Free Minor Closed Classes and the Kuratowski theorem.

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Abstract

Free-minor closed classes [2] and free-planar graphs [3] are considered. Versions of Kuratowski-like theorem for free-planar graphs and Kuratowski theorem for planar graphs are considered.

We are using usual definitions of the graph theory [1]. Considering graph topologically and Kuratowski theorem, we use the notion of minor following the theory of Robertson and Seymour[2]. We say, that a graph G is a minor of a graph H, denoting it by $G \prec H$, if G can be obtained from H by edge contractions from a subgraph of H, i.e. G can be obtained by vertex deletions, edge deletions and edge contractions from H.

A class of graphs A is minor closed, if from $G \in A$ and $H \prec G$ follow that $H \in A$.

The set of forbidden minors of a class A is denoted by $F(A)$ which is equal to $\lfloor\{G \mid G \notin A\}\rfloor$, where $\lfloor B\rfloor$ contains only minimal minors of B: $\lfloor B\rfloor \triangleq \{G \mid H \in B \wedge H \prec G \Rightarrow H \cong G\}$.

$N_\circ(B)$ denotes the minor closed class with B as its set of forbidden minors, i.e. $N_\circ(B) \triangleq \{G \mid \forall H \in B : H \not\prec G\}$. In other words, we may say, that $N_\circ(B)$ is a minor closed class generated by its forbidden minors in B. For example, $N_\circ(K_5, K_{3,3})$ is the class of planar graphs, as it is stated by Kuratowski theorem.

Another interesting example is *free-planar graphs* [3]. A planar graph is called free-planar, if after adding an arbitrary edge it remains to be planar. In [3] without a proof is acclaimed, that the class of free-planar graphs is equal to $N_\circ(K_5^-, K_{3,3}^-)$, and its characterization in terms of the permitted 3-connected components is given. In this paper we give a proof of this characterization.

In [2] a generalization of the notion of free-planar graphs is suggested. We denote by $Free(A)$ the class of graphs that consist of all graphs which should belong to A after adding an arbitrary edge to them. It is easy to see, that, if A is minor closed, then $Free(A)$ is minor closed too [2]. Because of this we use to say, that $Free(A)$ is *free-minor-closed-class* for a minor closed class A.

In [2] Kratochvíl proved a theorem:

$$F(Free(A)) = \lfloor F(A)^- \cup F(A)^\odot \rfloor,$$

where $B^- \triangleq \{G - e \mid G \in B, e \in E(G)\}$ and $B^\odot \triangleq \{H \mid H \cong G \odot v, G \in B, v \in V(G)\}$ and operation $\odot$ [in its application $G \odot v$] denotes a non unique splitting of vertex v in G, which is the opposite operation to edge addition and its contraction [in result giving vertex v].

We may formulate the unproved statement of [3] as a theorem for class of planar graphs $Planar$:

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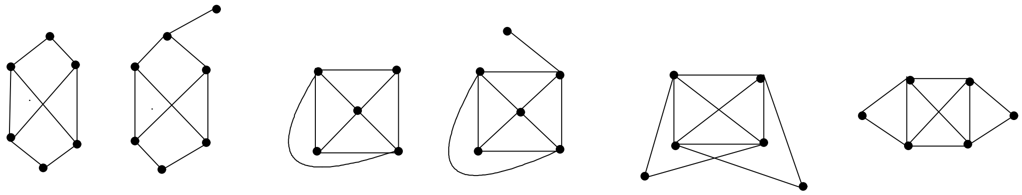


Figure 1: Graphs received applying the theorem of Kratochvíl to Kuratowski graphs.

Theorem 1. *$Free(Planar) = N_\circ(K_5^-, K_{3,3}^-)$.*

It is convenient to call the graphs $K_5^-, K_{3,3}^-$ – reduced Kuratowski subgraphs (or minors or graphs).

Now, direct application of the theorem of Kratochvíl gives the proof of theorem 1, that has been already shown in [2]. All possible graphs obtainable following the theorem are in fig. 1.

In [2] Kratochvíl suggested to prove Kuratowski's theorem from its weaker version for free-planar graphs. We do this here in two ways. One way – first specifying the class generated from reduced Kuratowski minors and then showing that it coincides with the class of free-planar graphs and then proving Kuratowski theorem itself. Second way – we prove Kuratowski theorem for free planar graphs directly, showing that with slight alteration this proof fits for a complete class of planar graphs too.

Let us set $FP = N_\circ(K_5^-, K_{3,3}^-)$ and start with proving, that graphs belonging to the class FP are free-planar, i.e. an extra edge does not make them nonplanar. Here we should explain how we are going to use Kuratowski theorem during the time we prove it. From a fact that G has a Kuratowski graph as minor we conclude that it is non-planar, i.e. we use the weak direction of Kuratowski theorem. Otherwise we conclude graphs planarity directly embedding it in the plane in cases when the graph is small or built up from 3-connected components in a certain way.

Theorem 2. *For $\forall G \in FP$ and $\forall e \notin E(G)$ $G + e$ is planar.*

Let us prove this theorem in several steps: firstly, enumerating by several theorems all possible graphs belonging to FP and thereafter, by direct check of each graph (or class of graphs) stating the assumption of the theorem.

Let us denote by ξ (see fig. 2) a particular graph $K_{2,3}$ with an extra hanging edge added to the vertex [s with hanging end t] of degree 2. Let vertices in $K_{2,3}$ of degree 3 be denoted x and y. Let the remaining vertices of degree 2 be u and v.

Let us denote by $m_i (i > 0)$ (see fig. 3) a graph, that actually is a multiedge of degree i with $i - 1$ (elementary) subdivided edges (naming it *i-multiedge*), e. g. $m_1 \cong K_2, m_2 \cong C_3, m_3 \cong K_4^-$.

Theorem 3. [Subgraph ξ theorem] *If G in FP is 3-connected, then ξ is not its minor.*

Let us first prove a lemma.

Lemma 4. *If G in FP is 3-connected, then m_4 is not its minor.*

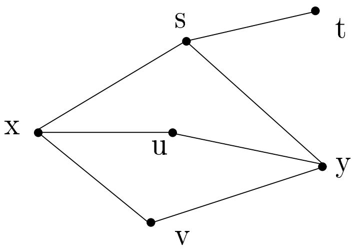


Figure 2: Graph ξ

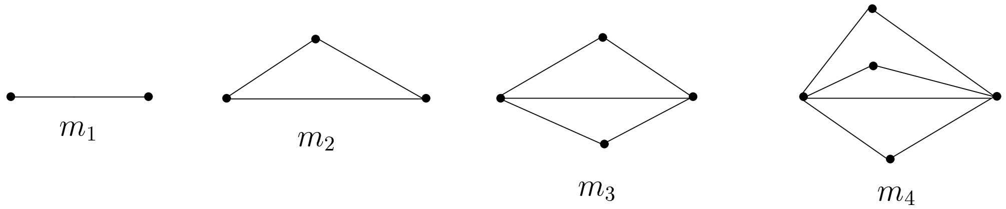


Figure 3: Graph $m_i, i = 1, ..., 4$

Proof. Let us assume, that G is 3-connected, has no one reduced Kuratowski minor, but has 4-multiedge as its minor. But, let us note, that m_4 as a minor is equivalent to K_5^- minus two incident edges at a vertex of degree four. Further, because of 3-connectivity, these absent edges should be recompensed by a chain [, uniting two vertices of degree two and going through the third one and avoiding vertices of degree three (condition of 3-connectivity)] (see fig. 4). Thus, existence of 4-multiedge implies existence of K_5^- too. □

Proof. [Proof of the subgraph ξ theorem] We can not unite t with any vertex outside the chain $x..s..y$, without giving $K_{3,3}^-$, nor unite t with x or y, because uniting t with, say, y

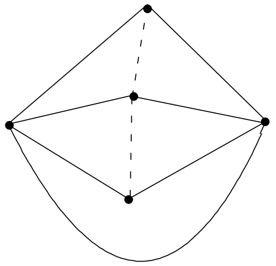


Figure 4: Graph K_5^- with two [dashed] eliminated edges is equal to m_4.

and contracting $x..s$, we get m_4. Furthermore, we can not unite t with vertices inside the chain $x..s..y$, because contracting the subchains of this chain from ends until the touch vertex and s we get m_4. Thus G can not have any minor isomorphic to ξ. □

The fact that m_4 is forbidden for graphs in FP can be formulated in the following assertion.

Corollary 5. [3-chain corollary] *Let G be 3-connected in FP. Then G is isomorphic to K_n, $n < 5$ or every pair of vertices are joined by 3 disjoined chains that contain all vertices of the graph and the remaining edges join inner vertices of different chains.*

Still, we need one more theorem that would help us to determine, which graphs belong to FP.

Theorem 6. *For every 3-connected $G \in FP$ there exists an edge e, that $G - e$ is outer planar.*

Proof. Let us assume G different from $K_n, n < 5$ and the theorem is not right, i.e. $G - e$ is not outer planar. Because of 3-chain corollary and 3-connectivity condition, arbitrary pair of vertices s and t are joined by just three chains, where all vertices are positioned on these chains. By the incorrectness assumption every of these chains contain at least one inner vertex, otherwise it should be outer planar. Let us denote these chains $s..x..t$, $s..y..t$ and $s..z..t$. Then, by the same arguments x and y join similar chains too. It is possible, supposing that all inner vertices of $s..z..t$ now are on $x..y$ which avoids s, t. But the same argument must be right also for a pair, say, x and z. It is impossible without giving K_5^-. □

Now we are ready to enumerate 3-connected graphs belonging to FP.

Theorem 7. [Prism- and wheel-graph theorem] *The only properly 3-connected graphs belonging to FP are the prism-graph [$\overline{C_6}$] and the wheel-graph [$W_k(k > 2)$].*

Proof. Let us assume G different from $K_n, n < 4$. Let us choose the edge $e = s$ (joining vertices s and t) that $G - e$ is outer planar. Then two chains $s..t$ contain all other vertices of the graph G. Let l be the length of the shortest of these chains. Case l=1 is not possible.

For l=2 all cases with the number of inner vertices on the other chain $i > 0$ are possible, giving graphs W_k $(k = i + 2 > 3)[wheelgraph]$.

Let the length of both chains be 3. This gives a possible graph $\overline{C_6}[prism - graph]$.

Let both chains be longer than 2 excluding both being equal to 3. Let the chains be $s..x_1..x_2..t$ and $s..y_1..y_2..y_3..t$. If we join x_2 with y_1 or y_2 then x_1 joined with y_3 would give $K_{3,3}^-$. By symmetry all other cases are excluded too. □

Up to now, we have considered the cases of 3-connective graphs in FP. Further, let us consider other cases and let us state, which edges in the 3-connected graphs eventually can be subdivided and which not in order to get different from 3-connected members of FP. Surely, by this reasoning we must get all non 3-connected graphs [3], because the edges that can be subdivided are just those [and only those], that can become virtual edges, when the graph is divided into 3-connected components.

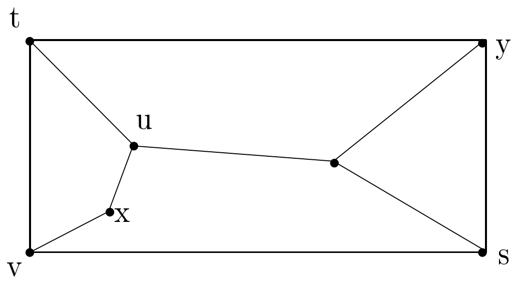


Figure 5: Prism graph with the triangle-edge subdivided, thus giving a minor $K_{3,3}^-$.

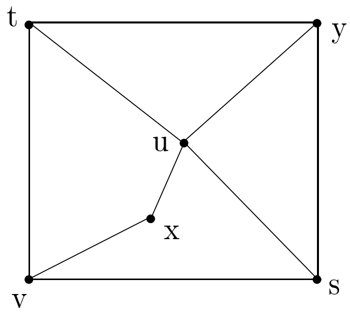


Figure 6: Wheel-graph with the spike-edge subdivided, thus giving a minor $K_{3,3}^-$.

Theorem 8. [Prism graph edge-subdivision theorem] *The edges of the triangles in the prism-graph are the only edges that can not be subdivided to get new graphs belonging to FP.*

Proof. Putting a new vertex on an edge of a triangle of the prism-graph immediately gives $K_{3,3}^-$ as a minor. See fig. 5. [Names of the vertices in $K_{3,3}$ could be seen in the fig. 7]

Putting a new vertex (or new vertices) on an edge (or edges) that does not belong to triangle does not give $K_{3,3}^-$. [There does not exist a cycle with two non elementary bridges.] ☐

Theorem 9. [Wheel graph edge-subdivision theorem] *The spike edges in the prism-graph are the only edges that can not be subdivided to get new graphs belonging to FP.*

Proof. Putting a new vertex on a spike edge of the wheel-graph gives immediately $K_{3,3}^-$ as a minor. See fig. 6.

Putting a new vertex on a rim-edge does not give $K_{3,3}^-$. [Union of the new vertex with the center by an edge gives a wheel graph of a higher degree.] ☐

Theorem 10. [Tetrahedron edge-subdivision theorem] *Two edges of K_4 which subdivided gives $K_{3,3}^-$ as a minor can not in the same time be subdivided to get new graphs belonging to FP.*

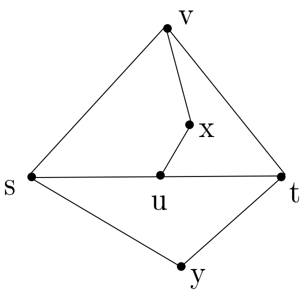


Figure 7: Tetrahedron-graph with two edges subdivided equals to $K_{3,3}^-$.

Proof. Trivially. See fig. 7. □

We are now ready to specify all the class of graphs FP by enumerating all possible graphs in it. In fact, we name all possible 3-connected graphs in FP with additionally telling which edges in them might become virtual as if the graphs that are not 3-connected would be divided into 3-connected components.

Dealing with the 3-connected components, we must admit , that they are in general multigraphs [3].

Corollary 11. *Graphs or their 3-connected components that belong to FP are [3]:*
0) C_n or $m_n, n > 2$ with all edges possibly being virtual edges;
1) W_3 with spike edges possibly being virtual edges;
2) $W_k, k > 2$ with rim edges possibly being virtual edges;
3) $\overline{C_6}$ with possible virtual edges not belonging to triangles.

Proof. Dividing the graph into 3-connected components, possible virtual edges can be only these edges which can eventually be subdivided, to give possible new members of FP. □

Proof. Completion of the proof of theorem 2 Now, it can be immediately checked, that adding an edge to the properly 3-connected graphs of FP, i.e. prism-graph and wheel-graph, can not give a nonplanar graph. This does not need use of Kuratowski theorem because we infer planarity from direct implementation in the plane.

Further, looking through all cases of corollary 11, immediately can be checked, that subdividing edges in the mentioned graphs, as it is allowed by the 3 last theorems, and adding an extra edge, can not give a graph, that is not embeddable in the plane. □

Now the theorem is proved, saying that adding an edge to G from FP always gives a planar graph. We have proved that FP is a subset of the class of free-planar graphs. Let *Planar* be class of planar graphs. The result of theorem 2 can be expressed in the following lemma.

Lemma 12. $FP \subseteq Free(Planar)$.

Furthermore, we want to show that these sets in fact coincide. For this purpose, the following lemma is useful.

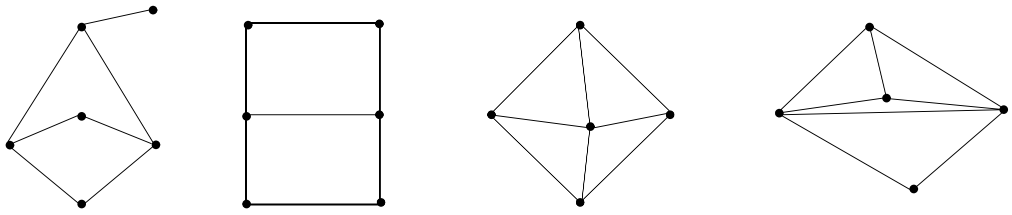


Figure 8: $K_{3,3}^-$ and K_5^- without an edge give four non isomorphic graphs

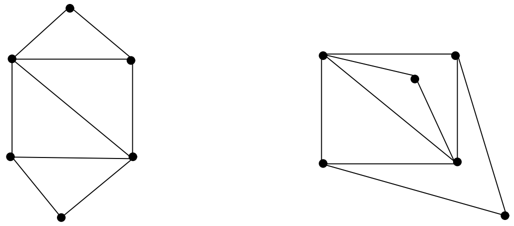


Figure 9: K_5^- with a split vertex gives two nonisomorphic graphs

Lemma 13. *$K_5^-, K_{3,3}^- \in F(Free(Planar))$.*

Proof. It is easy to see, that $K_5^-, K_{3,3}^-$ are forbidden in $Free(Planar)$ — addition of an appropriate edge gives a nonplanar graph. Further, the corresponding elimination of an edge in both graphs K_5^- and $K_{3,3}^-$, gives four possibilities for free planar graphs which are shown in fig. 8. The corresponding vertex split gives two non-trivial possibilities [see fig. 9]. $\square$

Further, from two facts, $F(FP) = \{K_5^-, K_{3,3}^-\}$ and $F(Free(Planar))$ is equal to $\{K_5^-, K_{3,3}^-, ...something\}$, there follows, that $Free(Planar) \subseteq FP$. Now, together with lemma 12 we might formulate, what may be called, the Kuratowski theorem for free-planar graphs.

Theorem 14 (Kuratowski-type theorem for free planar graphs).

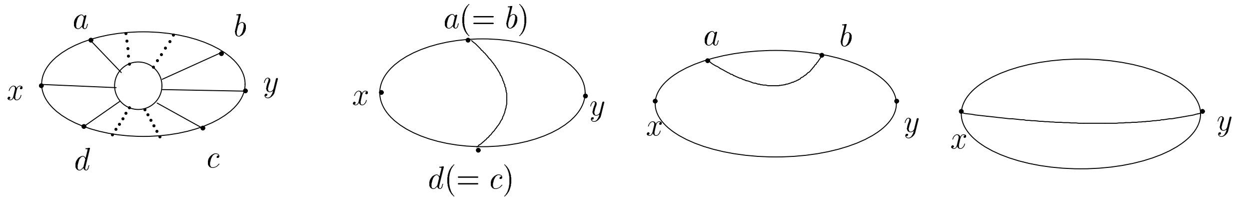
$$F(Free(Planar)) = \{K_5^-, K_{3,3}^-\}.$$

In fact, as we have already seen in the beginning, this theorem would be easy got using both traditional Kuratowski theorem and Kratochvíl's theorem [2], but now we did this proof without the use of these theorems.

Further we give a proof of the Kuratowski theorem for free-planar graphs, which serves as a proof for Kuratowski theorem for all class of planar graphs too.

Theorem 15 (Kuratowski theorem).

$$F(Planar) = \{K_5, K_{3,3}\}.$$



a) b) c) d)

Figure 10: A bridge $[x, a, b, y, c, d]$ with respect to a cycle with two distinguished vertices x and y: a) a bridge in general; b)a trivial screening bridge $[F, a, a, F, d, d]$; c) a trivial non-screening bridge $[F, a, b, F, F, F]$; d) edge x, y as a bridge with respect to C $[T, F, F, T, F, F]$.

Proof. Without loss of generality we suppose that graph G is two-connected.

Let us assume that theorem is not right and G is not free planar and it does not contain reduced Kuratowski minors. Then there is a cycle C with two vertices x, y on it and at least two bridges B_x and B_y that screen x from y on C and either they are not placeable on one side against C or they are connected [i.e. not placeable together] with an alternating [i.e. on one and other side of C] sequence $[B_1, ..., B_{2k}, k > 0]$ of non-screening [x from y] bridges. Finding of reduced Kuratowski minors would reprove the incorrectness assumption.

Let us describe bridge with sextet $[x, a, b, y, c, d]$, where values of it are either vertices on the cycle C or logical values $T(= true)$ or $F(= false)$ [see fig. 10]:

1) in place of $x(y)$ stands T if $x(y)$ is a leg[i.e. touch vertex to C] of the bridge, otherwise F;

2) $a(c)$ is nearest leg clockwise from $x(y)$, if different from $y(x)$, otherwise F;

3) $b(d)$ is nearest leg anticlockwise from $y(x)$, if different from $x(y)$, otherwise F;

The screening condition of bridge $[x, a, b, y, c, d]$ of x from y on C is – values a, b, c, d are not F. Non-screening bridges $B_i, [0 < i \leq 2k]$ are of a form $[x, a, b, y, F, F]$ or $[x, F, F, y, c, d]$ in general, but taken together with B_x and B_y in place of x and y should stand F.

There are three simple $[k = 0]$ cases and one non-simple $[k > 1]$ case to be considered:

1) In one of bridges, say B_x, both in x and y stand T. In this case K_5^- arises even when B_y is simple: $[F, a, a, F, c, c]$.

2) If $x(y)$ is T in both B_x and B_y, then K_5^- arises too: simplest case – both bridges are $[T, a, a, F, d, d]$ with minimal number of edges giving K_5^- with subdivided edge [by $y(x)$ on C].

3) Bridges B_x of form $[x, a_x, b_x, F, c_x, d_x]$ and B_y of form $[F, a_y, b_y, y, c_y, d_y]$ [where in x and (or) y may stand F] are not placeable on one side when legs' non intersecting condition –existence of two followingly specified pathes

$$x..a_1.b_1.a_2.b_2..y,$$

$$x..d_1.c_1.d_2.c_2..y.$$

– is not hold.

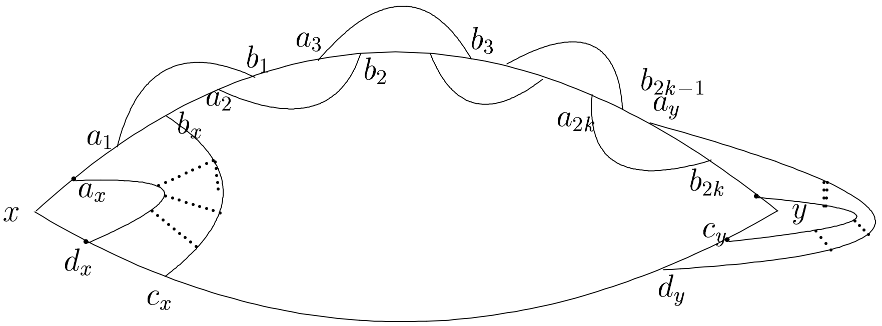


Figure 11: Case 4 in the proof of Kuratowski theorem

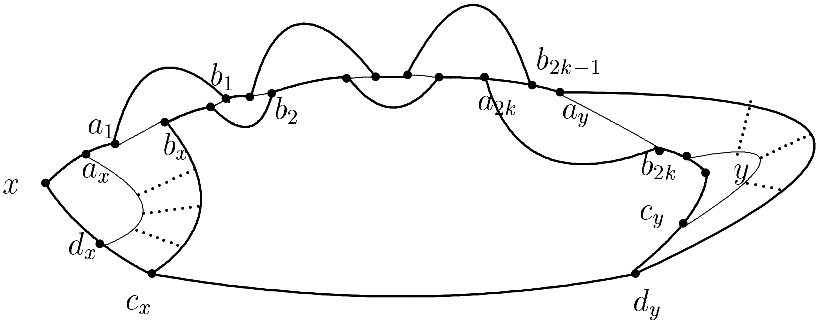


Figure 12: Minor $K_{3,3}$ bold: 1) cycle avoiding x and y, 2) chain through x outside and 3) chain through y inside

When this condition is not true, easy checkable $K^{-}_{3,3}$ arises.

4) Similarly as in case 3 bridges B_x and B_y can not be placed on one side of C, if alternating sequence of bridges, say of form, $[F, a_i, b_i, F, F, F]$ $[0 < i \leq 2k]$ join them when the condition – existence of path

$$x.a_1..b_x.a_2..b_1.a_3.. \;\; ... \;\; .a_{2k}..b_{2k-1}.a_y..b_{2k}.y$$

– is hold.

When the bridges joining condition is true, $K^{-}_{3,3}$ arises [see fig. 11]:

1) cycle

$$a_y..d_y..c_x..b_x.a_2.b_2.a_4. \;\; ... \;\; .b_{2k-2}.a_{2k}..b_{2k-1}.a_y;$$

2) a chain through x:

$$c_x..x.a_1..b_1.a_3.. \;\; ... \;\; .a_{2k-1}..b_{2k-1};$$

3) a chain through y:

$$d_y..y.b_{2k}..a_{2k} \quad .$$

It can be seen from fig. 11[and fig. 12 with $K^{-}_{3,3}$ bold] that both the cycle of supposed $K^{-}_{3,3}$ and the chain through y goes through even vertices belonging to, say, inner bridges of joining sequence of bridges. The chain through x goes through odd vertices, i.e. outer bridges of the sequence of joining bridges.

Thus G must have reduced Kuratowski graphs as its minors and $G+xy$ correspondingly – Kuratowski graph as its minor. This completes the proof of the Kuratowski theorem .

$\square$

It is easy to see that case 3 in the last proof is not necessary, i.e. it is equal to case 4 with $k = 0$.

References

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[2] J. Kratochvíl *About minor closed classes and the generalization of the notion of free-planar graphs*, personal communication, 1994, 2pp.

[3] D. Zeps. *The Triconnectivity Considered Dinamically*, KAM Series N 90-168, KAM MFF UK, Prague, 1990, 6pp.