# Adjointness Aspects of the Down-Set Functor

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**Abstract :** The down-set construction, when applied to the category of Boolean frames, can be viewed as a functor into the category of frames with frame homomorphisms subject to various conditions akin to openness. We prove that it has a right adjoint, which is then given by Booleanization, exactly for near openness and one other, closely related property; a similar result is obtained for the finitary case of pseudocomplemented distributive lattices. In addition, we present a characterization of the frames which are down-set frames of Boolean frames.

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When listing the cases of special conditions on frame homomorphisms in which Booleanization has a left adjoint ([**2**]) we mistakenly dismissed *near openness*, an error pointed out by J. Niederle ([**13**]) who established the existence of the left adjoint in this case by exhibiting the appropriate universal morphism. Since this did not explicitely describe the functor involved it seemed of interest to do so, and when we identified it as the standard down-set functor, important in many different contexts, this suggested a general investigation of the adjointness behaviour of the latter.

One of the features of the down-set functor $\mathfrak{D}$ considered for arbitrary frames is that $\mathfrak{D}(h)$ inherits various properties of a frame homomorphism $h$ akin to openness, such as openness itself (that is, being a complete Heyting algebra homomorphism - see [**10**]), near, feeble and weak openness, and several other properties expressed in terms of equations involving pseudocomplements. Thus, when applied to the category **BFrm** of Boolean frames, in which the homomorphisms are automatically open and consequently have any of the properties mentioned, $\mathfrak{D}$ can be viewed as a functor into **Frm(X)** where **X** indicates the property in question required of the homomorphisms.

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Regarding adjointness, we first prove that $\mathfrak{D}$ :**BFrm**→ **Frm(***nearly open***)**, and one slight variation, has a right adjoint, namely Boleanization. The question which of the other functors have a right adjoint as well has a very short answer, namely: none; proving this is the main aim of this paper.

In addition, we present a characterization of the range of $\mathfrak{D}$ for Boolean frames and their homomorphisms, showing that near openness plays a crucial role also in this context. Finally, we derive the counterparts of our adjointness results in the finitary case.

## 1. Openness and some of its variants

**1.1.** A *frame* is a complete lattice $L$ satisfying the distributivity condition

$$a \wedge \bigvee S = \bigvee \{a \wedge b \mid b \in S\}$$

for any $a \in L$ and $S \subseteq L$, and a *frame homomorphism* $h : L \to M$ between frames $L$ and $M$ is a map preserving all joins (including the bottom 0) and all finite meets (including the top 1).

A typical example of a frame is $\mathfrak{O}X$, the lattice of open sets of a topological space $X$; further, for any continuous map $f : X \to Y$, $\mathfrak{O}f : \mathfrak{O}Y \to \mathfrak{O}X$ sending $U$ to $f^{-1}(U)$ is a frame homomorphism. Another example is provided by the *complete Boolean algebras* and *complete Boolean homomorphisms*.

The category of frames and frame homomorphisms will be denoted by

$$\mathbf{Frm}.$$

For general background on frames we refer to [**8**] or [**15**].

**1.2.** By the distributivity above, the maps $x \mapsto x \wedge a$ of a frame into itself preserve suprema and hence have right adjoints; this makes each frame into a *Heyting algebra*, that is, a lattice with an additional operation $a \to b$ such that $a \wedge b \leq c$ iff $a \leq b \to c$.

The largest element of a frame $L$ meeting $a \in L$ in 0, that is, $a \to 0$, is called the *pseudocomplement* of $a$, and denoted by $a^*$. Thus we have

$$x \leq a^* \quad \text{iff} \quad x \wedge a = 0$$

from which we easily infer that

$$\begin{aligned} &a \leq b \;\; \text{implies} \;\; b^* \leq a^*, \\ &a \leq a^{**}, \quad a^* = a^{***}, \quad 0^* = 1, \quad 1^* = 0, \\ &x \wedge a = 0 \;\; \text{iff} \;\; x \wedge a^{**} = 0, \quad \text{and} \quad (a \vee a^*)^* = 0, \end{aligned} \tag{1.2.2}$$

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and also, slightly less easily, the well known fact

$$(a \wedge b)^{**} = a^{**} \wedge b^{**}. \tag{1.2.3}$$

**1.3.** For any frame $L$, the regular elements of $L$ are the $a \in L$ such that $a = a^{**}$; we let

$$\mathfrak{B}L = \{a \in L \mid a = a^{**}\}$$

and $\beta_L : L \to \mathfrak{B}L$ the map taking $a$ to $a^{**}$. $\mathfrak{B}L$ is a complete Boolean algebra, with the same finitary meets as in $L$ and join

$$a \sqcup b = (a \vee b)^{**}, \quad \bigsqcup S = (\bigvee S)^{**} \quad \text{for any} \quad a, b \in \mathfrak{B}L \quad \text{and} \quad S \subseteq \mathfrak{B}L,$$

and $\beta_L : L \to \mathfrak{B}L$ is a frame homomorphism. $\mathfrak{B}L$ is called the *Booleanization* of $L$ and goes back to the twenties (see [**6**]); its functorial aspects were studied in [**2**] and [**3**]).

For any frame homomorphism $h : L \to M$,

$$\overline{h} : \mathfrak{B}L \to \mathfrak{B}M$$

will be the map defined by $\overline{h}(a) = h(a)^{**}$ (see also [**4**], 4.2). Note that, in general, this only preserves finitary meets, including 1, and 0.

**1.4.** In [**4**] we presented a complete classification of the conditions on frame homomorphisms which are of categorial nature and can be expressed in terms of formulas concerning pseudocomplements. They are

$$\begin{array}{lllll}
\mathbf{A}: & h(a^*) = h(a)^* & \equiv\ h(a^{**}) = h(a)^{**}, & & \\
\mathbf{B}: & h(a^{**}) = h(a^*)^* & \equiv\ h(a^*) = h(a^{**})^*, & & \\
\mathbf{C}: & h(a)^* = h(a^*)^{**} & \equiv\ h(a)^* = h(a^{**})^* & \equiv & \\
\equiv & h(a)^{**} = h(a^*)^* & \equiv\ h(a)^{**} = h(a^{**})^{**} & \equiv & h(a^{**}) \le h(a)^{**}, \\
\mathbf{D}: & h(a^*)^* = h(a^{**})^{**} & \equiv\ h(a^{**})^* = h(a^*)^{**}, & & \\
\mathbf{E}: & h(a^*) = h(a^*)^{**} & \equiv\ h(a^{**}) = h(a^{**})^{**} & \equiv & h(a^{**}) \ge h(a)^{**}.
\end{array}$$

It should be noted that $\mathfrak{O}f$, for any continuous map $f : X \to Y$, satisfies **A** or **C** iff $f$ is *nearly open* or *weakly open*, respectively (see [**4**]), and hence one refers to the $h$ satisfying **A** or **C** as to *nearly*, or *weakly, open homomorphisms*, respectively.

Furthermore, we will consider the condition

$$\mathbf{M}: \quad h(\bigwedge S) = \bigwedge h[S] \quad \text{for all} \quad S \subseteq L,$$

the *open homomorphisms* (corresponding to open continuous maps - see [**10**], [**4**]) satisfying

$$\mathbf{O} : \quad \mathbf{M} \ \& \ (h(a \to b) = h(a) \to h(b) \ \text{ for all } \ a, b)$$

and the *feebly open* homomorphisms (corresponding to feebly open continuous maps - see [**5**], semi-open in [**11**]) satisfying

**FO** : there is a mapping $g : M \to L$ such that $g(b) \neq 0$ for $b \neq 0$,
and $h(a) \wedge b \leq h(c) \Rightarrow a \wedge g(b) \leq c$.

Also, we will deal with the combinations

$$\mathbf{X} = \mathbf{M} \ \& \ \mathbf{Y}$$

where **Y** is any of **A**, **A'**, **B**, **C**, **C'**, **D**, **E**, **FO** (for **A'** and **C'** see 1.6).

**1.5.** The conditions **A** - **E** can be reformulated in terms of the behaviour of the map $\overline{h}$ from 1.3. We have

THEOREM. *A frame homomorphisms $h : L \to M$ satisfies*

- **E** *iff* $h[\mathfrak{B}L] \subseteq \mathfrak{B}M$,
- **D** *iff* $\overline{h}$ *is a lattice homomorphism,*
- **B** *iff* $\overline{h}$ *is a lattice homomorphism and* $h[\mathfrak{B}L] \subseteq \mathfrak{B}M$,
- **C** *iff* $\overline{h}\beta_L = \beta_M h$, *in which case* $\overline{h}$ *is a frame homomorphism,*
- **A** *iff* $\overline{h}\beta_L = \beta_M h$ *and* $h[\mathfrak{B}L] \subseteq \mathfrak{B}M$, *in which case* $\overline{h}$ *is a frame homomorphism.*

PROOF: As the first is trivial and **B** = **E** & **D** and **A** = **E** & **C**, it suffices to prove the statements concerning **D** and **C**.

**D** : Let $h$ satisfy **D**. Then we have, for $a, b \in \mathfrak{B}L$

$$\overline{h}(a \sqcup b) = h((a \vee b)^{**})^{**} = h((a \vee b)^{*})^{*} = h(a^{*} \wedge b^{*})^{*} = (h(a^{*}) \wedge h(b^{*}))^{***} =$$
$$= (h(a^{*})^{**} \wedge h(b^{*})^{**})^{*} = (h(a^{**})^{*} \wedge h(b^{**})^{*})^{*} = (h(a)^{**} \vee h(b)^{**})^{**} = \overline{h}(a) \sqcup \overline{h}(b).$$

Conversely, if $\overline{h}$ preserves $\sqcup$ then in particular

$$1 = \overline{h}(a^{**}) \sqcup \overline{h}(a^{*}) = (h(a^{**})^{**} \vee h(a^{*})^{**})^{**}$$

which implies

$$h(a^{**})^{*} \wedge h(a^{*})^{*} = (h(a^{**})^{**} \vee h(a^{*})^{**})^{*} = 0$$

and hence $h(a^{**})^{*} \leq h(a^{*})^{**}$. The reverse inequality follows from the obvious $h(x^{*}) \leq h(x)^{*}$.

**C** : Clearly, $h$ satisfies **C** iff $\overline{h}\beta = \beta h$. Furthermore, given this equation, we have

$$\overline{h}(\bigsqcup S) = \overline{h}\beta_L(\bigvee S) = \beta_M h(\bigvee S) = \beta_M(\bigvee h[S]) \leq \beta_M(\bigvee \overline{h}[S]) = \bigsqcup \overline{h}[S]$$



for the category of pseudocomplemented lattices with the lattice homomorphisms satisfying **X** = **A**, **B**, **C**, **D**, **E** or **FO**. For **X** = **A**, **B**, **C** or **D** we then have a functor

$$\mathfrak{B} : \mathbf{PsD(X)} \to \mathbf{Bool}$$

where **Bool** is the category of Boolean algebras.

**7.3.** In order to adjust $\mathfrak{D}$ to the present finitary situation, arbitrary down-sets have to be replaced by the *finitely generated* ones, and accordingly we define

$$\mathfrak{D}_{\text{fin}} S = \{X \subseteq S \mid 0 \in X = \downarrow Y, \quad Y \text{ finite}\},$$

for any $S \in \mathbf{PsD}$ and

$$\mathfrak{D}_{\text{fin}} h = (X \mapsto \downarrow h[X]) : \mathfrak{D}_{\text{fin}} S \to \mathfrak{D}_{\text{fin}} S'$$

for any map $h : S \to S'$ in **PsD**. Given that $X = \downarrow Y$ for finite $Y$ implies that $\bigvee X$ exists, namely: $\bigvee X = \bigvee Y$, we then have the obvious analogue of Proposition 2.2.

**7.4.** Just as in 2.4 we can view $\mathfrak{D}_{\text{fin}}$ as a functor

$$\mathfrak{D}_{\text{fin}} : \mathbf{Bool} \to \mathbf{PsD(X)}$$

for any **X** in 7.2. This is obvious by the present version of 2.2 except for **FO**; in the frame context the latter was implied by **O** which we do not have here, but there is a simple direct proof: it is easy to check that

$$G = (X \mapsto \downarrow g[X]) : \mathfrak{D}_{\text{fin}} M \to \mathfrak{D}_{\text{fin}} L$$

for the $g : M \to L$ given by the definition of **FO** (see 1.4) provides the required map.

**7.5.** Since we are now only concerned with lattice homomorphisms, and since the definition of $\sigma_L$ in 3.1 is applicable to the present, midified context because the joins involved are finite joins, we readily obtain the following counterpart of 3.2:

THEOREM. *For any category $\mathcal{C}$ such that* $\mathbf{PsD(A)} \subseteq \mathcal{C} \subseteq \mathbf{PsD(B)}$, $\mathfrak{D}_{\text{fin}} : \mathbf{Bool} \to \mathcal{C}$ *has the Booleanization* $\mathfrak{B} : \mathcal{C} \to \mathbf{Bool}$ *as right adjoint.*

On the other hand we also have, in analogy with 4.6:

THEOREM. $\mathfrak{D}_{\text{fin}} : \mathbf{Bool} \to \mathbf{PsD(X)}$ *has no right adjoint for any* **X** *other than* **A** *or* **B**.

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with supercoherent $M$, and let $k : M \to \mathfrak{D}M$ be defined by

$$k(a) = \bigcup\{\downarrow s \mid s \in \downarrow a \cap \mathfrak{S}M\}.$$

Then $k$ is a frame homomorphism, preserving finitary meets precisely by supercoherence and joins by supercompactness, such that

$$\bigvee \cdot \mathfrak{D}h \cdot k = h$$

where $\mathfrak{D}h \cdot k$ is evidently supercoherent and easily checked to be the only supercoherent $f : L \to \mathfrak{D}M$ such that $\bigvee \cdot f = h$.

Of course, this makes $\mathfrak{D}$, as a functor on the category **Frm**, the coreflection functor into $\mathbf{SCohFrm}_0$ *with coreflection maps* $\bigvee : \mathfrak{D}L \to L$.

**6.6. Remark:** Obviously, 6.1, 6.2 and 6.5 are the precise counterparts of the more familiar case where $\mathbf{M}_0$ is replaced by the category of distributive lattices, and $\mathfrak{D}$ by the ideal lattice functor. In that situation, the notions corresponding to those introduced in 6.2 are the usual ones of compactness and coherence.

## 7. The finitary analogue

**7.1.** A *pseudocomplemented lattice* is a bounded distributive lattice with an extra unary operation $x \mapsto x^*$ for which

$$x \leq a^* \quad \text{iff} \quad x \wedge a = 0.$$

Since this was all that was needed in 1.2, any of the formulas from (1.2.2) and (1.2.3) hold in this more general context.

The category of pseudocomplemented lattices and bounded lattice homomorphisms will be denoted by

**PsD**.

In analogy with 1.3 we can construct the Booleanization $\beta_L : L \to \mathfrak{B}L = \{x \in L \mid x = x^{**}\}$ for any pseudocomplemented lattice. This time, of course, the Boolean algebra $\mathfrak{B}L$ is not necessarily complete, and $\beta_L$ is just a lattice homomorphism.

**7.2.** All the requirements **A** - **E** of 1.4 make sense here and we have the full analogue of 1.5, in the present context with "frame homomorphism" replaced by "lattice homomorphism". Similarly, **FO** still makes perfect sense.

In analogy with 1.8 we put

**PsD(X)**



for any $S \subseteq \mathfrak{B}L$. □

**1.6.** Comparing the characterizations in 1.5 we note there are two obvious conditions missing which may be required of $\overline{h}$, namely

$$\textbf{A'} : \quad \overline{h} \text{ is a frame homomorphism and } h[\mathfrak{B}L] \subseteq \mathfrak{B}M,$$

and

$$\textbf{C'} : \quad \overline{h} \text{ is a frame homomorphism.}$$

Obviously

$$\textbf{A} \Rightarrow \textbf{A'} \Rightarrow \textbf{B} \quad \text{and} \quad \textbf{C} \Rightarrow \textbf{C'} \Rightarrow \textbf{D}.$$

Proposition. *None of these implications can be reversed.*

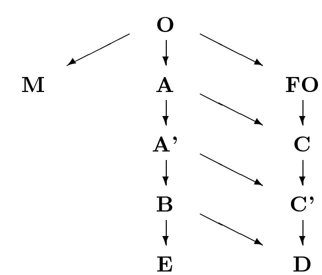
Proof: In [**4**] we presented an example showing that **B** does not imply **A**, and since it is finite, it also shows that **A'** does not imply **A**. Next, consider any extremally disconnected space $X$ with open-closed $V_i \subseteq X$ whose union is not closed, and let $h : L \to M$ be the identical embedding of $L = \mathfrak{O}X$ into the power set $M$ of $X$. As $M$ is Boolean, $\overline{h}[\mathfrak{B}L] \subseteq \mathfrak{B}M = M$ and as the regular elements of $L$ are exactly the open-closed sets in $X$, $\overline{h}$ preserves finite joins. On the other hand, however,

$$\overline{h}(\bigsqcup V_i) = h(\overline{\bigcup V_i}) = \overline{\bigcup V_i} \neq \bigcup V_i = \bigsqcup \overline{h}(V_i).$$

This example also shows that **D** does not imply **C'**.

Finally, if **C'** implied **C** we would have **A'** ≡ **E** & **C'** ⇒ **E** & **C** ⇒ **A** which we already know is not the case. □

**1.7.** The following diagram, in which all arrows indicate irreversible implications, summarizes the situation:




**1.8.** The category of all frames and all frame homomorphisms satisfying **X** where **X** is any of the conditions mentioned in 1.4 and 1.6 (including **M** and the combinations **M** & **Y**) will be denoted by

**Frm(X)**.

If **X** implies **C'** or **A'** we have a functor

$$\mathfrak{B} : \mathbf{Frm(X)} \to \mathbf{BFrm}$$

defined by $\mathfrak{B}(h) = \overline{h}$, where **BFrm** is the full subcategory of **Frm** given by the Boolean frames, that is, the complete Boolean algebras.

## 2. The down-set functor

**2.1.** Recall that on the category of frames we have the familiar down-set functor

$$\mathfrak{D} : \mathbf{Frm} \to \mathbf{Frm}$$

defined by

$$\mathfrak{D}L = \{X \subseteq L \mid 0 \in X = {\downarrow}X\} \quad \text{and} \quad \mathfrak{D}(X) = {\downarrow}h[X]$$

where ${\downarrow}X = \{y \in L \mid y \leq x \text{ for some } x \in X\}$. We note that $\mathfrak{D}$ may more generally be viewed as a functor

$$\mathfrak{D} : \mathbf{M}_0 \to \mathbf{Frm}$$

on the category $\mathbf{M}_0$ of bounded semilattices and bounded semilattice homomorphisms. There it is an important fact that the bounded homomorphisms $d_S = (x \mapsto {\downarrow}x) : S \to \mathfrak{D}S$ are the universal bounded semilattice homomorphisms, in analogy with the well-known situaion for semilattices and semilattice homomorphisms in general, provided by the down-set construction *including* the empty set (see [**8**]).

**2.2.** Proposition. *The pseudocomplements in $\mathfrak{D}L$ are given by the formula*

$$X^* = {\downarrow}(\bigvee X)^*.$$

Proof: $X \cap Y = \{0\}$ iff for all $x \in X$ and $y \in Y$, $x \wedge y = 0$, that is, iff for each $y \in Y$, $y \wedge \bigvee X = 0$ and hence $y \leq (\bigvee X)^*$. □

Corollary. *The regular elements in $\mathfrak{D}L$ are exactly the ${\downarrow}a$ with regular $a \in L$; in particular, for Boolean $L$, they are all ${\downarrow}a$, $a \in L$.*

**2.3.** Proposition. *For any open frame homomorphism $h : L \to M$, $\mathfrak{D}h$ is open.*

Proof: Let $h$ satisfy **M** so that it has a left adjoint $g : M \to L$ and define $G : \mathfrak{D}L \to \mathfrak{D}M$ by setting $G(Y) = {\downarrow}g[Y]$. $G$ is then clearly order preserving, and we will show it is a left adjoint to $\mathfrak{D}h$, that is

$$G(Y) \subseteq X \quad \text{iff} \quad Y \subseteq \mathfrak{D}h(X) = {\downarrow}h[X],$$



Proposition. *$\mathfrak{D}$ induces an equivalence between $\mathbf{M}_0$ and $\mathbf{SCohFrm}_0$ with equivalence inverse $\mathfrak{S} : \mathbf{SCohFrm}_0 \to \mathbf{M}_0$ for which $\mathfrak{S}L$ is the bounded meet semi-lattice of all supercompact elements of $L$, with natural isomorphisms $\mathfrak{D}\mathfrak{S} \to \mathrm{Id}$ and $\mathrm{Id} \to \mathfrak{S}\mathfrak{D}$ given by $\bigvee : \mathfrak{D}(\mathfrak{S}L) \to L$ and ${\downarrow} : S \to \mathfrak{S}(\mathfrak{D}S)$, respectively.*

**6.3.** One obtains a slightly different theory if supercompactness is defined *without* the restriction $S \neq \emptyset$, which is the stuation considered in [**1**]. In this sense, 0 is never supercompact whereas it is trivially so in the present setting.

**6.4.** Theorem. *The frames isomorphic to $\mathfrak{D}B$ for some Boolean frame $B$ are exactly those supercoherent frames $L$ in which the supercompact elements are the same as the regular ones, that is, $\mathfrak{S}L = \mathfrak{B}L$. Further, the homomorphisms between such frames that correspond by $\mathfrak{D}$ to maps in* **BFrm** *are exactly the nearly open supercoherent homomorphisms.*

Proof: It is inherent in 6.2 that the supercompact elements of *any* $\mathfrak{D}M$ are precisely the principal down-sets but if $M$ is a Boolean frame these are also precisely the regular elements of $\mathfrak{D}M$ (see 2.2). Conversely, if $L$ is supercoherent such that $\mathfrak{S}L = \mathfrak{B}L$ then $L \cong \mathfrak{D}(\mathfrak{S}L) = \mathfrak{D}B$ for the Boolean frame $B = \mathfrak{B}L$.

Concerning maps, since any $\mathfrak{D}h : \mathfrak{D}B \to \mathfrak{D}B'$ is actually open for $h : B \to B'$ in **BFrm** by 2.3, it is sufficient to show that any nearly open supercoherent homomorphism $f : \mathfrak{D}B \to \mathfrak{D}B'$ is in fact such a $\mathfrak{D}h$. Now 6.2 provides a *bounded semilattice homomorphism* $h : B \to B'$ such that $f = \mathfrak{D}h$, and we have to show that this is in fact a frame homomorphism - meaning that it preserves arbitrary joins. Without a loss of generality it is enough to prove this just for the $S \in \mathfrak{D}B$, and for these we have

$${\downarrow}h(\bigvee S) = f({\downarrow}\bigvee S) = f(S^{**}) = f(S)^{**} = {\downarrow}\bigvee f(S) = {\downarrow}\bigvee \mathfrak{D}h(S) = {\downarrow}\bigvee h[S]$$

so that $h(\bigvee S) = \bigvee h[S]$. □

**6.5.** The unit $\mathfrak{D}J \to \mathrm{Id}$ for the adjunction in 6.1 is given by $X \mapsto \bigvee X$. There is another aspect of the join map in our context:

$\bigvee : \mathfrak{D}L \to L$ *is the universal homomorphism to $L$ from supercoherent frames.*

To see this consider the commuting diagram

$$\begin{array}{ccc} \mathfrak{D}L & \xrightarrow{\bigvee} & L \\ {\scriptstyle \mathfrak{D}h}\uparrow & & \uparrow{\scriptstyle h} \\ \mathfrak{D}M & \xrightarrow{\bigvee} & M \end{array}$$

Theorem. *For any category $\mathcal{C}$ such that* $\mathbf{SRFrm(O)} \subseteq \mathcal{C} \subseteq \mathbf{SRFrm(B)}$, $\mathcal{C} = \mathbf{SRFrm(A)}$ *iff*

(1) $\mathfrak{D} : \mathbf{BFrm} \to \mathcal{C}$ *has a right adjoint, and*
(2) *for any frame homomorphisms $h : L \to M$ and onto $g : N \to L$, $h \in \mathcal{C}$ whenever $g$ and $hg$ belong to $\mathcal{C}$.*

Proof: ($\Rightarrow$) We have to check that $\mathbf{SRFrm(A)}$ satisfies (2). For any $a \in L$, let $a = g(x)$ since $g$ is onto. Then

$$h(a^*) = h(g(x)^*) = h(g(x^*)) = (hg)(x)^* = h(a)^*$$

since $g$ and $hg$ satisfy $\mathbf{A}$.

($\Leftarrow$) By the proof of 4.4, (1) implies that $\mathcal{C} \subseteq \mathbf{SRFrm(A)}$ so that we only have to prove the reverse inclusion. Now, for any $h : L \to M$ in $\mathbf{SRFrm(A)}$, we have the homomorphism $\overline{h} : \mathfrak{B}L \to \mathfrak{B}M$ and hence the commuting square

$$\begin{array}{ccc} \mathfrak{D}\mathfrak{B}L & \xrightarrow{\mathfrak{D}\overline{h}} & \mathfrak{D}\mathfrak{B}M \\ {\scriptstyle \varepsilon_L}\downarrow & & \downarrow{\scriptstyle \varepsilon_M} \\ L & \xrightarrow[h]{} & M \end{array}.$$

Here, $\varepsilon_L$ and $\varepsilon_M$ belong to $\mathcal{C}$ as does $\mathfrak{D}\overline{h}$, being open; hence $h\varepsilon_L = \varepsilon_M \mathfrak{D}\overline{h}$ belongs to $\mathcal{C}$, and since $\varepsilon_L$ is onto by semiregularity this shows $h \in \mathcal{C}$. ☐

**Remark:** Note that semiregularity and the above condition (2) drastically limit the possibilities inherent in the extension of Theorem 3.2.

## 6. The range of $\mathfrak{D}$

**6.1.** Here we add some observations concerning the nature of the frames $\mathfrak{D}L$. In particular we will present a simple characterization of the frames that are $\mathfrak{D}B$ for a Boolean frame $B$ and of the corresponding homomorphisms $\mathfrak{D}h$.

Recall the functor $\mathfrak{D} : \mathbf{M}_0 \to \mathbf{Frm}$ from 2.1, and note this is a left adjoint to the embedding $J : \mathbf{Frm} \subseteq \mathbf{M}_0$ forgetting joins, by the universality property mentioned there.

**6.2.** In analogy with [**1**] we call $a \in L$ *supercompact* whenever $a \le \bigvee S$, $S \neq \emptyset$, implies $a \le t$ for some $t \in S$, and $L$ itself *supercoherent* if

(1) it is generated by its supercompact elements, and
(2) the meet of any finite set of supercompact elements is supercompact.

Further, a frame homomorphism $h : L \to M$ is called *supercoherent* provided $h(a)$ is supercompact for each supercompact $a \in L$, and $\mathbf{SCohFrm}_0$ is the corresponding subcategory of $\mathbf{Frm}$. Now we obviously have



thus proving $\mathbf{M}$ for $\mathfrak{D}h$. If $G(Y) \subseteq X$ then, for any $y \in Y$, $g(y) = x \in X$, hence $y \le h(x)$ and therefore $y \in \mathfrak{D}h(X)$, showing that $Y \subseteq \mathfrak{D}h(X)$. Conversely, if this holds then, for any $x \in G(Y)$, $x \le g(y)$ for some $y \in Y$, but also $y \le h(z)$ for some $z \in X$ so that $x \le gh(z) \le z$ and therefore $x \in X$, showing that $G(Y) \subseteq X$.

Further, let $h$ also preserve the Heyting operation. To prove the same for $\mathfrak{D}h$ we only have to show that $\mathfrak{D}h(X) \to \mathfrak{D}h(Y) \subseteq \mathfrak{D}h(x \to Y)$ as the reverse inclusion holds for any frame homomorphism. For this, note first that the Heyting operation in $\mathfrak{D}L$ is easily seen to be given by the formula

$$X \to Y = \bigcup \{\downarrow \bigwedge_{x \in X} (x \to \tau(x)) \mid \tau \in Y^X\}$$

where $Y^X$ is the set of all maps from $X$ to $Y$. Now, if $a \in \downarrow h[X] \to \downarrow h[Y]$ then $\downarrow a \cap \downarrow h[X] \subseteq \downarrow h[Y]$ and hence, for each $x \in X$, $a \wedge h(x) \le h(y)$ for some $y \in Y$ so that $a \le h(x) \to h(y) = h(x \to y)$; it follows that

$$a \le \bigwedge_{x \in X} h(x \to \tau(x)) = h(\bigwedge_{x \in X} (x \to \tau(x)))$$

for some $\tau \in Y^X$ and therefore $a \in \downarrow h[X \to Y]$ by the above formula for $X \to Y$. ☐

**Note :** From 2.2 and the part of the above proof concerning $\mathbf{M}$ we easily deduce that, moreover,

> *if $h : L \to M$ satisfies any of the conditions $\mathbf{X}$ listed in 1.8 then so does $\mathfrak{D}h : \mathfrak{D}L \to \mathfrak{D}M$.*

**2.4.** As any frame homomorphism between Boolean frames is automatically open, the down-set functor can be viewed as $\mathfrak{D} : \mathbf{BFrm} \to \mathbf{Frm(O)}$ and consequently also as

$$\mathfrak{D} : \mathbf{BFrm} \to \mathbf{Frm(X)}$$

for any of the conditions $\mathbf{X}$ in 1.8.

The main aim of this article is to determine which of these have right adjoints.

## 3. Existence of adjoints

**3.1.** For any frame $L$ and any Boolean frame $B$, define maps

$$\sigma_L : \mathfrak{D}\mathfrak{B}L \to L \quad \text{and} \quad \delta_B : B \to \mathfrak{B}\mathfrak{D}B$$

by putting

$$\sigma_L(X) = \bigvee X, \quad \delta_B(b) = \downarrow b$$

where the latter is justified by 2.2. Then we have

LEMMA. *All the $\sigma_L$ are frame homomorphisms satisfying* **A** *and all the $\delta_B$ are isomorphisms.*

PROOF: $\sigma_L$ obviously preserves all joins, $\sigma_L(\mathfrak{B}L)=1$, and

$$\sigma_L(X)\cap\sigma_L(Y)=\bigvee X\cap\bigvee Y=\bigvee\{x\wedge y\mid x\in X,\ y\in Y\}\subseteq$$
$$\subseteq\bigvee(X\cap Y)=\sigma_L(X\cap Y)\subseteq\sigma_L(X)\cap\sigma_L(Y).$$

Also, $\sigma_L$ satisfies **A** by 2.2. Similarly, $\delta_B$ is one-one onto by 2.2, and obviously an isomorphism since $a\le b$ iff $\downarrow a\subseteq\downarrow b$. □

**3.2.** THEOREM. *For any category $\mathcal{C}$ such that* $\mathbf{Frm(A)}\subseteq\mathcal{C}\subseteq\mathbf{Frm(A')}$, $\mathfrak{D}:\mathbf{BFrm}\to\mathcal{C}$ *has the Booleanization* $\mathfrak{B}:\mathcal{C}\to\mathbf{BFrm}$ *as right adjoint.*

PROOF: We will show that the $\sigma$ and $\delta$ from 3.1 are the units of the adjunction. First, it is easy to check that they constitute natural transformations $\mathfrak{DB}\to\mathrm{Id}$ and $\mathrm{Id}\to\mathfrak{BD}$, making the latter a natural equivalence. Next, for the composite

$$\mathfrak{D}B\xrightarrow{\mathfrak{D}\delta_B}\mathfrak{DBD}B\xrightarrow{\sigma_{\mathfrak{D}B}}\mathfrak{D}B,$$

we have

$$\sigma_{\mathfrak{D}B}(\mathfrak{D}\delta_B(\downarrow b))=\sigma_{\mathfrak{D}B}(\downarrow\delta_B(b))=\delta_B(b)=\downarrow b$$

and since the $\downarrow b$ generate $\mathfrak{D}B$, the composite is the identity. Finally take the composite

$$\mathfrak{B}L\xrightarrow{\delta_{\mathfrak{B}L}}\mathfrak{BDB}L\xrightarrow{\mathfrak{B}\sigma_L}\mathfrak{B}L.$$

For any $a=a^{**}\in L$, $(\downarrow a)^{**}=\downarrow a$ by 2.2 and hence

$$\mathfrak{B}\sigma_L(\delta_{\mathfrak{B}L}(a))=\mathfrak{B}\sigma_L(\downarrow a)=\sigma_L(\downarrow a)=a.\quad\square$$

The above proof makes it obvious that we also have the following extension of the theorem:

> *For any category $\mathcal{C}$ such that* $\mathbf{Frm(O)}\subseteq\mathcal{C}\subseteq\mathbf{Frm(A)}$, $\mathfrak{D}\mathbf{BFrm}\to\mathcal{C}$ *has* $\mathfrak{B}:\mathcal{C}\to\mathbf{BFrm}$ *as a right adjoint iff all* $\sigma_L:\mathfrak{DB}L\to L$ *belong to* $\mathcal{C}$.

**3.3. Remark:** Note that the existence of a left adjoint to Booleanization in the cases of **Frm(A)** and **Frm(A')** was proved in [**13**]; now the adjoint is explicitely described as the down-set functor.



have intersection $\{\emptyset\}$ while $\bigcup\mathcal{H}_n=\mathbb{R}$ for any $n$. In all, this leaves **X**=**A** and **X**=**A'** as claimed. □

## 5. The semiregular case

**5.1.** An $a\in L$ will be called *s-regular* if it is a join of regular elements. Recall that a frame is called *semiregular* if each of its elements is s-regular. In particular, every regular frame is semiregular.

The category of semiregular frames and all their frame homomorphisms will be denoted by **SRFrm**, and we will use the symbol

$$\mathbf{SRFrm(X)}$$

in the sense analogous to that of 1.8. Note that by 2.2 we can view the down-set functors also as

$$\mathfrak{D}:\mathbf{BFrm}\to\mathbf{SRFrm(X)}$$

since $U=\bigcup\{\downarrow b\mid b\in U\}$ for each $U\in\mathfrak{D}B$.

**5.2.** PROPOSITION. *The following are equivalent for any frame homomorphism $h$:*

(1) *$h$ satisfies* **A'**,
(2) *for each s-regular $a$, $h(a^*)=h(a)^*$,*
(3) *for each s-regular $a$, $h(a^{**})=h(a)^{**}$.*

PROOF: (1)⇒(3): If $a=\bigvee a_i$ with regular $a_i$ we have

$$h(a^{**})=\overline{h}(\bigsqcup a_i)=\bigsqcup\overline{h}(a_i)=(\bigvee h(a_i))^{**}=h(a)^{**}.$$

(3)⇒(1): Any $a\in\mathfrak{B}L$ is regular and hence $h(a)=h(a^{**})=h(a)^{**}$, showing that $h[\mathfrak{B}L]\subseteq\mathfrak{B}M$. On the other hand, for any $a_i\in\mathfrak{B}L$,

$$\overline{h}(\bigsqcup a_i)=h((\bigvee a_i)^{**})^{**}=h(\bigvee a_i)^{**}=(\bigvee h(a_i))^{**}=\bigsqcup\overline{h}(a_i).$$

(2)⇒(3): Trivial.
(3)⇒(2): For any s-regular $a$, $a\vee a^*$ is s-regular so that

$$(h(a)\vee h(a^*))^{**}=h(a\vee a^*)^{**}=h((a\vee a^*)^{**})=1,$$

hence

$$h(a)^*\wedge h(a^*)^*=(h(a)\vee h(a^*))^*=0,$$

and therefore

$$h(a)^*\le h(a^*)^{**}=h(a^*)\le h(a)^*,$$

the middle equality because $a^*$ is regular. □

**5.3.** Thus, in the case of semiregular frames, $\mathbf{A}\equiv\mathbf{A'}$. Consequently, Theorem 4.6 reduces to a single case of **X** admitting the adjunction. In fact we can do better. We have

**4.5.** For any category $\mathcal{C}$ as in the previous proposition, $\mathfrak{R}L$ can be replaced by $\mathfrak{B}L$ via the isomorphisms $\lambda_L^{-1}$, and consequently

$$\kappa_{F,L}(\xi_a^L) = \mu_a^{\mathfrak{B}L}$$

for any $a \in \mathfrak{B}L$.

Proposition. *For any $L \in \mathcal{C}$, the adjunction unit $\varepsilon_L : \mathfrak{D}\mathfrak{B}L \to L$ is given by taking joins in $L$.*

Proof: Following $\varepsilon_L = \kappa^{-1}_{\mathfrak{B}L,L}(\mathrm{id}_{\mathfrak{B}L})$ through the commuting square

$$\begin{array}{ccc} \mathcal{C}(\mathfrak{D}\mathfrak{B}L, L) & \xrightarrow{\kappa_{\mathfrak{B}L,L}} & \mathbf{BFrm}(\mathfrak{B}L, \mathfrak{B}L) \\ \downarrow & & \downarrow \\ \mathcal{C}(\mathfrak{D}F, L) & \xrightarrow[\kappa_{F,L}]{} & \mathbf{BFrm}(F, \mathfrak{B}L) \end{array}$$

induced by $\mu_a^{\mathfrak{B}L}$ for any $a \in \mathfrak{B}L$ we obtain

$$\kappa_{F,L}(\xi_a^L) = \mu_a^{\mathfrak{B}L} = \kappa_{F,L}(\varepsilon_L \mathfrak{D}\mu_a^{\mathfrak{B}L}),$$

hence $\xi_a^L = \varepsilon_L \mathfrak{D}\mu_a^{\mathfrak{B}L}$, and therefore

$$\varepsilon_L(\downarrow a) = \varepsilon_L \mathfrak{D}\mu_a^{\mathfrak{B}L}(\downarrow\diamond) = \xi_a^L(\downarrow\diamond) = a.$$

Consequently

$$\varepsilon_L(U) = \varepsilon_L(\bigcup\{\downarrow a \mid a \in U\}) = \bigvee\{\varepsilon_L(\downarrow a) \mid a \in U\} = \bigvee U$$

for any $U \in \mathfrak{D}\mathfrak{B}L$. □

**4.6.** Theorem. *For* **X** *as in 1.8, none of the down-set functors* $\mathfrak{D}$ : **BFrm** → **Frm(X)** *has a right adjoint except for* **X**=**A** *and* **X**=**A'**.

Proof: Assuming the existence of a right adjoint, the natural equivalence of 4.4 shows that

$$|\mathcal{C}(\mathfrak{D}F, L_n)| = |\mathbf{BFrm}(F, \mathfrak{R}L_n)| = |\mathfrak{R}L_n|$$

has to be a power of 2, being the number of elements of a finite Boolean algebra, and by 4.2 this eliminates any **X** between **D** and **FO** & **M** and **X** = **E**. It follows that **X** has to imply **B** so that we may assume **X** implies **A'** and $\mathfrak{R} = \mathfrak{B}$ by 4.4, with the adjunction unit $\mathfrak{D}\mathfrak{B}L \to L$ given by taking joins in $L$ by 4.5. This excludes any **X**= **Y** & **M** because in general the latter map does not preserve arbitrary meet: In $L = \mathfrak{O}(\mathbb{R})$ where $\mathbb{R}$ is the real line, the down-sets

$$\mathcal{H}_n = \{U \in \mathfrak{B}\mathfrak{O}(\mathbb{R}) \mid \mathrm{diam}(U) < \frac{1}{n}\}, \quad n = 1, 2, \cdots$$



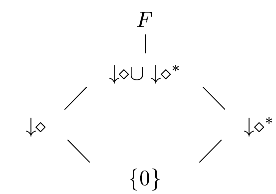
## 4. Non-existence of adjoints

**4.1.** In the following, $F$ will be the free Boolean algebra on one generator $\diamond$, and for any Boolean algebra $B$ and $x \in B$

$$\mu_x^B : F \to B$$

will be the homomorphism taking $\diamond$ to $x$. Note that

$\mathfrak{D}F$ =




**4.2.** For any category $\mathcal{C}$, $\mathcal{C}(-,-)$ will be the usual functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to$ **Set** and $|\mathcal{C}(A,B)|$ is the cardinality of the set $\mathcal{C}(A,B)$.

Further, $L_n$ will be the chain of $n+1$ elements.

Lemma. *For any category $\mathcal{C}$ such that* **Frm(FO & M)** $\subseteq \mathcal{C} \subseteq$ **Frm(D)** *we have* $|\mathcal{C}(\mathfrak{D}F, L_n)| = 2n$. *On the other hand, if* $\mathcal{C} =$ **Frm(E)** *then* $|\mathcal{C}(\mathfrak{D}F, L_2)| = 3$.

Proof: For any frame homomorphism $h : \mathfrak{D}F \to L_n$, $h(\downarrow\diamond) = 0$ or $h(\downarrow\diamond^*) = 0$ since $h(\downarrow\diamond) \wedge h(\downarrow\diamond^*) = 0$ and $L_n$ is a chain, and if $h$ satisfies **D** one cannot have both because otherwise

$$1 = 0^* = h((\downarrow\diamond)^*)^* = h((\downarrow\diamond)^{**})^{**} = h(\downarrow\diamond)^{**} = 0,$$

a contradiction. Thus either $h(\downarrow\diamond) > 0$ and $h(\downarrow\diamond^*) = 0$ or $h(\downarrow\diamond) = 0$ and $h(\downarrow\diamond^*) > 0$, and any non-zero element of $L_n$ can occur here, providing $2n$ such $h$ in **Frm(D)**. To see that they in fact belong to $\mathcal{C}$ we show that they satisfy **FO & M**. If $h(\downarrow\diamond) > 0$ let $g : L_n \to \mathfrak{D}F$ map 0 to 0 and all the other elements to $\downarrow\diamond$. Then, for any $x, z \in \mathfrak{D}F$ and $y \in L_n$, $x \wedge g(y) \not\leq z$ implies $y > 0$ and $x \cap \downarrow\diamond \not\leq z$, hence $z$ is 0 or $\downarrow\diamond^*$ and $x \geq \downarrow\diamond$, showing that $h(z) = 0$ and $h(x) \wedge y > 0$ and therefore $h(x) \wedge y \not\leq h(z)$. Since **M** is trivial here by finiteness this proves the claim if $h(\downarrow\diamond) > 0$, and the case that $h(\downarrow\diamond^*) > 0$ follows by symmetry.

In $\mathcal{C} =$ **Frm(E)** we have exactly one more $\mathfrak{D}F \to L_2$ besides the two described above, namely the one which takes *both* $\downarrow\diamond$ and $\downarrow\diamond^*$ to 0. □

**4.3.** For any frame $L$, if $a \in \mathfrak{B}L$ we let $\xi_a : \mathfrak{D}F \to L$ be the homomorphism such that

$$\xi_a(\downarrow\diamond) = a, \quad \xi_a(\downarrow\diamond^*) = a^*, \quad \xi_a(\downarrow\diamond \cup \downarrow\diamond^*) = a \vee a^*.$$

Lemma. *Every frame homomorphism $h : \mathfrak{D}F \to L$ which satisfies* **B** *is a $\xi_a$ for some $a \in \mathfrak{B}L$, and each of these is open.*

Proof: For $a = h(\downarrow\!\diamond)$ we obtain from **B** and the properties of pseudocomplementation in $\mathfrak{D}F$ that

$$a^* = h(\downarrow\!\diamond)^* = h((\downarrow\!\diamond)^{**})^* = h((\downarrow\!\diamond)^*) = h(\downarrow\!\diamond^*)$$

and hence also

$$a^{**} = h((\downarrow\!\diamond)^*)^* = h((\downarrow\!\diamond)^{**}) = h(\downarrow\!\diamond) = a,$$

showing that $a \in \mathfrak{B}L$ and $h = \xi_a$.

Further, each $\xi_a$ preserves all meets by finiteness. On the other hand, since the Heyting $\to$ generally satisfies the conditions

$$\begin{array}{ll} x \to y = 1 \ \text{ if } \ x \leq y, & x \to x^* = x^* \\ 1 \to x = x \quad \text{and} & x^* \to x = x^{**}, \end{array}$$

and since $\xi_a(\downarrow\!\diamond)^* = \xi((\downarrow\!\diamond)^*)$, the preservation of $\to$ only has to be checked for

$$(\downarrow\!\diamond \cup \downarrow\!\diamond^*) \to x = (\downarrow\!\diamond \to x) \wedge (\downarrow\!\diamond^* \to x)$$

which is easily done by going through all possible cases. □

**4.4.** Let $\mathcal{C}$ now be one of the categories in 1.8 and assume that $\mathfrak{D} : \mathbf{BFrm} \to \mathcal{C}$ has a right adjoint $\mathfrak{R}$, with adjunction maps

$$\mathcal{C}(\mathfrak{D}B, L) \xrightarrow[\cong]{\kappa_{B,L}} \mathbf{BFrm}(B, \mathfrak{R}L) \quad (B \in \mathbf{BFrm},\ L \in \mathcal{C}).$$

Proposition. *If $\mathcal{C}$ is a subcategory of* **Frm(B)** *then the morphisms $L \to M$ effect frame homomorphisms $\mathfrak{B}L \to \mathfrak{B}M$ and $\mathfrak{R}$ is naturally equivalent to the resulting functor $\mathcal{C} \to \mathbf{BFrm}$.*

Proof: Let $\lambda_L : \mathfrak{B}L \to \mathfrak{R}L$ be the map such that

$$\lambda_L(a) = (\kappa_{F,L}(\xi_a^L))(\diamond)$$

with $\xi_a^L : \mathfrak{D}F \to L$ as in 4.3. This is one-one onto by 4.3, and we first establish that it is an isomorphism by showing it preserves binary meet. For this, let $G$ be the free Boolean algebra on two generators $\diamond_1$ and $\diamond_2$, define $\psi : G \to \mathfrak{R}L$ by letting $\psi(\diamond_i) = \lambda_L(a_i)$ for fixed but arbitrarily chosen $a_1$ and $a_2$ in $\mathfrak{B}L$, and put $\varphi = \kappa_{G,L}^{-1}(\psi) : \mathfrak{D}G \to L$. Then, following $\varphi$ through the commuting square

$$\begin{array}{ccc} \mathcal{C}(\mathfrak{D}G, L) & \xrightarrow{\kappa_{G,L}} & \mathbf{BFrm}(G, \mathfrak{R}L) \\ \downarrow & & \downarrow \\ \mathcal{C}(\mathfrak{D}F, L) & \xrightarrow[\kappa_{F,L}]{} & \mathbf{BFrm}(F, \mathfrak{R}L) \end{array}$$



induced by any $\mu_x^G : F \to G$ we see that

$$(*) \qquad \kappa_{F,L}(\varphi \cdot \mathfrak{D}\mu_x^G) = \psi \cdot \mu_x^G.$$

In particular for $x = \diamond_i$ this implies

$$\kappa_{F,L}(\varphi \cdot \mathfrak{D}\mu_{\diamond_i}^G)(\diamond) = \psi(\diamond_i) = \lambda_L(a_i) = \kappa_{F,L}(\xi_{a_i}^L)(\diamond),$$

and since $\kappa_{F,L}$ is one-one we obtain

$$\varphi \cdot \mathfrak{D}\mu_{\diamond_i}^G = \xi_{a_i}^L.$$

Consequently

$$\begin{aligned} \varphi(\mathfrak{D}\mu_{\diamond_1\wedge\diamond_2}^G(\downarrow\!\diamond)) &= \varphi(\downarrow\!(\diamond_1 \wedge \diamond_2)) = \varphi(\downarrow\!\diamond_1 \cap \downarrow\!\diamond_2) = \varphi(\downarrow\!\diamond_1) \wedge \varphi(\downarrow\!\diamond_2) = \\ &= \varphi(\mathfrak{D}\mu_{\diamond_1}^G(\downarrow\!\diamond)) \wedge \varphi(\mathfrak{D}\mu_{\diamond_2}^G(\downarrow\!\diamond)) = \xi_{a_1}^L(\downarrow\!\diamond) \wedge \xi_{a_2}^L(\downarrow\!\diamond) = a_1 \wedge a_2, \end{aligned}$$

and by 4.3 this shows that we also have

$$\varphi \cdot \mathfrak{D}\mu_{\diamond_1\wedge\diamond_2}^G = \xi_{a_1\wedge a_2}^L.$$

Finally, using this together with $(*)$ for $x = a_1 \wedge a_2$ we have

$$\begin{aligned} \lambda_L(a_1 \wedge a_2) &= \kappa_{F,L}(\xi_{a_1\wedge a_2}^L)(\diamond) = \kappa_{F,L}(\varphi \cdot \mathfrak{D}\mu_{\diamond_1\wedge\diamond_2}^G)(\diamond) = \psi\mu_{\diamond_1\wedge\diamond_2}^G(\diamond) = \\ &= \psi(\diamond_1 \wedge \diamond_2) = \psi(\diamond_1) \wedge \psi(\diamond_2) = \lambda_L(a_1) \wedge \lambda_L(a_2), \end{aligned}$$

as claimed.

To complete the proof consider the commuting square

$$\begin{array}{ccc} \mathcal{C}(\mathfrak{D}F, L) & \xrightarrow{\kappa_{F,L}} & \mathbf{BFrm}(F, \mathfrak{R}L) \\ \downarrow & & \downarrow \\ \mathcal{C}(\mathfrak{D}F, M) & \xrightarrow[\kappa_{F,M}]{} & \mathbf{BFrm}(F, \mathfrak{R}M) \end{array}$$

induced by any homomorphism $h : L \to M$ in $\mathcal{C}$. Then, for any $a \in \mathfrak{B}L$,

$$\mathfrak{R}(h\kappa_{F,L}(\xi_a^L)) = \kappa_{F,M}(h\xi_a^L) = \kappa_{F,M}(\xi_{h(a)}^M)$$

and hence

$$\mathfrak{R}h(\lambda_L(a)) = \lambda_M(h(a)).$$

It follows that the map $\mathfrak{B}L \to \mathfrak{B}M$, effected by $h$ in virtue of **B**, is a frame homomorphism, and that the resulting functor on $\mathcal{C}$ is naturally equivalent to $\mathfrak{R}$. □