On the signed domination in graphs*

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Abstract

We prove a conjecture of Füredi and Mubayi: For any graph G on n vertices with minimum degree r, there exists a two-coloring of the vertices of G with colors $+1$ and -1, such that the closed neighborhood of each vertex contains more $+1$'s than -1's, and altogether the number of 1's does not exceed the number of -1's by more than $O(n/\sqrt{r})$. As a construction by Füredi and Mubayi shows, this is asymptotically tight. The proof uses the partial coloring method from combinatorial discrepancy theory.

Let G be a (simple, undirected) graph on n vertices. For a vertex $v \in V(G)$, the *closed neighborhood* $N[v]$ of v is the set consisting of v and all of its neighbors. A *signed domination function* of G is any function $\chi : V(G) \to \{-1, +1\}$ such that for every vertex $v \in V(G)$, we have $\chi(N[v]) > 0$ (here and in the sequel, we use the notation $\chi(S) = \sum_{x \in S} \chi(x)$ for a subset S of the domain of χ). The *signed domination number* of G, $\gamma_s(G)$, is defined as

$$\gamma_s(G) = \min\{\chi(V(G)) : \chi \text{ is a signed domination function of } G\}.$$

This variant of the usual domination number was introduced by Dunbar et al. [4] in the early 1990s. Several researchers have studied estimates

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for the largest possible value of $\gamma_s(G)$ for r-regular n-vertex graphs (or for n-vertex graphs of minimum degree r) in dependence on n and on r (see [5] for references). Recently Füredi and Mubayi [5] proved, by a simple probabilistic argument, that for any n-vertex graph G of minimum degree r, $\gamma_s(G) \leq \left(2\sqrt{\frac{\log r}{r}} + \frac{1}{r}\right) n$ holds. They also constructed an r-regular graph on $4r$ vertices with $\gamma_s(G) \geq \frac{1}{2}\sqrt{r} - O(1)$, which shows that the upper bound is asymptotically nearly tight, up to the factor of $\sqrt{\log r}$. They conjectured that the lower bound is in fact asymptotically optimal, i.e. that all n-vertex graphs of minimum degree r have signed domination number $O(n/\sqrt{r})$. For the special case of r-regular n-vertex graphs, they derived this conjecture from a long-standing conjecture of Beck and Fiala [3] in discrepancy theory.

Here we prove the Füredi–Mubayi conjecture, using the so-called partial coloring method from combinatorial discrepancy theory (invented by Beck [2] and refined by Spencer [7]). We acutally prove the result in a more general form, for hypergraphs. As is usual in discrepancy theory, by a *coloring* of a set X we mean a mapping $\chi : X \to \{-1, +1\}$. We show

Theorem 1 *For any hypergraph $(X, \mathcal{S})$, with $|X| = |\mathcal{S}| = n$ and with $|S| \geq r$ for all $S \in \mathcal{S}$, there exists a coloring $\chi : X \to \{-1, +1\}$ such that $\chi(X) = O(\frac{n}{\sqrt{r}})$ and $\chi(S) > 0$ for all $S \in \mathcal{S}$.*

For the proof, we need to recall some concepts and results from combinatorial discrepancy theory. Let us define a *partial coloring* to be a mapping $\chi : X \to \{-1, 0, +1\}$, and a let *substantial* partial coloring be a partial coloring χ with $\chi(x) \neq 0$ for at least $\frac{1}{2}|X|$ points $x \in X$.

We will need the following auxiliary result.

Lemma 2 *Let $\mathcal{S}$ be a system of m sets on an n-point set X, $m \geq n$. Then there exists a substantial partial coloring $\chi : X \to \{-1, 0, +1\}$ with $\chi(X) = 0$ and with*

$$|\chi(S)| \leq C\sqrt{|S| \log \tfrac{2m}{n}}$$

for all $S \in \mathcal{S}$, where C is a sufficiently large constant.

Spencer [7] proved a very similar result, but without the condition $\chi(X) = 0$ and with the bound $|\chi(S)| = O\left(\sqrt{n \log(2m/n)}\right)$, i.e. without taking the set sizes into account. It is easy to modify Spencer's proof, or its technically simplified version involving entropy as in Alon and Spencer [1], to prove Lemma 2. For reader's convenience, we recall a general result of [6] on the existence of partial colorings, which implies Lemma 2 by a simple calculation (which we leave to the reader).

Proposition 3 (Entropy method) *Let $\mathcal{S}$ be a set system on an n-point set X, and let a number $\Delta_S > 0$ be given for each $S \in \mathcal{S}$. Suppose that*

$$\sum_{S \in \mathcal{S}} h\left(\frac{\Delta_S}{\sqrt{|S|}}\right) \leq \frac{n}{5}$$

holds, where the function $h(\lambda)$ can be estimated by

$$h(\lambda) \leq g(\lambda) = \begin{cases} K e^{-\lambda^2/9} & \text{if } \lambda > 0.1 \\ K \ln(\lambda^{-1}) & \text{if } \lambda \leq 0.1 \end{cases}$$

with an absolute constant K. Then there exists a substantial partial coloring $\chi : X \to \{\pm 1\}$ such that $|\chi(S)| < \Delta_S$ for all $S \in \mathcal{S}$.

Finally, we need a lemma of Füredi and Mubayi concerning ℓ-transversals. An *ℓ-transversal* of a hypergraph $(X, \mathcal{S})$ is a set $T \subseteq X$ such that $|T \cap S| \geq \ell$ for all $S \in \mathcal{S}$. The lemma is proved by a simple probabilistic argument.

Lemma 4 ([5]) *Let $(X, \mathcal{S})$ be a hypergraph with n vertices and m edges, such that all edges have size at least s, and let $\ell \leq \frac{s}{2}$. Then there exists an ℓ-transversal for $(X, \mathcal{S})$ of size at most*

$$\frac{2\ell}{s} n + \frac{\ell}{e^{\ell/4}} m.$$

Proof of Theorem 1. Throughout the proof, we may assume that both n and r are sufficiently large (for otherwise we may set $\chi(x) = 1$ for all $x \in X$).

The coloring χ is produced by an iterative procedure. Put $X_1 = X$ and execute the following step for $i = 1, 2, \ldots$ until the coloring χ is fully defined.

At the beginning of the ith step, we suppose that $X_i \subseteq X$ has already been defined and the values of χ have been determined on all of $X \setminus X_i$. If $n_i = |X_i|$ is smaller than $n/\sqrt{r}$, we put $\chi(x) = 1$ for all $x \in X_i$ and the procedure is finished.

Next, we describe the ith step supposing that $n_i > n/\sqrt{r}$. We begin with a rough outline and then we fill in the details. Let $\mathcal{S}_i$ be the set system $\mathcal{S}$ restricted to X_i. We first find a suitable small enough subset (transversal) $T_i \subseteq X_i$, which intersects all "large" sets in $\mathcal{S}_i$ in sufficiently many points, and we put $\chi(x) = 1$ for all $x \in T_i$. Then we let $\mathcal{S}_i'$ be the set system $\mathcal{S}_i$ restricted to the set $X_i' = X_i \setminus T_i$ and we apply Lemma 2 to $(X_i', \mathcal{S}_i')$, obtaining a substantial partial coloring χ_i. For some sets $S \in \mathcal{S}$, the value of $\chi_i(S \cap X_i')$ may be negative (although the magnitude is controlled by Lemma 2), but we make sure that this negative contribution to $\chi(S)$ is compensated by T_i (for "large" sets S) or by T_1 (for "small" sets S).

To finish the ith step, we let Y_i be the set of all points of X_i' where χ_i is nonzero, and we define $\chi(x) = \chi_i(x)$ for $x \in Y_i$. Finally we put $X_{i+1} = X_i' \setminus Y_i$ and we continue with the $(i+1)$st step.

Let us describe the choice of the transversal T_i. We put $r_i = r\frac{n_i}{n}$, and for $j = 0, 1, 2, \ldots$, we define $s_{ij} = 2^j r_i$. Let

$$\mathcal{S}_{ij} = \{S \in \mathcal{S}_i : s_{ij} \leq |S| < 2s_{ij}\}$$

(the $\mathcal{S}_{ij}$'s contain all "large" sets of $\mathcal{S}_i$; also note that $\mathcal{S}_{ij} = \emptyset$ for $j > \log n$, say, and so although we formally let j run to infinity, we are really considering only finitely many values of j). We put $\ell_{ij} = C_1\sqrt{s_{ij}\log(2n/n_i)}$, with an absolute constant C_1 much larger than the C from Lemma 2. We have $n_i \geq n/\sqrt{r}$, and so $s_{ij} \geq \sqrt{r}$, $\log(2n/n_i) = O(\log r)$, and since r is sufficiently large, we may assume $\ell_{ij} \leq \frac{1}{2}s_{ij}$. We apply Lemma 4 with $\ell = \ell_{ij}$ and $s = s_{ij}$ to the set system $\mathcal{S}_{ij}$, which has n_i points and at most n sets. This gives us an ℓ_{ij}-transversal $T_{ij} \subseteq X_i$ for the set system $\mathcal{S}_{ij}$ with

$$|T_{ij}| \leq \frac{2\ell_{ij}}{s_{ij}}\, n_i + \frac{\ell_{ij}}{e^{\ell_{ij}/4}} n.$$

We further note that $\ell_{ij} \geq \sqrt{s_{ij}} \geq 2^{j/2} r^{1/4}$. We can estimate

$$\frac{\ell_{ij}}{e^{\ell_{ij}/4}}\, n < \frac{n}{\ell_{ij}^4} \leq \frac{n}{2^{2j} r}$$

and

$$\frac{\ell_{ij}}{s_{ij}} n_i \quad = \quad O\left(\frac{\sqrt{\log(2n/n_i)}}{\sqrt{s_{ij}}}\, n_i\right)$$

$$
\begin{aligned}
&= O\left(\frac{\sqrt{\log(2n/n_i)}}{2^{j/2}\sqrt{r}\sqrt{n_i/n}}\, n_i\right) \\
&= O\left(\frac{n}{2^{j/2}\sqrt{r}}\left(\frac{n_i}{n}\right)^{1/3}\right).
\end{aligned}
$$

We put $T_i = \bigcup_{j=0}^{\infty} T_{ij}$. By the above estimates, we have

$$
|T_i| \le \sum_{j=0}^{\infty} |T_{ij}| = O\left(\frac{n}{\sqrt{r}}\left(\frac{n_i}{n}\right)^{1/3} + \frac{n}{r}\right). \tag{1}
$$

As was announced above, having selected T_i, we put $X_i' = X_i \setminus T_i$, we let $\mathcal{S}_i'$ be $\mathcal{S}_i$ restricted to X_i', and we apply Lemma 2 to the system $(X_i', \mathcal{S}_i')$. Since we have n_i, n in the role of n, m in the lemma, the resulting substantial partial coloring χ_i satisfies $\chi_i(X_i') = 0$ and $|\chi_i(S)| \le C\sqrt{|S| \log(2n/n_i)}$ for all $S \in \mathcal{S}_i'$.

This finishes the description of the ith step of the coloring procedure. Since we have $n_{i+1} \le \frac{1}{2} n_i$, the procedure finishes in $q+1 = O(\log r)$ steps (the last, $(q+1)$-st step colors the remaining at most $n/\sqrt{r}$ points by 1's). After the last step, we obtain a full coloring $\chi : X \to \{-1, +1\}$. It remains to show that this χ has the desired properties. Since $\chi(Y_i) = 0$ for all i, we have, using (1),

$$
\chi(X) \le \frac{n}{\sqrt{r}} + \sum_{i=1}^{q} |T_i| = O\left(\frac{n}{\sqrt{r}}\right)\left(1 + \sum_{i=1}^{q}\left(\frac{n_i}{n}\right)^{1/3}\right) + q \cdot O\left(\frac{n}{r}\right) = O\left(\frac{n}{\sqrt{r}}\right).
$$

Let $S \in \mathcal{S}$ be a set, and let $I = I(S)$ be the set of all indices i such that $|S \cap X_i| \ge r_i$ (i.e. such that S participated in some of the set systems $\mathcal{S}_{ij}$). Moreover, for $i \in I$, let $j(i)$ be the index with $S \in \mathcal{S}_{i,j(i)}$. Note that for $i \in I$, we have $|S \cap T_i| \ge \ell_{i,j(i)}$ and $|\chi_i(S \cap Y_i)| \le C\sqrt{2 s_{i,j(i)} \log(2n/n_i)} \le \frac{1}{2}\, \ell_{i,j(i)}$. Further, by the condition $|S| \ge r = r_1$ in the theorem, we have $1 \in I$ and $j(1) = 0$ (for all S). We get

$$
\begin{aligned}
\chi(S) &\ge \sum_{i \in I} |S \cap T_i| - \sum_{i=1}^{q} |\chi_i(S \cap Y_i)| \\
&\ge \sum_{i \in I}\Big(|S \cap T_i| - |\chi_i(S \cap Y_i)|\Big) - \sum_{i \notin I} |\chi(S \cap Y_i)|\,.
\end{aligned}
$$

By the above considerations, each of the summands in the first sum above is nonnegative and, moreover, the summand for $i = 1$ is at least $\frac{1}{2}\ell_{1,j(1)} = \frac{C_1}{2}\sqrt{r}$. The second sum can be estimated by

$$\sum_{i=1}^{q} C\sqrt{r_i \log \tfrac{2n}{n_i}} \leq C\sqrt{r} \cdot \sum_{i=1}^{q} \left(\frac{n_i}{n}\right)^{1/3} = O(\sqrt{r}),$$

with the constant of proportionality independent of C_1. Hence, for large C_1, we have $\chi(S) > 0$. This finishes the proof of Theorem 1. $\square$

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