Bandwidth of chain graphs

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Abstract

We show that there is an $O(n^2 \log n)$ algorithm to compute the bandwidth of a chain graph. Here n is the number of vertices in the graph.

1 Introduction

The BANDWIDTH problem for a graph is that of labeling its vertices with distinct integers so that the maximum difference across an edge is minimized.

A bipartite graph $G = (X, Y, E)$ is called a *chain graph* if the neighborhoods of the vertices in X form a chain, i.e., if there is an ordering of the vertices of X, say $[x_1, ..., x_p]$, such that $N(x_1) \supseteq N(x_2) \supseteq \ldots \supseteq N(x_p)$. It is easy to see that then the neighborhoods of vertices in Y also form a chain [12].

In this note we show that the bandwidth of chain graphs can be computed in polynomial time.

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2 Preliminaries

Let $G = (V, E)$ be a graph. We denote by n the number of vertices of G. We denote a bipartite graph as $G = (X, Y, E)$ where X and Y are the two color classes of G. We denote $|X| = p$ and $|Y| = q$ throughout.

Definition 1 *A* layout L *of a graph* G *is a 1-1 mapping from* V *into* $\{1, ..., |V|\}$. *The* width *of* L *is defined as* $b(G, L) = \max\{|L(u) - L(v)| \mid (u, v) \in E\}$. *The* bandwidth *of* G *is*

$$bw(G) = \min\{b(G, L) \mid L \textit{ is a layout of } G\}.$$

In general, computing the bandwidth of a graph is NP-complete [9].

As far as the fixed parameter cases are concerned we can report the following results. It can be checked in linear time if the bandwidth of a graph is at most two [3], and for general k there is an $O(n^k)$ algorithm to check if the bandwidth of a graph is at most k [5]. This could well be 'best possible' since the problem is $w[t]$-hard for every t [2].

Classes where the bandwidth is computable in polynomial time are the theta graphs [10] and caterpillars with hairs of length at most two [1]. However, for caterpillars with hairs of length at most three the problem remains NP-complete [8].

For this paper one non trivial graph class, for which the bandwidth can be computed efficiently, plays a crucial role. This is the class of interval graphs. It was shown in that the bandwidth of an interval graph can be computed efficiently in a greedy manner [11] in $O(n \log n)$ time, when the interval model of the graph is given.

Recall that a graph is *chordal* if it does not contain an induced cycle of length at least four as an induced subgraph. A special kind of chordal graphs are the interval graphs (see, e.g., [4]). We shall use the characterization of [7] for interval graph. An *asteroidal triple* (abreviated AT) of a graph is an independent set of three vertices x, y and z, such that between any two of them there exists a path P such that no vertex of P is adjacent to the third vertex of the triple. In [7] it is shown that a graph G is an interval graph if and only if G is chordal and does not contain an asteroidal triple.

In [6] it is shown that the bandwidth of a graph is equal to its *proper pathwidth*. The proper pathwidth of a graph G is defined as the minimum

[9] Papadimitriou, C.H., The NP-completeness of the bandwidth minimization problem, *Computing*, **16**, (1976), pp. 263–270.

[10] Peck, G. W. and A. Shastri, Bandwidth of theta graphs with short paths, *Discrete Mathematics* **103**, (1992), pp. 177–187.

[11] Sprague, A. P., An $O(n \log n)$ algorithm for bandwidth of interval graphs, *SIAM Journal on Discrete Mathematics*, **7** (1994), pp. 213–220.

[12] Yannakakis, M., Node deletion problems on bipartite graphs, *SIAM J. Comput.*, **10**, (1981), pp. 310–327.

cardinality of a maximum clique of a proper interval supergraph of G decreased by one.

A chordal graph H is called a *triangulation* of a graph G if H and G have the same set of vertices, and G is a subgraph of H.

Definition 2 *A bipartite graph $G = (X, Y, E)$ is called a* chain graph *if the neighborhoods of the vertices of the color class X form a* chain*, i.e., the vertices of X can be ordered $[x_1, \ldots, x_p]$ such that $N(x_1) \supseteq N(x_2) \supseteq \ldots \supseteq N(x_p)$.*

Clearly, if $G = (X, Y, E)$ is a chain graph then there exists an ordering $[x_1, \ldots, x_p]$ of X and an ordering $[y_1, \ldots, y_q]$ of Y such that $N(x_1) \supseteq \ldots \supseteq N(x_p)$ and $N(y_1) \subseteq \ldots \subseteq N(y_q)$.

Henceforth we assume these orderings of the vertices is given. If this is not the case, they can easily be computed in linear time by sorting the vertices according to their degrees.

3 Bandwidth of chain graphs

Let $G = (X, Y, E)$ be a chain graph, with vertices of X and Y ordered $[x_1, \ldots, x_p]$ and $[y_1, \ldots, y_q]$ such that $N(x_1) \supseteq \ldots \supseteq N(x_p)$ and $N(y_1) \subseteq \ldots \subseteq N(y_q)$. We assume that the graph is connected so that $Y = N(x_1)$ and $X = N(y_q)$. (If G is not connected the bandwidth is just the maximum bandwidth over all connected components.)

Definition 3 *Let $H_0, H_1, \ldots, H_{p-1}$ be the following graphs. H_0 is obtained from G by making a clique of X. For $i = 1, \ldots, p-1$, let ℓ be the smallest index such that x_{i+1} is adjacent to y_ℓ. The graph H_i is obtained from G by making a clique of $\{x_1, \ldots, x_i\}$ and of $\{y_\ell, \ldots, y_q\}$.*

Theorem 1 *$H_0, \ldots, H_{p-1}$ are interval graphs.*

Proof. The claim is clearly true for H_0.

Let $i \geq 1$, and let ℓ be the smallest index such that x_{i+1} and y_ℓ are adjacent. Notice that H_i is a split graph, i.e., $S = \{x_1, \ldots, x_i\} \cup \{y_\ell, \ldots, y_q\}$ induces a clique in H_i and $I = \{x_{i+1}, \ldots, x_p\} \cup \{y_1, \ldots, y_{\ell-1}\}$ induces an independent set in H_i. Hence the graph H_i is chordal.

Assume that H_i has an AT x, y an z. Notice that for two vertices in $\{x_{i+1}, \ldots, x_p\}$, or for two in $\{y_1, \ldots, y_{\ell-1}\}$, one neighborhood is contained in the other, and hence they cannot both be in an AT. The only other possibility is that one vertex say x of the AT is in $\{x_{i+1}, \ldots, x_p\}$, one vertex, say y, is in $\{y_1, \ldots, y_{\ell-1}\}$ and z is in S. But then every path from x to y passes through S and hence contains a neighbor of z. □

Theorem 2 *Every triangulation H of G has one of $H_0, \ldots, H_{p-1}$ as a subgraph.*

Proof. Let H be a triangulation of G. If X induces a clique in H then H_0 is a subgraph of H and if Y is a clique of H then H_1 is a subgraph of H. Assume neither is the case.

Take k maximal and t minimal such that $\{x_1, \ldots, x_k\}$ and $\{y_t, \ldots, y_q\}$ form cliques in H. We claim that x_{k+1} is not adjacent to y_{t-1}. If this were the case, then $(\{x_1, \ldots, x_{k+1}\}, \{y_{t-1}, \ldots, y_q\})$ would induce a complete bipartite subgraph in G. But every triangulation of a complete bipartite graph must be such that at least one of the color classes is a clique (otherwise there would be a chordless 4-cycle). This contradicts the fact that k is maximal and t is minimal.

Now let ℓ be the minimal index such that x_{k+1} and y_ℓ are adjacent. Then it follows that $\ell \geq t$. This proves that H_k is a subgraph of H. □

Corollary 1 *The set of all minimal triangulations of G is contained in $\{H_0, \ldots, H_{p-1}\}$.*

Theorem 3 *There exists an $O(n^2 \log n)$ time algorithm computing the bandwidth of a chain graph $G = (X, Y, E)$ with n vertices.*

Proof. Let H be triangulation of G into a proper interval graph such that the bandwidth of G is equal to the clique number minus one of H.

By Theorem 2, one of the interval graphs $H_0, \ldots, H_{p-1}$, say H_k is a subgraph of H. Since H_k is a triangulation of G we have $bw(G) \leq bw(H_k) \leq bw(H) = bw(G)$.

It is easy to see that an interval model for each H_i can be computed in linear time. Using the algorithm of [11], our algorithm computes the bandwidth of each H_i in time $O(n \log n)$. The minimum value gives, by the above argument, the bandwidth of G. □

4 Conclusion

In this note we have given an easy algorithm which computes the bandwidth of a chain graph with n vertices in $O(n^2 \log n)$ time. Chain graphs form a proper subclass of the class of bipartite permutation graphs. It would be interesting to know whether our results can be extended to the class of all (bipartite) permutation graphs.

References

[1] Assmann, S. F., G. W. Peck, M. Syslo and J. Zak, The bandwidth of caterpillars with ahirs of length 1 and 2, *SIAM Journal on Algebraic and Discrete Methods* **2**, (1981), pp. 387–393.

[2] Bodlaender, H., M. R. Fellows and M. T. hallet, Beyond NP-completenss for problems of bounded width: Hardness for the W hierarchy, *Proceedings of the* 26^{th} *STOC*, 1994, pp. 449–458.

[3] Garey, M. R., R. L. Graham, D. S. Johnson and D. E. Knuth, Complexity results for bandwidth minimization, *SIAM Journal on Applied Mathematics* **34**, (1978), pp. 477–495.

[4] Golumbic, M. C., *Algorithmic graph theorey and perfect graphs*, Academic Press, New York, 1980.

[5] Gurari, E. and I. H. Sudborough, Improved dynamic programming algorithms for the bandwidth minimization and the mincut linear arrangement problem, *Journal of Algorithms* **5**, (1984), pp. 531–546.

[6] Kaplan, H. and R. Shamir, Pathwidth, bandwidth and completion problems to proper interval graphs with small cliques. Manuscript, 1993.

[7] Lekkerkerker, C. E. and J. Ch. Boland, Representation of a finite graph by a set ofS intervals on a real line, *Fund. Math.*, **51**, (1962), pp. 45–64.

[8] Monien, B., The bandwidth minimization problem for caterpillars with hairs of length 3 is NP-complete, *SIAM Journal on Algebraic and Discrete methods* **7**, (1986), pp. 505–512.