

On the signed domination in graphs*

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Abstract

We prove a conjecture of Füredi and Mubayi: For any graph G on n vertices with minimum degree r , there exists a two-coloring of the vertices of G with colors $+1$ and -1 , such that the closed neighborhood of each vertex contains more $+1$'s than -1 's, and altogether the number of 1 's does not exceed the number of -1 's by more than $O(n/\sqrt{r})$. As a construction by Füredi and Mubayi shows, this is asymptotically tight. The proof uses the partial coloring method from combinatorial discrepancy theory.

Let G be a (simple, undirected) graph on n vertices. For a vertex $v \in V(G)$, the *closed neighborhood* $N[v]$ of v is the set consisting of v and all of its neighbors. A *signed domination function* of G is any function $\chi : V(G) \rightarrow \{-1, +1\}$ such that for every vertex $v \in V(G)$, we have $\chi(N[v]) > 0$ (here and in the sequel, we use the notation $\chi(S) = \sum_{x \in S} \chi(x)$ for a subset S of the domain of χ). The *signed domination number* of G , $\gamma_s(G)$, is defined as

$$\gamma_s(G) = \min\{\chi(V(G)) : \chi \text{ is a signed domination function of } G\}.$$

This variant of the usual domination number was introduced by Dunbar et al. [4] in the early 1990s. Several researchers have studied estimates

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for the largest possible value of $\gamma_s(G)$ for r -regular n -vertex graphs (or for n -vertex graphs of minimum degree r) in dependence on n and on r (see [5] for references). Recently Füredi and Mubayi [5] proved, by a simple probabilistic argument, that for any n -vertex graph G of minimum degree r , $\gamma_s(G) \leq \left(2\sqrt{\frac{\log r}{r}} + \frac{1}{r}\right)n$ holds. They also constructed an r -regular graph on $4r$ vertices with $\gamma_s(G) \geq \frac{1}{2}\sqrt{r} - O(1)$, which shows that the upper bound is asymptotically nearly tight, up to the factor of $\sqrt{\log r}$. They conjectured that the lower bound is in fact asymptotically optimal, i.e. that all n -vertex graphs of minimum degree r have signed domination number $O(n/\sqrt{r})$. For the special case of r -regular n -vertex graphs, they derived this conjecture from a long-standing conjecture of Beck and Fiala [3] in discrepancy theory.

Here we prove the Füredi–Mubayi conjecture, using the so-called partial coloring method from combinatorial discrepancy theory (invented by Beck [2] and refined by Spencer [7]). We actually prove the result in a more general form, for hypergraphs. As is usual in discrepancy theory, by a *coloring* of a set X we mean a mapping $\chi : X \rightarrow \{-1, +1\}$. We show

Theorem 1 *For any hypergraph (X, \mathcal{S}) , with $|X| = |\mathcal{S}| = n$ and with $|S| \geq r$ for all $S \in \mathcal{S}$, there exists a coloring $\chi : X \rightarrow \{-1, +1\}$ such that $\chi(X) = O\left(\frac{n}{\sqrt{r}}\right)$ and $\chi(S) > 0$ for all $S \in \mathcal{S}$.*

For the proof, we need to recall some concepts and results from combinatorial discrepancy theory. Let us define a *partial coloring* to be a mapping $\chi : X \rightarrow \{-1, 0, +1\}$, and let a *substantial* partial coloring be a partial coloring χ with $\chi(x) \neq 0$ for at least $\frac{1}{2}|X|$ points $x \in X$.

We will need the following auxiliary result.

Lemma 2 *Let \mathcal{S} be a system of m sets on an n -point set X , $m \geq n$. Then there exists a substantial partial coloring $\chi : X \rightarrow \{-1, 0, +1\}$ with $\chi(X) = 0$ and with*

$$|\chi(S)| \leq C\sqrt{|S| \log \frac{2m}{n}}$$

for all $S \in \mathcal{S}$, where C is a sufficiently large constant.

Spencer [7] proved a very similar result, but without the condition $\chi(X) = 0$ and with the bound $|\chi(S)| = O\left(\sqrt{n \log(2m/n)}\right)$, i.e. without taking the set sizes into account. It is easy to modify Spencer's proof, or its technically

By the above considerations, each of the summands in the first sum above is nonnegative and, moreover, the summand for $i = 1$ is at least $\frac{1}{2}\ell_{1,j(1)} = \frac{C_1}{2}\sqrt{r}$. The second sum can be estimated by

$$\sum_{i=1}^q C\sqrt{r_i \log \frac{2n}{n_i}} \leq C\sqrt{r} \cdot \sum_{i=1}^q \left(\frac{n_i}{n}\right)^{1/3} = O(\sqrt{r}),$$

with the constant of proportionality independent of C_1 . Hence, for large C_1 , we have $\chi(S) > 0$. This finishes the proof of Theorem 1. \square

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References

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simplified version involving entropy as in Alon and Spencer [1], to prove Lemma 2. For reader's convenience, we recall a general result of [6] on the existence of partial colorings, which implies Lemma 2 by a simple calculation (which we leave to the reader).

Proposition 3 (Entropy method) *Let \mathcal{S} be a set system on an n -point set X , and let a number $\Delta_S > 0$ be given for each $S \in \mathcal{S}$. Suppose that*

$$\sum_{S \in \mathcal{S}} h\left(\frac{\Delta_S}{\sqrt{|S|}}\right) \leq \frac{n}{5}$$

holds, where the function $h(\lambda)$ can be estimated by

$$h(\lambda) \leq g(\lambda) = \begin{cases} Ke^{-\lambda^2/9} & \text{if } \lambda > 0.1 \\ K \ln(\lambda^{-1}) & \text{if } \lambda \leq 0.1 \end{cases}$$

with an absolute constant K . Then there exists a substantial partial coloring $\chi : X \rightarrow \{\pm 1\}$ such that $|\chi(S)| < \Delta_S$ for all $S \in \mathcal{S}$.

Finally, we need a lemma of Füredi and Mubayi concerning ℓ -transversals. An ℓ -transversal of a hypergraph (X, \mathcal{S}) is a set $T \subseteq X$ such that $|T \cap S| \geq \ell$ for all $S \in \mathcal{S}$. The lemma is proved by a simple probabilistic argument.

Lemma 4 ([5]) *Let (X, \mathcal{S}) be a hypergraph with n vertices and m edges, such that all edges have size at least s , and let $\ell \leq \frac{s}{2}$. Then there exists an ℓ -transversal for (X, \mathcal{S}) of size at most*

$$\frac{2\ell}{s}n + \frac{\ell}{e^{\ell/4}}m.$$

Proof of Theorem 1. Throughout the proof, we may assume that both n and r are sufficiently large (for otherwise we may set $\chi(x) = 1$ for all $x \in X$).

The coloring χ is produced by an iterative procedure. Put $X_1 = X$ and execute the following step for $i = 1, 2, \dots$ until the coloring χ is fully defined.

At the beginning of the i th step, we suppose that $X_i \subseteq X$ has already been defined and the values of χ have been determined on all of $X \setminus X_i$. If $n_i = |X_i|$ is smaller than n/\sqrt{r} , we put $\chi(x) = 1$ for all $x \in X_i$ and the procedure is finished.

Next, we describe the i th step supposing that $n_i > n/\sqrt{r}$. We begin with a rough outline and then we fill in the details. Let \mathcal{S}_i be the set system \mathcal{S} restricted to X_i . We first find a suitable small enough subset (transversal) $T_i \subseteq X_i$, which intersects all “large” sets in \mathcal{S}_i in sufficiently many points, and we put $\chi(x) = 1$ for all $x \in T_i$. Then we let \mathcal{S}'_i be the set system \mathcal{S}_i restricted to the set $X'_i = X_i \setminus T_i$ and we apply Lemma 2 to (X'_i, \mathcal{S}'_i) , obtaining a substantial partial coloring χ_i . For some sets $S \in \mathcal{S}$, the value of $\chi_i(S \cap X'_i)$ may be negative (although the magnitude is controlled by Lemma 2), but we make sure that this negative contribution to $\chi(S)$ is compensated by T_i (for “large” sets S) or by T_1 (for “small” sets S).

To finish the i th step, we let Y_i be the set of all points of X'_i where χ_i is nonzero, and we define $\chi(x) = \chi_i(x)$ for $x \in Y_i$. Finally we put $X_{i+1} = X'_i \setminus Y_i$ and we continue with the $(i+1)$ st step.

Let us describe the choice of the transversal T_i . We put $r_i = r \frac{m}{n}$, and for $j = 0, 1, 2, \dots$, we define $s_{ij} = 2^j r_i$. Let

$$\mathcal{S}_{ij} = \{S \in \mathcal{S}_i : s_{ij} \leq |S| < 2s_{ij}\}$$

(the \mathcal{S}_{ij} 's contain all “large” sets of \mathcal{S}_i ; also note that $\mathcal{S}_{ij} = \emptyset$ for $j > \log n$, say, and so although we formally let j run to infinity, we are really considering only finitely many values of j). We put $\ell_{ij} = C_1 \sqrt{s_{ij} \log(2n/n_i)}$, with an absolute constant C_1 much larger than the C from Lemma 2. We have $n_i \geq n/\sqrt{r}$, and so $s_{ij} \geq \sqrt{r}$, $\log(2n/n_i) = O(\log r)$, and since r is sufficiently large, we may assume $\ell_{ij} \leq \frac{1}{2}s_{ij}$. We apply Lemma 4 with $\ell = \ell_{ij}$ and $s = s_{ij}$ to the set system \mathcal{S}_{ij} , which has n_i points and at most n sets. This gives us an ℓ_{ij} -transversal $T_{ij} \subseteq X_i$ for the set system \mathcal{S}_{ij} with

$$|T_{ij}| \leq \frac{2\ell_{ij}}{s_{ij}} n_i + \frac{\ell_{ij}}{e^{\ell_{ij}/4}} n.$$

We further note that $\ell_{ij} \geq \sqrt{s_{ij}} \geq 2^{j/2} r^{1/4}$. We can estimate

$$\frac{\ell_{ij}}{e^{\ell_{ij}/4}} n < \frac{n}{\ell_{ij}^4} \leq \frac{n}{2^{2j} r}$$

and

$$\frac{\ell_{ij}}{s_{ij}} n_i = O\left(\frac{\sqrt{\log(2n/n_i)}}{\sqrt{s_{ij}}} n_i\right)$$

$$\begin{aligned} &= O\left(\frac{\sqrt{\log(2n/n_i)}}{2^{j/2} \sqrt{r} \sqrt{n_i/n}} n_i\right) \\ &= O\left(\frac{n}{2^{j/2} \sqrt{r}} \left(\frac{n_i}{n}\right)^{1/3}\right). \end{aligned}$$

We put $T_i = \bigcup_{j=0}^{\infty} T_{ij}$. By the above estimates, we have

$$|T_i| \leq \sum_{j=0}^{\infty} |T_{ij}| = O\left(\frac{n}{\sqrt{r}} \left(\frac{n_i}{n}\right)^{1/3} + \frac{n}{r}\right). \quad (1)$$

As was announced above, having selected T_i , we put $X'_i = X_i \setminus T_i$, we let \mathcal{S}'_i be \mathcal{S}_i restricted to X'_i , and we apply Lemma 2 to the system (X'_i, \mathcal{S}'_i) . Since we have n_i, n in the role of n, m in the lemma, the resulting substantial partial coloring χ_i satisfies $\chi_i(X'_i) = 0$ and $|\chi_i(S)| \leq C \sqrt{|S| \log(2n/n_i)}$ for all $S \in \mathcal{S}'_i$.

This finishes the description of the i th step of the coloring procedure. Since we have $n_{i+1} \leq \frac{1}{2}n_i$, the procedure finishes in $q+1 = O(\log r)$ steps (the last, $(q+1)$ -st step colors the remaining at most n/\sqrt{r} points by 1's). After the last step, we obtain a full coloring $\chi : X \rightarrow \{-1, +1\}$. It remains to show that this χ has the desired properties. Since $\chi(Y_i) = 0$ for all i , we have, using (1),

$$\chi(X) \leq \frac{n}{\sqrt{r}} + \sum_{i=1}^q |T_i| = O\left(\frac{n}{\sqrt{r}}\right) \left(1 + \sum_{i=1}^q \left(\frac{n_i}{n}\right)^{1/3}\right) + q \cdot O\left(\frac{n}{r}\right) = O\left(\frac{n}{\sqrt{r}}\right).$$

Let $S \in \mathcal{S}$ be a set, and let $I = I(S)$ be the set of all indices i such that $|S \cap X_i| \geq r_i$ (i.e. such that S participated in some of the set systems \mathcal{S}_{ij}). Moreover, for $i \in I$, let $j(i)$ be the index with $S \in \mathcal{S}_{i,j(i)}$. Note that for $i \in I$, we have $|S \cap T_i| \geq \ell_{i,j(i)}$ and $|\chi_i(S \cap Y_i)| \leq C \sqrt{2s_{i,j(i)} \log(2n/n_i)} \leq \frac{1}{2} \ell_{i,j(i)}$. Further, by the condition $|S| \geq r = r_1$ in the theorem, we have $1 \in I$ and $j(1) = 0$ (for all S). We get

$$\begin{aligned} \chi(S) &\geq \sum_{i \in I} |S \cap T_i| - \sum_{i=1}^q |\chi_i(S \cap Y_i)| \\ &\geq \sum_{i \in I} (|S \cap T_i| - |\chi_i(S \cap Y_i)|) - \sum_{i \notin I} |\chi(S \cap Y_i)|. \end{aligned}$$