Cycle bases for lattices of matroids with no Fano dual minor and their one-element extensions

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Abstract

In this paper we study the question of existence of a basis consisting only of cycles for the lattice $\mathbb{Z}(\mathcal{M})$ generated by the cycles of a binary matroid $\mathcal{M}$. We show that, if $\mathcal{M}$ has no Fano dual minor, then any set of fundamental circuits can be completed to a cycle basis of $\mathbb{Z}(\mathcal{M})$; moreover, for any one-element extension $\mathcal{M}'$ of such matroid $\mathcal{M}$, any cycle basis for $\mathbb{Z}(\mathcal{M})$ can be completed to a cycle basis for $\mathbb{Z}(\mathcal{M}')$.

1 Introduction

Let $\mathcal{M} = (E, \mathcal{C})$ be a binary matroid on a (finite) set E with cycle space $\mathcal{C}$; that is, $\mathcal{C}$ is a family of subsets of E which is closed under taking symmetric differences, whose members are called the *cycles* of $\mathcal{M}$. The minimal nonempty cycles are called *circuits*; then every nonempty cycle is a disjoint union of circuits. In this paper we consider the lattice[1] $\mathbb{Z}(\mathcal{M})$ generated by the cycles (or circuits) of $\mathcal{M}$; that is,

$$\mathbb{Z}(\mathcal{M}) := \{\sum_{C \in \mathcal{C}} \lambda_C \chi^C \mid \lambda_C \in \mathbb{Z} \ \forall C \in \mathcal{C}\}.$$

[1]The set $\mathbb{Z}(\mathcal{M})$ is clearly a *lattice*; that is, a discrete subgroup of $\mathbb{R}^E$.

The lattice $\mathbb{Z}(\mathcal{M})$ is called the *cycle lattice* of $\mathcal{M}$. More specifically, we are interested in the conjecture proposed by Hochstättler and Loebl [7] whether every cycle lattice admits a basis consisting only of cycles (then called a *cycle basis*). This problem has been considered by Gallucio and Loebl ([4],[5]) who showed that it has a positive answer for graphic matroids and, more generally, binary matroids with no F_7^* minor. This conjecture is, in fact, a special case of a more general problem posed by Deza, Grishukhin and Laurent ([2],[3]) in the setting of Delaunay polytopes; see Corollary 1.3 below for details.

The proof of Gallucio and Loebl [5] in the case of binary matroids with no F_7^* minor relies on Seymour's decomposition results for these matroids ([12]). We give here in Section 2 a short elementary proof for their result. Moreover, we show that a cycle basis can be obtained from any fundamental set of circuits (basis over $GF(2)$) by adding some circuits that are the binary sum of two fundamental circuits. We then consider in Section 3 one-element extensions of matroids with no F_7^* minor. We show that their cycle lattice also admits a cycle basis consisting of circuits which can be obtained by extending a cycle basis of the cycle lattice of the original matroid. Hence in both cases we construct a cycle basis consisting of circuits.

Note that a matroid $\mathcal{M}$ without an F_7^* minor has the lattice of circuits property; that is, its cycle lattice contains all even integer vectors (assuming $\mathcal{M}$ is cosimple). It remains an open problem to determine whether the cycle lattice of any binary matroid with the lattice of circuits property has a cycle basis.

In what follows, we introduce some further definitions and preliminaries that are needed in the paper. Our notation and terminology are fairly standard and can be found, e.g., in the textbooks by Oxley [11] or Welsh [14].

Definitions. Let $\mathcal{M} = (E, \mathcal{C})$ be a binary matroid. Setting

$$\mathcal{C}^* := \{D \subseteq E : |D \cap C| \in 2\mathbb{Z}\ \forall C \in \mathcal{C}\},$$

then $\mathcal{M}^* := (E, \mathcal{C}^*)$ is also a binary matroid, called the *dual* of $\mathcal{M}$; the members of $\mathcal{C}^*$ are also called the *cocycles* of $\mathcal{M}$. The minimal nonempty cocycles are called the *cocircuits* of $\mathcal{M}$. Every nonempty cycle is a disjoint union of circuits. A set $I \subseteq E$ is *independent* in $\mathcal{M}$ if it contains no circuit; the maximum cardinality of an independent set is the *rank* of $\mathcal{M}$.

Let T be a maximal independent set in $\mathcal{M}$. For $e \in E \setminus T$, we let C_e denote the *fundamental circuit* of e with respect to T (that is, C_e is the unique circuit such that $e \in C_e \subseteq T \cup \{e\}$); then $(C_e \mid e \in E \setminus T)$ is called the *fundamental set of circuits* of T.

An element $e \in E$ is a *coloop* of $\mathcal{M}$ if $\{e\}$ is a cocircuit and two elements $e, f \in E$ are said to be *coparallel* if $\{e, f\}$ is a cocircuit. A *coparallel class* P is a maximal subset of E whose elements are pairwise coparallel; it is said to be trivial if $|P| = 1$ and nontrivial otherwise. The matroid $\mathcal{M}$ is said to be *cosimple* if every cocircuit has cardinality ≥ 3.

The cycle space of a binary matroid $\mathcal{M}$ on E can be realized as the set of solutions $x \in \{0,1\}^E$ of a linear equation of the form: $Mx \equiv 0$ (modulo 2), where M is a binary matrix whose columns are indexed by E; such matrix M is called a *representation matrix* of $\mathcal{M}$.

The *Fano matroid* F_7 is the matroid on $E := \{1, \ldots, 7\}$ represented by the matrix:

$$
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1
\end{pmatrix}
\end{array}
\tag{1.1}
$$

and the *Fano dual matroid* is the dual F_7^* of F_7. For $r \geq 2$, the *projective space* $\mathcal{P}_r$ is the binary matroid represented by the $r \times (2^r - 1)$ matrix whose columns are all nonzero $0,1$-vectors of length r; hence, $\mathcal{P}_r$ has rank r and $\mathcal{P}_3 = F_7$.

Let $\mathcal{M} = (E, \mathcal{C})$ be a binary matroid and let $e \in E$. Setting

$$\mathcal{C} \backslash e := \{C \in \mathcal{C} \mid e \notin C\}, \; \mathcal{C}/e := \{C \setminus \{e\} \mid C \in \mathcal{C}\},$$

then $\mathcal{M} \backslash e := (E \setminus \{e\}, \mathcal{C} \backslash e)$ and $\mathcal{M}/e := (E \setminus \{e\}, \mathcal{C}/e)$ are binary matroids obtained from $\mathcal{M}$ by, respectively, *deleting* and *contracting* e. A *minor* of $\mathcal{M}$ is any binary matroid $\mathcal{N}$ that can be obtained from $\mathcal{M}$ by a sequence of deletions and/or contractions.

For a set $A \subseteq E$, we let χ^A denote its incidence vector with $\chi^A_e := 1$ if $e \in A$ and $\chi^A_e := 0$ if $e \in E \setminus A$. Moreover, for a finite subset $X \subseteq \mathbb{R}^E$, we set: $\mathbb{Z}(X) := \{\sum_{x \in X} \lambda_x x \mid \lambda_x \in \mathbb{Z} \; \forall x \in X\}$.

Preliminary Results. Let $\mathcal{M} = (E, \mathcal{C})$ be a binary matroid. Then, the following holds obviously for all $x \in \mathbb{Z}(\mathcal{M})$:

(i) $\displaystyle\sum_{e \in D} x_e$ is even for all cocircuits $D \in \mathcal{C}^*$,

(ii) $x_e = 0$ if e is a coloop of $\mathcal{M}$ and

(iii) $x_f = x_g$ if f and g are coparallel in $\mathcal{M}$.

Following Goddyn [6], we say that $\mathcal{M}$ has the *lattice of circuits property*[2] [3] if the above necessary conditions (i)-(iii) are already sufficient for ensuring membership of any $x \in \mathbb{Z}^E$ in $\mathbb{Z}(\mathcal{M})$. As can be easily verified, $\mathcal{M}$ has the lattice of circuits property if and only if $2\chi^P \in \mathbb{Z}(\mathcal{M})$ for every coparallel class P of $\mathcal{M}$ or, equivalently (for $\mathcal{M}$ cosimple), if the dual lattice[4] $(\mathbb{Z}(\mathcal{M}))^*$ is contained in $\frac{1}{2}\mathbb{Z}^E$. Note that F_7^* does not have the lattice of circuits property (since $\mathbb{Z}(F_7^*) \subseteq 4\mathbb{Z}^7$).

The dimension of the cycle lattice $\mathbb{Z}(\mathcal{M})$ is denoted by dim $\mathcal{M}$ and is given by

$$\dim \mathcal{M} = |E| - N - \sum_{P \text{ coparallel class}} (|P| - 1),$$

where N denotes the number of coloops of $\mathcal{M}$. Clearly, for the purpose of studying the cycle lattice $\mathbb{Z}(\mathcal{M})$, we may restrict ourselves to the case when $\mathcal{M}$ is cosimple; then $\mathbb{Z}(\mathcal{M})$ is full-dimensional.

An early result of Cunningham [1] asserts that, if $\mathcal{M}$ has no F_7^* minor then, for every element e, there exist two circuits C, C' such that $C \cap C' = \{e\}$ if and only if every cocircuit containing e has cardinality ≥ 3. This implies that every binary matroid with no F_7^* minor has the lattice of circuits property. Lovász and Seress ([9],[10]) study the lattice of circuits property in detail. In particular, they characterize the binary matroids having the lattice of circuits property and they show that, if $\mathcal{M}$ is cosimple with no $\mathcal{P}_{r+1}^*$ minor, then $2^{r-1}\mathbb{Z}^E \subseteq \mathbb{Z}(\mathcal{M})$, i.e, $(\mathbb{Z}(\mathcal{M}))^* \subseteq \frac{1}{2^{r-1}}\mathbb{Z}^E$.

Let $P \subseteq \mathbb{R}^k$ be a full-dimensional polytope with set of vertices V_P admitting the origin as a vertex. Then, P is said to be a *Delaunay polytope* if it satisfies the following conditions: (i) the set $L := \mathbb{Z}(V_P)$ is a lattice; (ii) P is inscribed on a sphere S with, say, center c and radius r (that is, $\| x - c \| = r$ for all $x \in V_P$); and (iii) $\| x - c \| \geq r$ for all $x \in L$, with equality if and only if $x \in V_P$. (Here, $\| x \|$ denotes the Euclidean norm of $x \in \mathbb{R}^k$.) Deza, Grishukhin and Laurent ([2],[3]) posed the following question:

[2]This terminology reflects the analogy with the *sums of circuits property* considered by Seymour [13], a binary matroid having the latter property if the cone generated by its cycles is completely described by some 'obvious necessary' linear conditions.

[3]Lovász and Seress [9] call a binary matroid $\mathcal{M}$ *Eulerian* if its dual $\mathcal{M}^*$ has the lattice of circuits property.

[4]The *dual* L^* of a lattice $L \subseteq \mathbb{R}^k$ consists of the vectors $x \in \mathbb{R}^k$ such that $x^T y \in \mathbb{Z}$ for all $y \in L$.

Given a Delaunay polytope P, is it always possible to find a basis of the lattice $L = \mathbb{Z}(V_P)$ consisting only of vertices of P ?

No example of a Delaunay polytope is known for which this question has a negative answer. On the other hand, a basis consisting only of vertices has been constructed for several concrete instances of Delaunay polytopes in [2]. Observe that the question addressed in this paper of existence of a cycle basis for any cycle lattice arises as a special instance of the above problem. Indeed, as we show below, the polytope $\mathrm{conv}(\mathcal{C})$ which is defined as the convex hull of the incidence vectors of the cycles of $\mathcal{M}$, is a Delaunay polytope in the lattice $\mathbb{Z}(\mathcal{M})$.

Lemma 1.2. *Let X be a finite full-dimensional subset of $\{0,1\}^k$ and set $Y := \mathbb{Z}(X) \cap \{0,1\}^k$. Then, the polytope $\mathrm{conv}(Y)$ is a Delaunay polytope.*

PROOF. The polytope $P := \mathrm{conv}(Y)$ is full-dimensional in $\mathbb{R}^k$ since X is full-dimensional. Assertion (i) holds obviously since Y consists of integer vectors. Let S denote the sphere in $\mathbb{R}^k$ with center $c := (\frac{1}{2}, \ldots, \frac{1}{2})$ and radius $r := \frac{1}{2}\sqrt{|E|}$. Then all points of Y lie on the sphere S, i.e., (ii) holds. We have that $\| x - c \|^2 - r^2 = \sum_{e \in E} x_e(x_e - 1) \geq 0$ for all $x \in \mathbb{Z}(X)$. Moreover, equality holds if and only if $x \in \{0,1\}^k$, i.e., if $x \in Y$; thus, (iii) holds. This shows that P is a Delaunay polytope. ∎

Note that, with the notation of Lemma 1.2, it may happen that the set Y contains strictly the set X. For such an example, let X consist of the unit vectors in $\mathbb{R}^k$ in which case $Y = \{0,1\}^k$. On the other hand, if X consists of the incidence vectors of the cycles of a binary matroid $\mathcal{M}$, then equality[5] $X = Y$ holds. Indeed, the only $(0,1)$-vectors in the cycle lattice $\mathbb{Z}(\mathcal{M})$ are the incidence vectors of cycles (which follows essentially from the fact that $(\mathcal{M}^*)^* = \mathcal{M}$).

Corollary 1.3. *Let $\mathcal{M} = (E, \mathcal{C})$ be a cosimple binary matroid. Then, the polytope $\mathrm{conv}(\mathcal{C})$ which is defined as the convex hull of the incidence vectors of the cycles of $\mathcal{M}$, is a Delaunay polytope.* ∎

[5]Note that the equality: $\mathbb{Z}(X) \cap \{0,1\}^k = X$ for a finite subset $X \subseteq \{0,1\}^k$, does not imply that X is closed under taking binary sums (i.e., X is not necessarily the cycle space of a binary matroid). For a counterexample, let $X \subseteq \{0,1\}^3$ consist of the vectors $(0,0,0)$, $(1,1,0)$ and $(1,0,1)$.

2 Cycle basis for matroids with no F_7^* minor

We indicate here a very simple method for constructing a cycle basis of the cycle lattice of any binary matroid with no F_7^* minor. The method can also be applied to some other binary matroids, for instance, for projective spaces. As was observed earlier, we can restrict our attention to cosimple binary matroids.

Let $\mathcal{M} = (E, \mathcal{C})$ be a cosimple binary matroid. Let T be a maximal independent set in $\mathcal{M}$, $\overline{T} := E \setminus T$ and, for $e \in \overline{T}$, let C_e denote its fundamental circuit. So, $|T| = r$, $|E \setminus T| = m - r$ if $\mathcal{M}$ has rank r and $|E| = m$. Clearly, $r \leq m - 1$ since $\mathcal{M}$ has no coloop. If $r = 0$, then $C_e = \{e\}$ (for $e \in E$) are the fundamental circuits. If $r = m - 1$ then E is the unique fundamental circuit and thus the unique nonempty cycle. Therefore, in both cases, $\mathbb{Z}(\mathcal{M})$ has trivially a basis of cycles (consisting of fundamental circuits). Thus we can now assume that $1 \leq r \leq m - 2$. We start with an easy observation.

Lemma 2.1. *Let $\mathcal{A}$ be a set of vectors in $\mathbb{Z}^E$. If $\mathcal{A}$ generates (over $\mathbb{Z}$) the fundamental circuits and the vectors $2\chi^e$ (for $e \in T$), then $\mathcal{A}$ generates all elements in $\mathbb{Z}(\mathcal{M})$.*

Proof. Let C be a cycle in M. Then, C is the symmetric difference of the fundamental circuits C_e for $e \in C \cap \overline{T}$. Hence, $x := \chi^C - \sum_{e \in C \cap \overline{T}} \chi^{C_e}$ is an even vector (i.e., $x \in 2\mathbb{Z}^E$) which is zero on $\overline{T}$. Hence, $x \in \mathbb{Z}(\mathcal{A})$ by the assumption, which implies that $\chi^C \in \mathbb{Z}(\mathcal{A})$ too. ▮

The following result will be the main tool for constructing a cycle basis.

Theorem 2.2. *Let $\mathcal{M}$ be a cosimple binary matroid with no F_7^* minor. Let T be a maximal independent set in $\mathcal{M}$ and assume that $|T| \geq 1$ and $|\overline{T}| \geq 2$. Then there exist two fundamental circuits C and C' such that $|C \cap C'| = 1$.*

Proof. Let C_x and C_y (where $x, y \in \overline{T}$) be two fundamental circuits for which $C_x \cap C_y \neq \emptyset$ and $|C_x \cap C_y|$ is minimum. Such circuits do exist. (Indeed, let $e \in T$, let C be a fundamental circuit containing e (which exists since e is not a coloop), and let $x \in C \cap \overline{T}$; then then there exists another fundamental circuit C' containing e, since e and x are not coparallel.) If $|C_x \cap C_y| = 1$ we are done. Else, let $e, f \in C_x \cap C_y$, $e \neq f$. As e and f are not coparallel, there exists a fundamental circuit C_z such that, for instance, $e \in C_z$, $f \notin C_z$. Then, there exists an element $g \in C_z \cap (C_x \setminus C_y)$ since, by our minimality assumption, $C_z \cap C_x \not\subseteq C_x \cap C_y \setminus \{f\}$. Similarly, there exists an element $h \in C_z \cap (C_y \setminus C_x)$. Set $X := \overline{T} \setminus \{x, y, z\}$ and $Y := T \setminus \{e, f, g, h\}$ and consider the matroid $\mathcal{M}' := \mathcal{M} \backslash X / Y$ (delete X and contract Y). Then, $\mathcal{M}'$ coincides with the Fano dual matroid F_7^*, as the circuits C_x, C_y, C_z have the following form:

$$\begin{array}{c}
\\ C_x \\ C_y \\ C_z
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc} e & f & g & h & Y & x & y & z & X \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 0 & * & 1 & 0 & 0 & 0\ldots0 \\
1 & 1 & 0 & 1 & * & 0 & 1 & 0 & 0\ldots0 \\
1 & 0 & 1 & 1 & * & 0 & 0 & 1 & 0\ldots0
\end{pmatrix}
\end{array}.$$

This contradicts our assumption that $\mathcal{M}$ has no F_7^* minor. ∎

Corollary 2.3. *Let $\mathcal{M}$ be a cosimple binary matroid with no F_7^* minor. Let T be a maximal independent set in $\mathcal{M}$ and assume that $r := |T| \geq 1$ and $|\overline{T}| \geq 2$. Then, the elements of T can be ordered as $e_1, \ldots, e_r$ in such a way that, for every $i = 1, \ldots, r$, there exist two fundamental circuits C_i, C'_i such that $e_i \in C_i \cap C'_i \subseteq \{e_1, \ldots, e_i\}$.*

PROOF. We show the result by induction on the size of the groundset. By Theorem 2.2, there exists an element $e_1 \in T$ and two fundamental circuits C_1, C'_1 (with respect to $\mathcal{M}$, T) such that $C_1 \cap C'_1 = \{e_1\}$. We are done if $|T| = 1$. Otherwise, we consider $\mathcal{M}' := \mathcal{M}/e_1$. Then, $T \setminus \{e_1\}$ is a maximal independent set in $\mathcal{M}'$. As $\mathcal{M}'$ is cosimple with no F_7^* minor, we deduce from the induction assumption that the elements of $T \setminus \{e_1\}$ can be ordered as $e_2, \ldots, e_r$ in such a way that, for every $i = 2, \ldots, r$, there exist two fundamental circuits D_i, D'_i (with respect to $\mathcal{M}'$, $T \setminus \{e_1\}$) such that $e_i \in D_i \cap D'_i \subseteq \{e_2, \ldots, e_i\}$. Then, $D_i = C_i \setminus \{e_1\}$, $D'_i = C'_i \setminus \{e_1\}$ where C_i, C'_i are circuits of $\mathcal{M}$. In fact, C_i, C'_i are also fundamental circuits with respect to T in $\mathcal{M}$ as each of them contains a unique element of $\overline{T}$. Moreover, $e_i \in C_i \cap C'_i \subseteq \{e_1, e_2, \ldots, e_i\}$. ∎

Corollary 2.4. *If $\mathcal{M}$ is a cosimple binary matroid of rank r with no F_7^* minor, then the lattice $\mathbb{Z}(\mathcal{M})$ has a basis consisting only of circuits of $\mathcal{M}$. Moreover, such a basis can be obtained by extending any set of fundamental circuits by r circuits that are the binary sum of two fundamental circuits. Therefore, $\mathcal{M}$ has the lattice of circuits property.*

PROOF. Let T be a maximal independent set in $\mathcal{M}$ and let $\{C_e \mid e \in \overline{T}\}$ be the associated set of fundamental circuits. The result is obvious if $|T| = 0$

or $|\overline{T}| \leq 1$. If $|T| \geq 1$ and $|\overline{T}| \geq 2$, then consider the set $\mathcal{B}$ consisting of the fundamental circuits C_e ($e \in \overline{T}$) together with the cycles $C_i \triangle C_i'$ (for $e_i \in T$) provided by Corollary 2.3. We have $|\mathcal{B}| = |E|$. Note that necessarily the symmetric difference of two intersecting fundamental circuits is a circuit. Furthermore, by Corollary 2.3, we obtain that $2\chi^e \in \mathbb{Z}(\mathcal{M})$ for each $e \in T$. Subtracting these vectors from $2\chi^{C_e}$ ($e \in \overline{T}$) shows that $\mathcal{M}$ has the lattice of circuits property and Lemma 2.1 implies that $\mathcal{B}$ generates the full lattice $\mathbb{Z}(\mathcal{M})$. ∎

We close this section with several remarks on possible further applications of the above construction method, as well as its limits and open questions.

Let us first consider the projective space $\mathcal{P}_r$, defined on the set $E := GF(2)^r \setminus \{0\}$ and whose cycles are the linearly dependent (over $GF(2)$) subsets of E. As we now see, the above construction method applies very easily for finding a cycle basis of the lattice $\mathbb{Z}(\mathcal{P}_r)$. Indeed, let $T := \{e_1, \ldots, e_r\}$ be a maximal independent set in $\mathcal{P}_r$. We can suppose that $r \geq 3$ (else there is obviously a basis of cycles). Then, every element $e_i \in T$ is the intersection of two fundamental circuits. For instance, the two fundamental circuits $\{e_1, e_2, e_1 \oplus e_2\}$ and $\{e_1, e_3, e_1 \oplus e_3\}$ meet in e_1. Therefore, the conclusion of Theorem 2.2 holds and, thus, the cycle lattice of $\mathcal{P}_r$ has a basis consisting of cycles.

We remark that the cycle lattice of the dual $\mathcal{P}_r^*$ of the projective space also has a cycle basis; this is obvious since the nonempty cycles of $\mathcal{P}_r^*$ are linearly independent.

On the other hand, the above construction method does not apply to the Fano dual matroid $P_3^* = F_7^*$. Indeed, suppose that F_7 is represented by the matrix from relation (1.1). Then, the set $T := \{4, 5, 6, 7\}$ is a maximal independent set in F_7^*. The fundamental circuits with respect to T are the sets $\{1, 4, 5, 7\}$, $\{2, 4, 6, 7\}$ and $\{3, 5, 6, 7\}$. Hence, their pairwise intersections have size 2 and, thus, the result from Theorem 2.2 does not hold for F_7^*.

In fact, the technique from the next section will apply to the matroid F_7^* since F_7^* can be seen as a one-element extension of a graphic matroid.

Finally, let us note again that the question of existence of a cycle basis for the cycle lattice remains open for general matroids with the lattice of circuits property. We make here some relevant remarks. Let $\mathcal{M} = (E, \mathcal{C})$ be a cosimple binary matroid, let T be a maximal independent subset of E and let C_e ($e \in \overline{T}$) be the corresponding fundamental circuits. Consider the matrix W whose rows are the incidence vectors of the sets $C_e \cap C_f$ (for $e, f \in \overline{T}$). Lovász and Seress [9] show that $\mathcal{M}$ has the lattice of circuits property if and only if the matrix W has full column rank $|E|$ over $GF(2)$. If we could find a set I of pairs (e,f) $(e \neq f \in \overline{T})$ for which the submatrix W_0 with rows C_e $(e \in \overline{T})$ and $C_e \cap C_f$ $((e,f) \in I)$ has its determinant equal to 1, then the set $\{C_e\ (e \in \overline{T}),\ C_e \triangle C_f\ ((e,f) \in I)\}$ would be a cycle basis of $\mathbb{Z}(\mathcal{M})$. Note that, by the result of Lovász and Seress, such a submatrix exists which has its determinant equal to 1 (modulo 2) ! What we have shown in Corollary 2.3 is that this goal of finding such submatrix W_0 with determinant 1 can be achieved when $\mathcal{M}$ has no F_7^* minor.

3 One-element extensions of matroids with no Fano dual minor

Given a binary matroid $\mathcal{M}$ on a set E, a *one-element extension* of $\mathcal{M}$ is any binary matroid $\mathcal{N}$ on $E \cup \{t\}$ (where t is an additional element not belonging to E) such that $\mathcal{N} \backslash t = \mathcal{M}$. We show in this section that, if $\mathcal{M}$ is a matroid with no F_7^* minor, then the cycle lattice of any one-element extension of $\mathcal{M}$ also admits a cycle basis, obtained by extending any cycle basis of the cycle lattice of $\mathcal{M}$. In the special case when $\mathcal{M}$ is a graphic matroid this yields a cycle basis of the cycle lattice of a graft matroid. Goddyn [6] posed the question of describing the cycle lattice of graft matroids.

Before stating the result and its proof, we introduce some notation. Let $\mathcal{M}$ be a binary matroid on E and let Σ be a subset of E. A set $A \subseteq E$ is said to be Σ-*even* (resp. Σ-*odd*) if $|A \cap \Sigma|$ is even (resp. odd). We may simply say that A is even or odd if there is no ambiguity in the choice of the sign set Σ. We denote by $\mathcal{M}_\Sigma$ the binary matroid on $E \cup \{t\}$ (t is an additional element not belonging to E) whose cycles are:
- the cycles of $\mathcal{M}$ and
- the sets $(C \triangle \Sigma) \cup \{t\}$, where C is a cycle of $\mathcal{M}$.

Then the cocycles of $\mathcal{M}_\Sigma$ are:
- the even cocycles of $\mathcal{M}$ and
- the sets $D \cup \{t\}$ where D is an odd cocycle of $\mathcal{M}$.

For convenience, let us call any set $C \triangle \Sigma$ a *join* of $\mathcal{M}$ and $(C \triangle \Sigma) \cup \{t\}$ an *augmented join*. (We borrow this terminology from the subcase of graphs; see below for details.) Thus, the set Σ itself is a join and, obviously, $\mathcal{M}_\Sigma = \mathcal{M}_{\Sigma'}$ if Σ' is another join.

If A is a 0-1 matrix representing $\mathcal{M}$, we obtain a representation matrix for $\mathcal{M}_\Sigma$ by adding a new column to A corresponding to element t; this new column has a one precisely in the rows of A that have an odd number of 1's in the columns indexed by Σ.

Clearly, the matroids $\mathcal{M}_\Sigma$ for varying $\Sigma \subseteq E$ are nothing but all possible one-element extensions of $\mathcal{M}$.

When $\mathcal{M}$ is a graphic matroid, the matroid $\mathcal{M}_\Sigma$ is also known under the name of *graft matroid* (cf. [8], [12]). Let A be the node-edge incidence of a graph $G = (V, E)$, $\Sigma \subseteq E$ and let $T \subseteq V$ be the set of nodes having an odd degree in the subgraph (V, Σ). (Hence, $|T|$ is even.) Then, the sets $C \triangle \Sigma$ (C cycle of G) are precisely the T-joins of G. We remind that an edge set $F \subseteq E$ is a *T-join* if the nodes of T are precisely those nodes that are incident to an odd number of edges in F. Then, a cut $\delta(S)$ is Σ-odd precisely if it is an odd T-cut in the usual sense; that is, if $|S \cap T|$ is odd.

Note that the Fano dual matroid F_7^* is, in fact, a graft matroid. Indeed, F_7^* can be seen as a one-element extension of the graphic matroid of the complete bipartite graph $K_{2,3}$, taking for Σ the set of edges adjacent to a node of degree 3. Hence, the results of this section apply for constructing a cycle basis of $\mathbb{Z}(F_7^*)$. Moreover, this example shows that the one-element extension of a matroid with the lattice of circuits property needs not have this property. We will give in Corollary 3.4 a characterization of the one-element extensions of matroids with no F_7^* minor having the lattice of circuits property.

It is useful to observe how the deletion and contraction operations apply to the one-element extension matroid $\mathcal{M}_\Sigma$. Suppose that $\mathcal{M}$ is a matroid on E, $\Sigma \subseteq E$ and $\mathcal{M}_\Sigma$ is cosimple. Clearly, $\mathcal{M}_\Sigma \backslash t = \mathcal{M}$. If $f \in E$ and $f \notin \Sigma$, then $\mathcal{M}_\Sigma \backslash f = (\mathcal{M} \backslash f)_\Sigma$. (Note that, if Σ contains f, one can replace Σ by another join $\Sigma' = \Sigma \triangle C$, where C is a cycle of $\mathcal{M}$ containing f (which exists since f is not a coloop).) Moreover, $\mathcal{M}_\Sigma / f = (\mathcal{M}/f)_{\Sigma \backslash \{f\}}$ for any $f \in E$.

We consider in this section the cycle lattice $\mathbb{Z}(\mathcal{M}_\Sigma)$ of the one-element extension of a matroid $\mathcal{M}$. We show that, if $\mathcal{M}$ has no Fano dual minor, then this lattice admits a cycle basis $\mathcal{B}$. Moreover, such a basis can be constructed by adding to any cycle basis $\mathcal{B}_\mathcal{M}$ of $\mathbb{Z}(\mathcal{M})$ (obtained, for instance, by applying the procedure proposed in Section 2) a suitable set $\mathcal{B}'$ of augmented joins.

Theorem 3.1. *Let $\mathcal{M}$ be a matroid on E with no F_7^* minor, let $\Sigma \subseteq E$, and let $\mathcal{M}_\Sigma$ be the corresponding one-element extension of $\mathcal{M}$. Then, for* *every basis $\mathcal{B}_{\mathcal{M}}$ of $\mathbb{Z}(\mathcal{M})$ composed of circuits of $\mathcal{M}$, there exists a basis $\mathcal{B}$ of the lattice $\mathbb{Z}(\mathcal{M}_\Sigma)$ composed of circuits of $\mathcal{M}_\Sigma$ and such that $\mathcal{B}_{\mathcal{M}} \subseteq \mathcal{B}$. Moreover, $4\chi^P \in \mathbb{Z}(\mathcal{M}_\Sigma)$ for every coparallel class P of $\mathcal{M}_\Sigma$.*

PROOF. The proof is by induction on $|E|$. As was already observed earlier, we may assume without loss of generality that the matroid $\mathcal{M}_\Sigma$ is cosimple. Moreover, we may assume that the matroid $\mathcal{M}$ is not cosimple. Indeed,

Claim 1. *If $\mathcal{M}$ is cosimple then Theorem 3.1 holds. Moreover, $\mathcal{M}_\Sigma$ has the lattice of circuits property.*

PROOF OF CLAIM 1. Assume that $\mathcal{M}$ is cosimple. Then, $\dim \mathcal{M}_\Sigma = |E| + 1 = \dim \mathcal{M} + 1$. Set $\mathcal{B} := \mathcal{B}_{\mathcal{M}} \cup \{\Sigma \cup \{t\}\}$. Let B be an arbitrary join. From $\chi^{B\cup\{t\}} + \chi^{\Sigma\triangle B} - \chi^{\Sigma\cup\{t\}} = 2\chi^{B\setminus\Sigma}$, we deduce that $\chi^{B\cup\{t\}} \in \mathbb{Z}(\mathcal{B})$ if and only if $2\chi^{B\setminus\Sigma} \in \mathbb{Z}(\mathcal{B})$. But, $2\chi^{\{e\}} \in \mathbb{Z}(\mathcal{B}_{\mathcal{M}})$ for every $e \in E$ as $\mathcal{M}$ is cosimple (and has the lattice of circuits property). Therefore, $\mathcal{B}$ spans every augmented join and $2\mathbb{Z}^{E\cup\{t\}} \subseteq \mathbb{Z}(\mathcal{B})$. If $\Sigma \cup \{t\}$ is a circuit of $\mathcal{M}_\Sigma$, then we are done as $\mathcal{B}$ is composed only of circuits. Otherwise, $\Sigma \cup \{t\}$ is a disjoint union of circuits out of which a unique one is of the form $A \cup \{t\}$ for some join A. Then, replacing $\Sigma \cup \{t\}$ in $\mathcal{B}$ by $A \cup \{t\}$, we obtain a basis of $\mathbb{Z}(\mathcal{M}_\Sigma)$ composed only of circuits. ∎

From now on we assume that $\mathcal{M}$ is not cosimple. As $\mathcal{M}_\Sigma$ is cosimple, this implies that $\mathcal{M}$ has no coloop, but $\mathcal{M}$ contains some cocircuits of size 2, all of them being odd. Say, $\mathcal{M}$ contains $N_0 \geq 1$ (odd) cocircuits $D_1, \dots, D_{N_0}$ of size 2. Note that every cocycle of size 3 in $\mathcal{M}$ is a cocircuit (as $\mathcal{M}$ has no coloop). We use the following notation: $D_i := \{f_i, f_i^+\}$ for $i = 1, \dots, N_0$; $E_0 := \{f_1, f_1^+, \dots, f_{N_0}, f_{N_0}^+\}$; $F_0 := \{f_i \mid i = 1, \dots, N_0\}$; and we can suppose that $\Sigma \cap E_0 = \{f_i^+ \mid i = 1, \dots, N_0\}$.

The basic idea of the proof is as follows. Consider the matroid $\mathcal{N} := \mathcal{M}\setminus f$ obtained by deleting an element $f \in F_0$ in $\mathcal{M}$. We remind that $f \notin \Sigma$. By the induction assumption, we can find a basis of the lattice $\mathbb{Z}(\mathcal{N}_\Sigma)$ of the form: $\mathcal{B}_{\mathcal{N}} \cup \mathcal{B}'$, where $\mathcal{B}_{\mathcal{N}}$ is a basis of cycles of the lattice $\mathbb{Z}(\mathcal{N})$ and $\mathcal{B}'$ is composed of augmented joins of $\mathcal{M}$ not containing element f. If

$$\dim \mathcal{M}_\Sigma - \dim \mathcal{M} \geq \dim \mathcal{N}_\Sigma - \dim \mathcal{N}. \tag{3.2}$$

then a strategy can be to look for a set $\mathcal{B}^*$ of suitable augmented joins (containing f) such that the set $\mathcal{B} := \mathcal{B}_\mathcal{M} \cup \mathcal{B}' \cup \mathcal{B}^*$ forms a basis of $\mathbb{Z}(\mathcal{M}_\Sigma)$. However, relation (3.2) does not hold in general. As we see below, we have to apply the induction assumption to a smaller matroid $\mathcal{N}' := \mathcal{M}\backslash f/X$ where some further elements are contracted. The elements to contract will arise from the cocircuits of size 3 containing f and contained in F_0. We now give the details of the proof.

We begin with a detailed analysis of the dimensions of the cycle lattices of matroids $\mathcal{M}$, $\mathcal{M}_\Sigma$, $\mathcal{N}$, $\mathcal{N}_\Sigma$. Here, $\mathcal{N} := \mathcal{M}\backslash f$ where f is a given element of F_0 and Σ is a join of $\mathcal{M}$ not containing f. Clearly, $\dim \mathcal{M} = |E| - N_0$, $\dim \mathcal{M}_\Sigma = |E| + 1$ and, thus,

$$\dim \mathcal{M}_\Sigma - \dim \mathcal{M} = N_0 + 1.$$

We now identify the coparallel classes in $\mathcal{N}$, $\mathcal{N}_\Sigma$. The cocircuits of size 3 containing f play an important role as they yield cocircuits of size 2 in $\mathcal{N}$ and, thus, contribute to the coparallel classes. The matroid $\mathcal{N}$ has $\{f^+\}$ as unique coloop and its nontrivial (i.e., of size ≥ 2) coparallel classes fall into one of the following categories:

1. $\{x, y\}$, where $\{x, y, f\}$ is a cocircuit in $\mathcal{M}$ with $x, y \notin E_0$.
2. $\{x, g, g^+\}$, where $\{x, f, g\}$ is a cocircuit in $\mathcal{M}$ with $x \notin E_0$, $g \in F_0$.
3. $\{g, g^+, h, h^+\}$, if $\{f, g, h\}$ is a cocircuit in $\mathcal{M}$ with $g, h \in F_0$.
4. $\{g, g^+\}$ for $g \in F_0$ not yet involved in one of the above classes.

Denote by N_1, N_2, N_3, N_4 the number of coparallel classes of respective types 1,2,3,4. Clearly,

$$N_4 = N_0 - 1 - N_2 - 2N_3,$$

$$\begin{aligned}\dim \mathcal{N} &= |E| - 1 - N_1 - 2N_2 - 3N_3 - N_4 - 1 \\ &= |E| - 1 - N_0 - N_1 - N_2 - N_3.\end{aligned}$$

Among the N_1 cocircuits of type 1, denote by N_1^e (resp. $N_1^o = N_1 - N_1^e$) the number of those cocircuits that are even (resp. odd). Besides $\{f^+, t\}$, the nontrivial coparallel classes of $\mathcal{N}_\Sigma$ are:

1. $\{x, y\}$, where $\{x, y, f\}$ is an even cocircuit of type 1.
2. The one of $\{x, g\}$, $\{x, g^+\}$ which is even, for each cocircuit $\{x, f, g\}$ of type 2.
3. The two (disjoint) even sets among $\{g, h\}$, $\{g^+, h\}$, $\{g, h^+\}$, and $\{g^+, h^+\}$ for each cocircuit $\{f, g, h\}$ of type 3.

Hence,

$$\dim \mathcal{N}_\Sigma = |E| - 1 - N_1^e - N_2 - 2N_3, \quad \text{and}$$

$$\dim \mathcal{N}_\Sigma - \dim \mathcal{N} = N_0 + N_1^o - N_3.$$

Relation (3.2) might be violated when $N_1^o > 0$. This leads us to considering the matroid $\mathcal{N}' := \mathcal{M}\backslash f/X$, where $\{x_i, y_i, f\}$ $(i = 1, \ldots, N_1^o)$ denote the N_1^o odd cocircuits of type 1 (with $x_i, y_i \notin E_0$) and $X := \{x_i \mid i = 1, \ldots, N_1^o\}$. Then, the coparallel classes of $\mathcal{N}'_\Sigma$ are $\{f^+, t\}$ together with the N_1^e classes $\{x, y\}$ (for $\{x, y, f\}$ even cocircuit with $x, y \notin E_0$), the N_2 classes of type 2 and the $2N_3$ classes of type 3. (By the notation $\mathcal{N}'_\Sigma$ we mean in fact $\mathcal{N}'_{\Sigma\backslash\{f\}}$.) Therefore,

$$\dim \mathcal{N}' = |E| - 1 - N_0 - N_1 - N_2 - N_3,$$

$$\dim \mathcal{N}'_\Sigma = |E| - 1 - N_1 - N_2 - 2N_3,$$

$$\dim \mathcal{N}'_\Sigma - \dim \mathcal{N}' = N_0 - N_3.$$

Applying the induction assumption, let $\mathcal{B}_{\mathcal{N}'} \cup \mathcal{B}''$ be a basis of $\mathbb{Z}(\mathcal{N}'_\Sigma)$, where $\mathcal{B}_{\mathcal{N}'}$ is a cycle basis of $\mathbb{Z}(\mathcal{N}')$ and $\mathcal{B}''$ consists of augmented joins of $\mathcal{N}'$. Every join of $\mathcal{N}'$ is of the form $A \setminus X$, where A is a join of $\mathcal{M}$ not containing f. Denote by $\mathcal{B}'$ the set of augmented joins of $\mathcal{M}$ defined from $\mathcal{B}''$ in this manner; so, $|\mathcal{B}'| = |\mathcal{B}''|$. Let $\mathcal{B}_\mathcal{M}$ be a cycle basis of $\mathbb{Z}(\mathcal{M})$. We establish some properties of the set $\mathcal{B}_\mathcal{M} \cup \mathcal{B}'$.

Claim 2. *The set $\mathcal{B}_\mathcal{M} \cup \mathcal{B}'$ spans (over $\mathbb{Z}$) all the augmented joins of $\mathcal{M}$ that do not contain f, as well as $4\chi^P$ for every coparallel class P of $\mathcal{N}'_\Sigma$.*

Proof of Claim 2. Let A be a join in $\mathcal{M}$ with $f \notin A$. Then, $A \setminus X$ is a join in $\mathcal{N}'$. Hence, $\chi^{(A\setminus X)\cup\{t\}}$ is spanned by $\mathcal{B}_{\mathcal{N}'} \cup \mathcal{B}''$. Say,

$$\chi^{(A\setminus X)\cup\{t\}} = \sum \lambda_i \chi^{(A_i\setminus X)\cup\{t\}} + \sum \mu_j \chi^{C_j\setminus X},$$

where the first sum runs over the augmented joins $A_i \cup \{t\}$ composing $\mathcal{B}'$ and the second sum involves cycles C_j of $\mathcal{M}\backslash f$ and all λ_i, μ_j's are integers. Consider the vector

$$u := \chi^{A\cup\{t\}} - \sum \lambda_i \chi^{A_i\cup\{t\}} - \sum \mu_j \chi^{C_j}.$$

By construction, every component of u indexed by an element in $(E\cup\{t\})\setminus X$ is zero. In particular, $u(t) = 1 - \sum \lambda_i = 0$. Consider one of the N_1^o odd cocircuits $\{f, x_i, y_i\}$. Then, $u(y_i) = 0$ and

$$u(x_i) = u(x_i) + u(y_i) = 1 - \sum \lambda_i - 2 \sum_{j | x_i \in C_j} \mu_j$$

is an even integer. Therefore, $u \in \mathbb{Z}(\mathcal{M}) = \mathbb{Z}(\mathcal{B}_{\mathcal{M}})$. This shows that $\chi^{A \cup \{t\}} \in \mathbb{Z}(\mathcal{B}_{\mathcal{M}} \cup \mathcal{B}')$.

Let P be a coparallel class of $\mathcal{N}'_{\Sigma}$. Then, $4\chi^P \in \mathbb{Z}(\mathcal{B}_{\mathcal{N}'} \cup \mathcal{B}'')$ by the induction assumption. Thus, keeping the same notation as above,

$$4\chi^P = \sum \lambda_i \chi^{(A_i \setminus X) \cup \{t\}} + \sum \mu_j \chi^{C_j \setminus X}.$$

Set again

$$u := 4\chi^P - \sum \lambda_i \chi^{A_i \cup \{t\}} - \sum \mu_j \chi^{C_j}.$$

Thus, $0 = u(t) = 4\chi^P(t) - \sum \lambda_i$ and

$$u(x_i) = u(x_i) + u(y_i) = 4(\chi^P(x_i) + \chi^P(y_i)) - \sum \lambda_i - 2 \sum_{j | x_i \in C_j} \mu_j$$

is an even integer. Therefore, $u \in \mathbb{Z}(\mathcal{M}) = \mathbb{Z}(\mathcal{B}_{\mathcal{M}})$ which shows that $4\chi^P \in \mathbb{Z}(\mathcal{B}_{\mathcal{M}} \cup \mathcal{B}')$. ∎

Note that if $2\chi^P \in \mathbb{Z}(\mathcal{B}_{\mathcal{N}'} \cup \mathcal{B}'')$ we even obtain that $2\chi^P \in \mathbb{Z}(\mathcal{B}_{\mathcal{M}} \cup \mathcal{B}')$.

We now show how to extend the set $\mathcal{B}_{\mathcal{M}} \cup \mathcal{B}'$ to a cycle basis of $\mathbb{Z}(\mathcal{M}_{\Sigma})$. For this, we have to find a set $\mathcal{B}^*$ of augmented joins of $\mathcal{M}$ (containing f) such that the set

$$\mathcal{B} := \mathcal{B}_{\mathcal{M}} \cup \mathcal{B}' \cup \mathcal{B}^*$$

forms a basis of $\mathbb{Z}(\mathcal{M}_{\Sigma})$. Necessarily, $|\mathcal{B}^*| = N_3 + 1$ as $|\mathcal{B}'| = N_0 - N_3$. We need an intermediary result.

Claim 3. *Let $\mathcal{F}$ be a cosimple binary matroid on $F := \{f, g_1, h_1, \dots, g_k, h_k\}$. Assume that $\mathcal{F}$ has no F_7^* minor and that the sets $\{f, g_i, h_i\}$ for $1 \le i \le k$ are cocycles of $\mathcal{F}$. Then, the set $\{g_i, h_i\}$ is a cycle of $\mathcal{F}$ for every $i = 1, \dots, k$.*

Proof of Claim 3. We show the result by induction on k. The result holds obviously for $k = 1$. Consider first the case $k = 2$. The set $\{g_1, h_1\}$ is a coparallel class in $\mathcal{F} \setminus f$ and, therefore[6], is a cycle or the intersection of two

[6]We use here the result from [1] that in a matroid with no F_7^* minor every coparallel class is a cycle or the intersection of two cycles.

cycles of $\mathcal{F}\backslash f$ (as $\mathcal{F}$ has no F_7^* minor); one of the two cycles does not contain g_2, h_2. Assume now $k \geq 3$ and that $\{g_1, h_1\}$ is not a cycle of $\mathcal{F}$. Let C_0 be a cycle containing f; say, $C_0 = \{f, g_1, \ldots, g_k\}$. Applying the induction assumption to matroid $\mathcal{F}/\{g_i, h_i\}$, we find that $\{g_1, h_1, g_i, h_i\}$ is a cycle of $\mathcal{F}$ for every $i \neq 1$. Set $C_1 := \{g_1, h_1, g_2, h_2\}$ and $C_2 := \{g_1, h_1, g_k, h_k\}$. The set $I := \{f, g_1, h_1, g_2, g_3, \ldots, g_{k-1}\}$ is independent in $\mathcal{F}$ and the sets $I \cup \{h_2\}$, $I \cup \{g_k\}$, $I \cup \{h_k\}$ are not independent as they contain, respectively, the cycles C_1, C_0 and $C_0 \triangle C_2$. Let J be a maximal independent set in $\mathcal{F}$ containing I and set $X := J \setminus \{f, g_1, h_1, g_2\}$ and $Y := F \setminus (J \cup \{h_2, g_k, h_k\})$. Then, the matroid $\mathcal{F}/X\backslash Y$ coincides with F_7^*. Note indeed that the incidence of $C_0, C_1, C_0 \triangle C_2$ on the elements of F is as follows:

$$\begin{array}{c} \\ C_1 \\ C_0 \\ C_0\triangle C_2 \end{array} \begin{array}{c} \begin{array}{ccccccccc} f & g_1 & h_1 & g_2 & X & h_2 & g_k & h_k & Y \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & * & 1 & 0 & 0 & 0\ldots0 \\ 1 & 1 & 0 & 1 & * & 0 & 1 & 0 & 0\ldots0 \\ 1 & 0 & 1 & 1 & * & 0 & 0 & 1 & 0\ldots0 \end{pmatrix} \end{array}$$

∎

We construct the set $\mathcal{B}^*$ in the following manner. Let $\{f, g_i, h_i\}$ ($i = 1, \ldots, N_3$) denote the cocircuits containing f and contained in F_0 and set $F_1 := \{f, g_1, h_1, \ldots, g_{N_3}, h_{N_3}\}$, $E_1 := F_1 \cup \{x^+ \mid x \in F_1\}$. By applying Claim 3 to the cosimple matroid $\mathcal{M}/(E \setminus F_1)$, we can find cycles C_i of $\mathcal{M}$ such that $C_i \cap E_1 = \{g_i, g_i^+, h_i, h_i^+\}$ for $i = 1, \ldots, N_3$. Let C be a cycle of $\mathcal{M}$ containing f and, thus, also f^+; to fix ideas suppose that $C \cap F_1 = \{f, g_1, \ldots, g_{N_3}\}$. We let $\mathcal{B}^*$ consist of the sets $(\Sigma \triangle C) \cup \{t\}$ and $(\Sigma \triangle C \triangle C_i) \cup \{t\}$ for $i = 1, \ldots, N_3$. In order to conclude the proof, it suffices to verify that $\mathcal{B} = \mathcal{B}_\mathcal{M} \cup \mathcal{B}' \cup \mathcal{B}^*$ spans all the augmented joins containing f and that $4\mathbb{Z}^{E \cup \{t\}} \subseteq \mathbb{Z}(\mathcal{B})$; this is done in the next claim.

Claim 4. *The set $\mathcal{B}$ spans (over $\mathbb{Z}$) all the augmented joins of $\mathcal{M}$ containing f and $4\mathbb{Z}^{E \cup \{t\}} \subseteq \mathbb{Z}(\mathcal{B})$.*

PROOF OF CLAIM 4. We first verify that $4\chi^{\{x\}} \in \mathbb{Z}(\mathcal{B})$ for all $x \in E \cup \{t\}$. We already know that $2\chi^{\{x\}} \in \mathbb{Z}(\mathcal{B})$ for every $x \in E \backslash E_0$ and $2\chi^{\{g, g^+\}} \in \mathbb{Z}(\mathcal{B})$ for every $g \in F_0$ (as $\mathcal{M}$ has the lattice of circuits property). By Claim 2, we obtain that $4\chi^{\{g\}}, 4\chi^{\{g^+\}} \in \mathbb{Z}(\mathcal{B})$ if $g \in F_0 \setminus \{f\}$ is not involved in one of the N_3 coparallel classes of type 3, i.e., if $g \notin F_1$. Consider the identity:

$$2\chi^{C \cap (C_i \triangle (\Sigma \cup \{t\}))} - 2\chi^{C \cap C_i} + 4\chi^{C \cap C_i \cap (\Sigma \cup \{t\})} = 2\chi^{C \cap (\Sigma \cup \{t\})}.$$

Now, $2\chi^{C\cap(\Sigma\cup\{t\})} = \chi^{\Sigma\cup\{t\}} + \chi^{C} - \chi^{C\triangle(\Sigma\cup\{t\})} \in \mathbb{Z}(\mathcal{B})$, $2\chi^{C\cap C_i} = \chi^{C} + \chi^{C_i} - \chi^{C\triangle C_i} \in \mathbb{Z}(\mathcal{B})$, and $2\chi^{C\cap(C_i\triangle(\Sigma\cup\{t\}))} = \chi^{C} + \chi^{C_i\triangle(\Sigma\cup\{t\})} - \chi^{C\triangle C_i\triangle(\Sigma\cup\{t\})} \in \mathbb{Z}(\mathcal{B})$. This implies that $4\chi^{C\cap C_i\cap(\Sigma\cup\{t\})} \in \mathbb{Z}(\mathcal{B})$, i.e., $4\chi^{\{g_i^+\}} \in \mathbb{Z}(\mathcal{B})$ and, thus, $4\chi^{\{g_i\}} \in \mathbb{Z}(\mathcal{B})$. From the identity:

$$2\chi^{(C\triangle C_i)\cap(\Sigma\cup\{t\})} - 2\chi^{C\cap(\Sigma\cup\{t\})} = \chi^{C\triangle C_i} - \chi^{C\triangle C_i\triangle(\Sigma\cup\{t\})} - \chi^{C} + \chi^{C\triangle(\Sigma\cup\{t\})},$$

we obtain that $2\chi^{\{h_i^+\}} - 2\chi^{\{g_i^+\}} \in \mathbb{Z}(\mathcal{B})$ and, therefore, that $4\chi^{\{h_i^+\}}, 4\chi^{\{h_i\}} \in \mathbb{Z}(\mathcal{B})$. Now, $4\chi^{\{t\}} \in \mathbb{Z}(\mathcal{B})$ follows immediately. So we have shown that $4\mathbb{Z}^{E\cup\{t\}} \subseteq \mathbb{Z}(\mathcal{B})$.

We finally verify that $\mathcal{B}$ spans every augmented join $B\cup\{t\}$ containing f. Note that $\chi^{B\cup\{t\}} \in \mathbb{Z}(\mathcal{B})$ if and only if $2\chi^{C\cap(B\cup\{t\})} \in \mathbb{Z}(\mathcal{B})$ because $\chi^{B\cup\{t\}} + \chi^{C} - \chi^{C\triangle(B\cup\{t\})} = 2\chi^{C\cap(B\cup\{t\})}$. To conclude, we use the identity:

$$\begin{aligned} 2\chi^{C\cap(B\cup\{t\})} &= 2\chi^{C\cap((B\cup\{t\})\triangle(\Sigma\cup\{t\}))} - 2\chi^{C\cap(\Sigma\cup\{t\})} + 4\chi^{C\cap(\Sigma\cup\{t\})\cap(B\cup\{t\})} \\ &= \chi^{\Sigma\triangle B} - \chi^{C\triangle\Sigma\triangle B} - \chi^{\Sigma\cup\{t\}} + \chi^{C\triangle(\Sigma\cup\{t\})} + 4\chi^{C\cap(\Sigma\cup\{t\})\cap(B\cup\{t\})}. \end{aligned}$$

∎

Let us consider separately the case when $N_3 = \emptyset$ and $2\chi^{P} \in \mathbb{Z}(\mathcal{B}_{\mathcal{M}} \cup \mathcal{B}')$ for each coparallel class P of $\mathcal{N}'_{\Sigma}$. From

$$\chi^{(\Sigma\cup\{t\})\triangle C} + \chi^{\Sigma\cup\{t\}} - \chi^{C} = 2\chi^{(\Sigma\cup\{t\})\setminus C}$$

we have that $2\chi^{\{t\}} \in Z(\mathcal{B}_{\mathcal{M}} \cup \mathcal{B}')$ and, also, $2\chi^{\{f\}} \in Z(\mathcal{B}_{\mathcal{M}} \cup \mathcal{B}')$ and $2\chi^{\{f^+\}} \in Z(\mathcal{B}_{\mathcal{M}} \cup \mathcal{B}')$.

Finally, in order to finish the proof of Theorem 3.1 it remains to adjust $\mathcal{B}$ so that it contains circuits only. By assumption, $\mathcal{B}_{\mathcal{M}} \subset \mathcal{B}$ consists of circuits. Each augmented join $B \in \mathcal{B} \setminus \mathcal{B}_{\mathcal{M}}$ which is not a circuit of $\mathcal{M}_{\Sigma}$ is a disjoint union of one augmented join $\overline{B}$ which is a circuit, and a cycle of $\mathcal{M}$. Replacing each B by $\overline{B}$ in $\mathcal{B} \setminus \mathcal{B}_{\mathcal{M}}$ yields the desired basis of circuits. This concludes the proof of Theorem 3.1. ∎

Remark 3.3. The assertion from Theorem 3.1 that $4\chi^{E\cup\{t\}} \subseteq \mathbb{Z}(\mathcal{M}_{\Sigma})$ if $\mathcal{M}$ is cosimple with no F_7^* minor also follows from the result of Lovaśz and Seress [10] which states that $4\mathbb{Z}^{E} \subseteq \mathbb{Z}(\mathcal{N})$ for any binary matroid $\mathcal{N}$ with no $\mathcal{P}_4^*$ minor. Indeed, one can verify that $\mathcal{M}_{\Sigma}$ does not contain $\mathcal{P}_4^*$ as a minor if $\mathcal{M}$ has no F_7^* minor.

Corollary 3.4. *Let $\mathcal{M}$ be a matroid on E with no F_7^* minor and let $\mathcal{M}_{\Sigma}$ be* *a one-element extension of $\mathcal{M}$. Then, $\mathcal{M}_\Sigma$ does not have the lattice of circuits property if and only if there exist elements $f_1, \ldots, f_k, f_1^+, \ldots, f_k^+ \in E$ ($k \geq 3$) such that the set $\{f_1, \ldots, f_k\}$ is a cocircuit of $\mathcal{M}$ while the sets $\{f_i, f_i^+\}$ ($i = 1, \ldots, k$) are odd cocircuits of $\mathcal{M}$.*

Proof. Suppose first that such cocircuits exist. We show that $\mathcal{M}_\Sigma$ does not have the lattice of circuits property by constructing a vector $x \in \mathbb{Q}^{E \cup \{t\}}$ with denominator 4 and such that $x(C) \in \mathbb{Z}$ for every cycle C of $\mathcal{M}_\Sigma$. For this, set $x(f_i) = x(f_i^+) = \frac{1}{4}$ ($i = 1, \ldots, k$) and $x(t) = 0, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}$ if k is congruent to 0,1,2,3 modulo 4, respectively. The converse implication follows from the proof of Theorem 3.1. ∎

We conclude this section with an observation on the limits of local constructions. It has been observed in [7] that, given a cycle basis of the cycle lattice of a contraction minor $\mathcal{M}/e$ of a binary matroid $\mathcal{M}$, it is in general not possible to extend it to a cycle basis of the cycle lattice of $\mathcal{M}$. We now give an example showing that the same holds if we are given a cycle basis of a deletion minor $\mathcal{M} \backslash e$ of $\mathcal{M}$.

Lemma 3.5. *Let $\mathcal{M}$ be a cosimple binary matroid on E and let $e \in E$. Assume that $\mathcal{M}$ has the lattice of circuits property and that $\mathcal{M} \backslash e$ does not have the lattice of circuits property. Then any cycle basis for $\mathbb{Z}(\mathcal{M})$ must have at least two cycles containing e. Therefore, if $\mathcal{M} \backslash e$ is cosimple, then no cycle basis of $\mathbb{Z}(\mathcal{M})$ exists that contains a cycle basis of $\mathbb{Z}(\mathcal{M} \backslash e)$.*

Proof. By the assumption, there exists a coparallel class P in $\mathcal{M} \backslash e$ such that $2\chi^P \in \mathbb{Z}(\mathcal{M}) \setminus \mathbb{Z}(\mathcal{M} \backslash e)$. This implies that any cycle basis of $\mathbb{Z}(\mathcal{M})$ must contain at least two cycles containing the element e. ∎

As an example of a matroid satisfying the conditions of Lemma 3.5, consider the matroid S_8 on the set $\{e_1, \ldots, e_8\}$ represented by the following matrix:

$$\begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then, $S_8 \backslash e_1 \sim F_7^*$, both F_7^* and S_8 are cosimple, and S_8 has the lattice of circuits property while F_7^* does not have it. To see that S_8 has the lattice of circuits property, one can note that S_8 is, in fact, a graft matroid and use Corollary 3.4. Indeed, S_8 is the graft matroid of the graph from Figure 1 taking $T := \{1,2,3,4\}$ (and labeling the edges $15, 25, 35, 45, 12, 13, 14$ as $e_1, \ldots, e_7$, respectively, and the additional element t as e_8). Therefore, S_8 admits a cycle basis by Theorem 3.1. (Alternatively, one can note that the set $T' := \{e_1, e_2, e_3, e_4\}$ is a maximal independent set in S_8 and that the pairwise intersections of fundamental circuits are the sets $\{e_1\}$ and $\{e_1, e_i\}$ for $i = 2, 3, 4$. This argument shows that S_8 has the lattice of circuits property and that a cycle basis for $\mathbb{Z}(S_8)$ can be constructed by applying the technique from Section 2.)

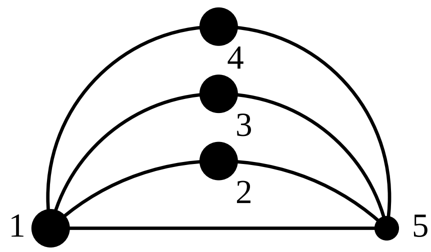

Figure 1: The matroid S_8 is a graft

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