreover, satisfies (notation from [**2**])

(S_2'): if $a \vee c = 1 \neq a, c$ then there are u, v such that $u \wedge v = 0$, $u \not\leq a$ and $v \not\leq c$.

Note that weakly Hausdorff is implied by Hausdorff (in the sense of Isbell) plus subfit (see [**8**]) and is incompatible with fit (this last must be folklore; by 5.4 in [**5**] and [**2**] it is incompatible already for spaces).

4.4. As for a T_1-space X, $\mathfrak{O}(X)$ is subfit, it is not generally true that a complete filter of a subfit L be minimal prime. On the other hand, it is easy to see that the complete filters of a regular L are minimal primes. The proposition below comes closer.

LEMMA. *Let a frame L be fit or weakly Hausdorff. Then each meet irreducible element is a co-atom.*

PROOF: I. Let L be fit and let b be meet irreducible. Let $b < a$. Take a c such that $a \vee c = 1$ and $c \to b \neq b$. As $c \wedge (c \to b) \leq b$ and $b < c \to b$, we have, by meet irreducibility $c \leq b$. Thus, $a = a \vee b \geq a \vee c = 1$.

II. Let L be weakly Hausdorff. Let $b < a < 1$. Then, by subfitness, there is a c' such that $b \vee c' \neq 1 = a \vee c'$. Set $c = b \vee c'$. Then $a \vee c = 1 \neq a, c$ and hence there are u, v such that $u \wedge v = 0$, $u \nleq a$ and $v \nleq c$. Then $u, v \nleq b$ and $b = (b \vee u) \wedge (b \vee v)$ is not meet irreducible. □

PROPOSITION. *If L is fit or weakly Hausdorff then each complete filter on L is minimal prime.*

PROOF: Let F be complete, hence $F = \{x \mid c \nleq b\}$ for a meet irreducible b. Recall 2.2. If y is such that $x \vee y \neq 1$ for all $x \leq b$ we have in particular $y \vee b \neq 1$ and hence, as $b \leq y \vee b$, $y \vee b = b$ and $y \leq b$. □

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