Almost-tiling the plane by ellipses *

Krystyna Kuperberg
Department of Mathematics
Auburn University
Auburn, AL 36849-5310
U.S.A.
e-mail: kuperkm@mail.auburn.edu

Wlodzimierz Kuperberg
Department of Mathematics
Auburn University
Auburn, AL 36849-5310
U.S.A.
e-mail: kuperwl@mail.auburn.edu

Jiří Matoušek
Department of Applied Mathematics
Charles University
Malostranské nám. 25, 118 00 Praha 1
Czech Republic
and
Institut für Theoretische Informatik
ETH Zürich, Switzerland
e-mail: matousek@kam.mff.cuni.cz

Pavel Valtr
Department of Applied Mathematics
Charles University
Malostranské nám. 25, 118 00 Praha 1
Czech Republic
and
DIMACS Center
Rutgers University, NJ, U.S.A.
e-mail: valtr@kam.mff.cuni.cz

Dedicated to our friend Imre Bárány on the occasion of his 50-th birthday.

Abstract

For any $\lambda > 1$ we construct a periodic and locally finite packing of the plane with ellipses whose λ-enlargement covers the whole plane. This answers a question of Imre Bárány. On the other hand, we show that if $\mathcal{C}$ is a packing in the plane with circular discs of radius at most 1, then its $(1 + 10^{-5})$-enlargement covers no square with side length 4.

*Research of K. Kuperberg was supported by NSF grant No. DMS-9704958. Research of W. Kuperberg was supported by NSF grant No. DMS-9704319. Research of J. Matoušek was supported by ETH Zürich, by Czech Republic Grant GAČR 0194/1996, and by Charles University grants No. 193,194/1996. Research of P. Valtr was supported by DIMACS Center, by Czech Republic Grant GAČR 0194/1996, and by Charles University grants No. 193,194/1996.

1 Introduction

Let $\mathcal{C}$ be a system (finite or infinite) of centrally symmetric convex bodies in $\mathbb{R}^d$ with disjoint interiors; we call such a $\mathcal{C}$ a *packing*. For a real number $\varepsilon > 0$ and for $C \in \mathcal{C}$, we let C^ε denote C enlarged by the factor $1+\varepsilon$ from its center, that is, $C^\varepsilon = (1+\varepsilon)(C-c)+c$, where c stands for the center of symmetry C. Let us call the closure of the set $C^\varepsilon \setminus C$ the *ε-ring* of C. We call the system $\mathcal{C}^\varepsilon = \{C^\varepsilon;\, C \in \mathcal{C}\}$ the $(1+\varepsilon)$*-enlargement* of $\mathcal{C}$.

For a class $\mathcal{C}_0$ of centrally symmetric convex bodies in $\mathbb{R}^d$, we define the *Bárány number* of $\mathcal{C}_0$ as the infimum of the numbers $\varepsilon > 0$ such that there exists a packing $\mathcal{C} \subseteq \mathcal{C}_0$ whose $(1+\varepsilon)$-enlargement covers the whole $\mathbb{R}^d$.

We observe that the system of all circular discs in the plane has Bárány number 0, since we can produce the desired packing for any $\varepsilon > 0$ by using larger and larger discs (add discs to the packing one by one, and in the ith step, choose the ith disc so that its ε-ring covers the disc of radius i around the origin). A different situation may occur if the diameter of the bodies in $\mathcal{C}_0$ is bounded.

Motivated by a problem concerning convex polytopes, Imre Bárány [1] raised a problem which in our terminology can be rephrased as follows: If $\mathcal{E}$ stands for the class of all ellipses of diameter at most 1, is the Bárány number of $\mathcal{E}$ zero?

In this paper we give a positive answer to this question:

Theorem 1 *For every $\lambda > 1$ there is a periodic packing of the plane with ellipses whose λ-enlargement is a covering.*

On the other hand, if we allow only discs of bounded radius, then Bárány's question has a negative answer:

Theorem 2 *Let $\mathcal{C}$ be a packing of the plane with circular discs of radius at most 1. Then $(1+10^{-5})$-enlargement of $\mathcal{C}$ covers no square with side length 4.*

Remarks. Our packing in Theorem 1 is locally finite and the details in the construction can be done so that all ellipses in the packing have diameter between $\varepsilon/10$ and 1, where $\varepsilon = \lambda - 1$ (however, their width varies from $\exp[-c/\varepsilon \log^2(1/\varepsilon)]$ to c and we need $\exp[c/\varepsilon \log(1/\varepsilon)]$ of them on each unit square). Our methods can be used to prove that Theorem 2 (possibly with a different positive constant instead of 10^{-5}) holds also in any dimension $d \geq 2$ and when $\mathcal{C}_0$ consists of convex bodies in $\mathbb{R}^d$ with a constant-bounded diameter and curvature. We do not prove these generalizations here, since the idea remains the same but the details become messy. The value $\varepsilon = 10^{-5}$ in Theorem 2 is certainly not the best possible one could get by our proof method, but it seems that a different method would be needed to determine the Bárány number for discs in the plane exactly or at least to prove a reasonable lower bound for its value.

2 Almost-tiling by ellipses

Throughout the construction, a number $\lambda > 1$ remains fixed. Choose an integer n such that the regular $2n$-polygon $P = P_{2n}$ circumscribed about a circular disk D is contained in the interior of the λ-enlargement of D. Denote two antipodal vertices of P by v^- and v^+. Suppose T is a triangle with a horizontal base B and vertex v above B. Then there is a (unique) polygon $P(T)$ satisfying the following conditions (see Fig. 1 which illustrates the case $n = 4$):

(i) $P(T) \subset T$; (ii) There is an affine transformation A such that $A(P) = P(T)$; (iii) $A(v^-)$ is the midpoint of B and $A(v^+) = v$; (iv) The angles at v of $P(T)$ and of T are equal.

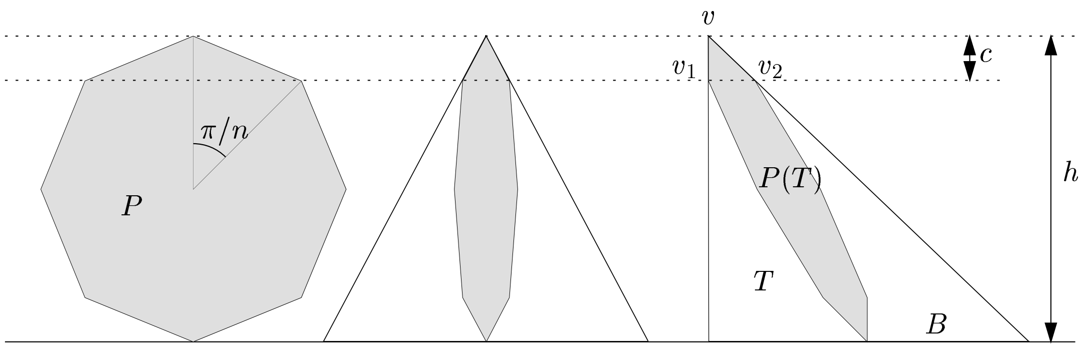


Figure 1: The affine copy of P properly inscribed in T.

We say that $P(T)$ is an *affine copy of P properly inscribed in T.* The existence and uniqueness of $P(T)$ follows from the fact that P itself, oriented so that one of its main diagonals is vertical, is properly inscribed in a triangle, and an affine transformation that sends this triangle onto T (top vertex onto top vertex and base onto base) determines $P(T)$ uniquely. Obviously, $P(T)$ contains an inscribed ellipse, namely $A(D)$, whose λ-enlargement contains a neighborhood of $P(T)$.

Observe the following property of the polygon $P(T)$:

(1) Let v_1 and v_2 be the vertices of $P(T)$ adjacent to v. Then the line v_1v_2 is parallel to B and partitions the height h of T at the ratio of

$$c : (h-c) = \left(1 - \cos\frac{\pi}{n}\right) : \left(1 + \cos\frac{\pi}{n}\right),$$

where c is the portion of h containing v (see Fig. 1).

It follows that

(2) For every vertex $w \neq v$ of $P(T)$ the distance from w to the line of B is less than or equal to μh, where $\mu < 1$ is a positive constant independent from T. Specifically,

$$\mu = \frac{1}{2}\left(1 + \cos\frac{\pi}{n}\right).$$

The construction continues with the following lemma:

Lemma. *If U is a neighborhood of a side of a triangle T, then there is a polygonal region W containing $T \setminus U$ and contained in T, which can be tiled by a finite collection of affine copies of P.*

Proof. Denote the vertices of T by a, b and c so that U is a neighborhood of ab. We introduce a rectangular coordinate system so that ab lies on the x-axis and the y-coordinate of c is positive. The affine copies of P used for the tiling will be referred to as *tiles.* We construct the tiling by an algorithm describing the successive tiles and their respective proper places. Let G_i denote the closure of the untiled part of T at the i-th stage of the construction ($i = 0, 1, 2, \ldots$), at which point i tiles have been put in place. Obviously, at the beginning, the number of tiles placed is 0 and $G_0 = T$. We define the first tile, P_1, to be an affine copy of P properly inscribed in T and we partition G_1 into a collection of triangles $\mathcal{T}_1 = \left\{T_1^1, T_1^2, \ldots, T_1^{2n-2}\right\}$ each of which has its base on ab and top vertex at some vertex of the first tile (see Figure 2).

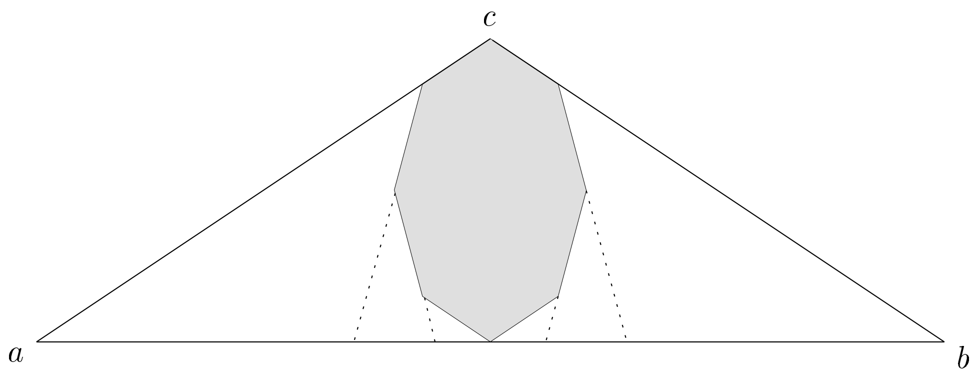


Figure 2: Partitioning G_1 into triangles.

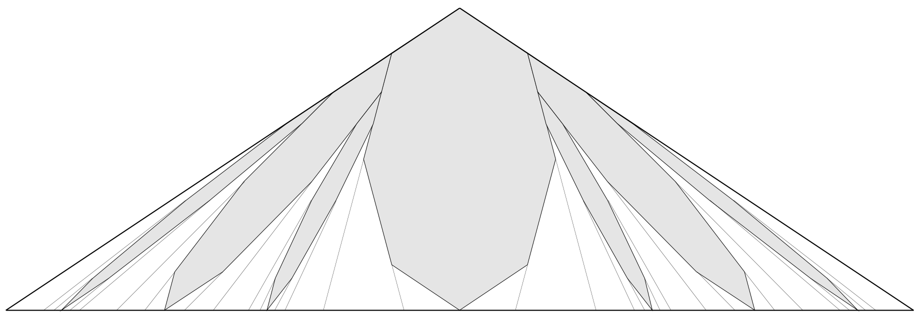


Figure 3: Tiling a triangle minus its base's neighborhood.

The formula for designing and placing the next (*i.e.*, the $(i+1)$-st) tile in T is:

Next Tile. Among all triangles of $\mathcal{T}_i$ choose a tallest one, *i.e.* one whose top vertex v has a maximum y-coordinate and call it $T_i^{\max}$. Let F be an affine transformation sending T onto $T_i^{\max}$ with $F(c) = v$ and define P_{i+1} to be $F(P_1)$, which is an affine copy of P properly inscribed in $T_i^{\max}$. Then define the partition $\mathcal{T}_{i+1}$ of G_{i+1} by replacing $T_i^{\max}$ with the images of the triangles in $\mathcal{T}_1$ under F. (Fig. 3 shows the tiling stage at $i = 7$).

Let now y_i be the maximum of the y-coordinates of points in the closure of G_i. Of course, y_i occurs at one of the vertices of G_i, thus at the top vertex of one of the triangles of $\mathcal{T}_i$. Obviously, $y_i > 0$, and $y_{i+1} \leq y_i$. Let M_i be the line $y = y_i$ and let m_i be the line $y = \mu y_i$, where $0 < \mu < 1$ is the constant described in (2). As we place the successive tile at the $(i+1)$-st stage of the construction, the top vertex of the tile eliminates one vertex of G_i lying on M_i, and, by (2), every non-top vertex of this tile lies below the line m_i. Thus, between the lines m_i and M_i, the set of vertices of G_{i+1} is obtained from the set of vertices of G_i by deleting one element. Therefore there exists an integer k such that no vertex of G_{i+k} lies above the line m_i. It follows that for every i there exists a k such that $y_{i+k} \leq \mu y_i$, and, consequently, $\lim_{i \to \infty} y_i = 0$. This implies that there is an integer k_0 such that all vertices of G_{k_0} lie in the neighborhood U of ab, and the proof of the lemma is complete.

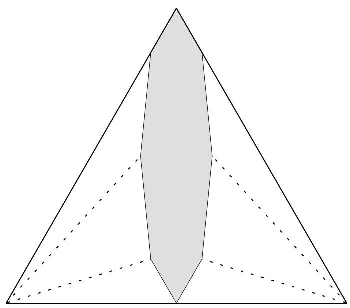


Figure 4: Triangulating the complement of the initial tile.

We now construct a periodic packing of the plane with ellipses. Begin with the familiar regular tiling of the plane with congruent equilateral triangles. Place in one of the triangles a properly inscribed affine copy of P, and call it the *initial tile*. Partition the remaining portion of the triangle into $2n-2$ smaller triangles as shown in Fig. 4. The ellipse inscribed in the initial tile, when homothetically λ-enlarged, covers a neighborhood of the tile, hence it covers a neighborhood of one edge of each of the smaller triangles. Tile each of the smaller triangles minus a neighborhood of the edge already covered, in the manner described in the proof of the Lemma (see Fig. 3). Finally, extend this pattern to all triangles of the regular tiling so that the initial tiles in each of them are translates of each other (see Figure 5).

Remark (István Talata). Instead of ellipses, affine images of an arbitrary

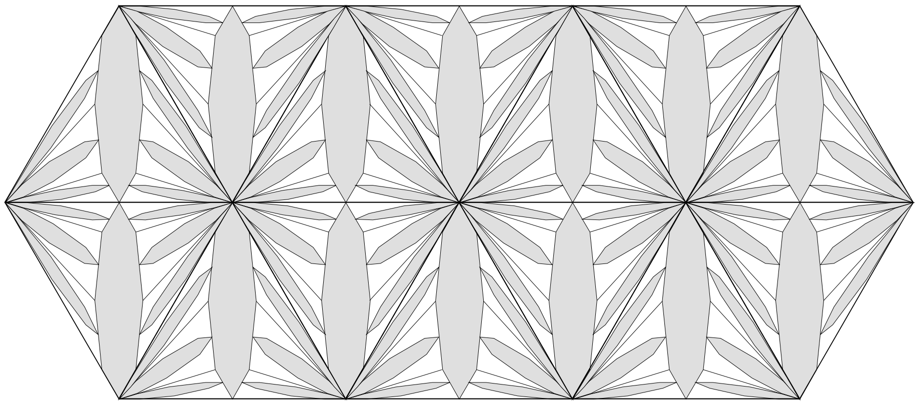


Figure 5: A periodic arrangement.

centrally symmetric convex domain can be used in the above construction, which would require minor modifications only. In other words, the Bárány number of the class of bounded-diameter affine images of a plane centrally symmetric convex domain is zero.

3 Packings with discs

Throughout this section, we have $\varepsilon = 10^{-5}$. Suppose for contradiction that there exists a packing $\mathcal{C}$ with discs of radius at most 1 such that its $(1+\varepsilon)$-enlargement covers a square S with side length 4. Let us say that a disc $C \in \mathcal{C}$ *bites* into a set $X \subseteq \mathbb{R}^2$ if $C^\varepsilon \cap X \neq \emptyset$. By induction, we are going to construct a sequence of compact sets $S = R_1 \supset R_2 \supset R_3 \ldots$ such that for each $n = 1, 2, \ldots$, no disc of $\mathcal{C}$ of radius greater than r_n bites into R_n, where $(r_n)_{n=1}^{\infty}$ is a decreasing sequence of real numbers tending to 0. Taking a point $x \in \bigcap_{n=1}^{\infty} R_n$ leads to a contradiction, since such an x cannot be covered by any C^ε with $C \in \mathcal{C}$.

Each of the regions R_n will be of one of two types, called the *square type* and the *crescent type*. We now describe the shape and the inductive hypothesis for these two types of regions.

A region R_n of the square type is a square of side $4r_n$, and we assume that no disc $C \in \mathcal{C}$ of radius greater than r_n bites into R_n. As a basis of the induction, we choose $r_1 = 1$ and we let $R_1 = S$.

A region R_n of the crescent type is defined using some disc $C_n \in \mathcal{C}$, and r_n is the radius of this C_n (see Fig. 6). We fix suitable constants[1] $\alpha = \frac{\pi}{16}$

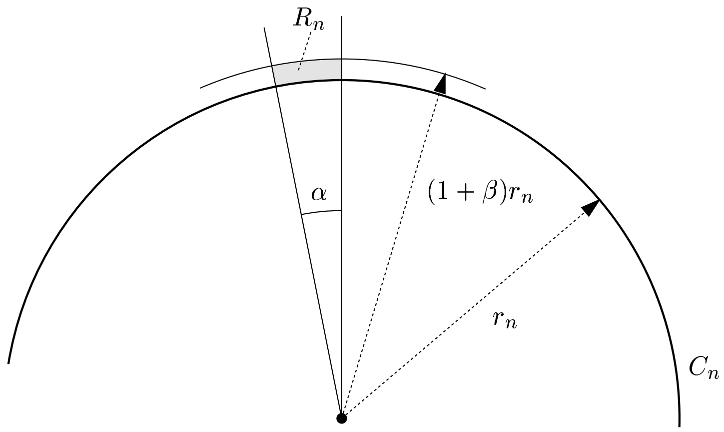
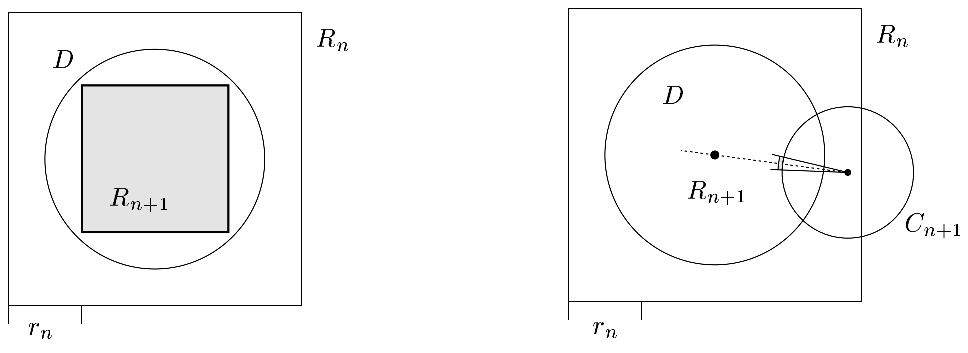


Figure 6: A crescent-type region R_n.

and $\beta = \frac{1}{16}$. Let c denote the center of C_n; then R_n is the intersection of an angle α with apex at c with the β-ring of C_n. We say that R_n is a *crescent of* C_n, and we call the halfline originating at c and dividing R_n into two equal parts the *axis of* R_n.

We describe how R_{n+1} is constructed from R_n. First, we treat the simpler case when R_n is of the square type. Let D be the disc of radius $\frac{3}{2}r_n$ centered at the center of the square R_n (Fig. 7). Choose $C_{n+1} \in \mathcal{C}$ as the disc of the largest radius that bites into D. If the radius of C_{n+1} is at most $r_n/2$, set $r_{n+1} = r_n/2$ and pick the region R_{n+1} as a square of side $4r_{n+1}$ inside the disc D, as in Fig. 7(a). Otherwise, let r_{n+1} be the radius of C_{n+1}. In this case, we pick R_{n+1} as a crescent of C_{n+1}. The axis of R_{n+1} is the halfline originating at the center of C_{n+1} and passing through the center of D; see Fig. 7(b) (if these centers happen to coincide then pick an arbitrary direction of the axis). This finishes the definition of R_{n+1}. Easy geometric considerations, whose details

[1] The choice of the constants in the proof is somewhat arbitrary. The goal, rather than trying to get the best value of ε, was to select them in such a way that realistic pictures can be drawn.

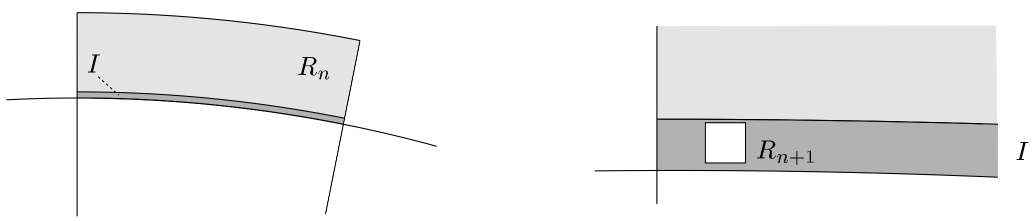


(a) (b)

Figure 7: The inductive step for a square-type region R_n.

we omit, show that thus constructed R_{n+1} satisfies the inductive hypothesis (i.e. no disc of radius larger than r_{n+1} bites into R_{n+1}).

It remains to discuss the inductive step from R_n to R_{n+1} for an R_n of the crescent type. For a simpler notation, we will measure distances in the units of r_n from now on, that is, we may assume $r_n = 1$. In this case, we let I denote the intersection of R_n with the $\frac{\beta}{16}$-ring of C_n (see Fig. 8(a)). Let C



(a) (b)

Figure 8: The region I (a), and the case of a very small r (b).

be the largest disc of $\mathcal{C}$ distinct from C_n biting into I, and let r be the radius of C. Here we distinguish three cases: $r \leq \frac{\beta}{80}$, $\frac{\beta}{80} < r \leq \frac{1}{8}$, and $\frac{1}{8} < r \leq 1$.

The case $r \leq \frac{\beta}{80}$. Here we set $r_{n+1} = \frac{\beta}{80}$ and we choose R_{n+1} as a square of side $\frac{\beta}{20}$ within I so that C_n doesn't bite into it, as in Fig. 8(b). This is a valid region of the square type.

The case $\frac{\beta}{80} < r \leq \frac{1}{8}$. Let c_n denote the center of C_n, and let c be the center of C. We set $C_{n+1} = C$, we define r_{n+1} as the radius of C_{n+1}, and we choose R_{n+1} as a crescent of C_{n+1} as follows (Fig. 9). The angle of the axis

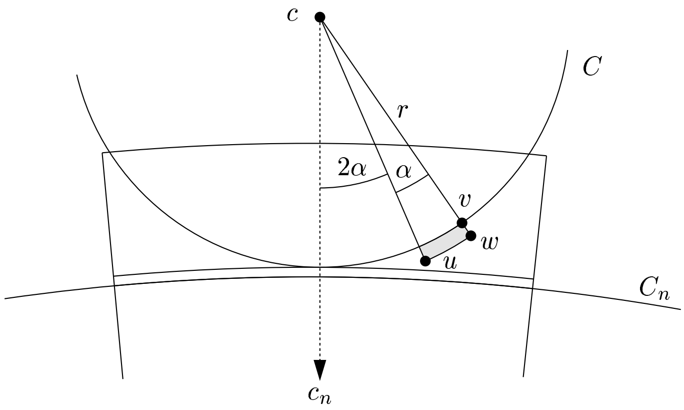


Figure 9: The case $\frac{\beta}{80} < r \leq \frac{1}{8}$.

of R_{n+1} and of the halfline cc_n is $\frac{5}{2}\alpha$ and R_{n+1} lies on the side of the segment $c_n c$ closer to the axis of R_n. To verify the inductive hypothesis for R_{n+1}, we first need to show that $R_{n+1} \subseteq R_n$. That is, we need to check that the point v in Fig. 9 has distance at most $1+\beta$ from c_n, that u has distance at least 1 from c_n, and that the point w cannot go beyond the side boundary of R_n. Let us check these conditions computationally. As for the first inequality, $|vc_n| \leq 1+\beta$, we use the cosine theorem for the triangle $c_n cv$:

$$|c_n v|^2 = |c_n c|^2 + |cv|^2 - 2|c_n c|.|cv| \cos 3\alpha = (|c_n c| - |cv|)^2 + 2|c_n c| \cdot |cv|(1 - \cos 3\alpha) \leq$$

$$\leq (1 + \frac{\beta}{16} + \frac{\varepsilon}{8})^2 + 2(1 + \frac{\beta}{16} + (1+\varepsilon)\frac{1}{8}) \cdot \frac{1}{8}(1 - \cos 3\alpha) = 1.055...$$

Thus, $|c_n v| < 1.03 < 1 + \beta = 1.0625$.

Similarly,

$$|c_n u|^2 = (|c_n c| - |cu|)^2 + 2|c_n c| \cdot |cu|(1 - \cos 2\alpha) > (1 - \beta r)^2 + 2 \cdot 1 \cdot r \cdot (1 - \cos 2\alpha) >$$

$$> 1 + 2r(1 - \cos 2\alpha - \beta) > 1 + \frac{\beta}{40}(1 - \cos 2\alpha - \beta) = 1.000021...$$

To show that $R_{n+1} \subseteq R_n$, it remains to verify that the point w cannot go beyond the side boundary of R_n. Suppose to a contrary that the angle cc_nw is more than $\frac{\alpha}{2}$. Then the distance of the point w to the line cc_n is at least $r_n \sin\frac{\alpha}{2} = \sin\frac{\alpha}{2}$. On the other hand, the same distance equals $(1+\beta)r\sin 3\alpha$. It follows that $\sin\frac{\alpha}{2} \le (1+\beta)r\sin 3\alpha \le (1+\beta)\frac{1}{8}\sin 3\alpha < \frac{1}{6}\sin 3\alpha$, which contradicts the concavity of the sine function in the interval $[0,\pi]$.

To verify the induction hypothesis for R_{n+1}, it remains to show that no disc C' of $\mathcal{C}$ with radius in the interval $(r,1]$ may bite into R_{n+1}. Since $r \ge \frac{\beta}{80}$ is not too small, any such C' biting into R_{n+1} would have to intersect C_n or C. C_n itself doesn't bite into R_{n+1}, since $|c_nu|^2 > 1.000021$ and thus $|c_nu| > 1+\varepsilon$.

The case $\frac{1}{8} < r \le 1$. Here we have the relatively large disc C biting into the region I (Fig. 10). Consider the circular arc a bounding the region I

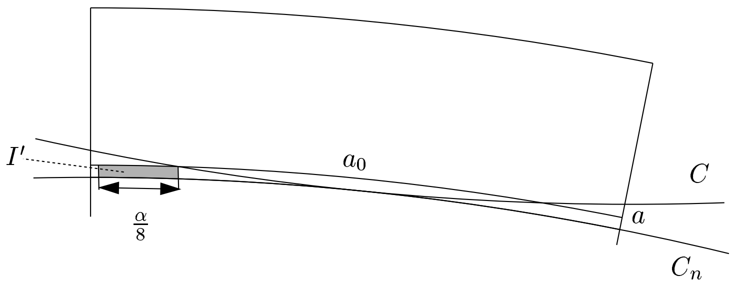


Figure 10: The case $\frac{1}{8} < r \le 1$.

from the outer side, and let a_0 be the portion of this arc contained in the the disc C^ε. Calculation shows that even if C has the largest possible radius 1 and touches C_n in the middle of the region R_n, one of the portions of $a \setminus a_0$ has angular length at least $\frac{\alpha}{8}$ (this extreme case is shown in Fig. 10). We thus select a portion I' of the region I of angular length $\frac{\alpha}{8}$, avoiding a_0 but adjacent to it.

What is the largest possible radius of a disc $C' \ne C_n$ of $\mathcal{C}$ that may bite into I'? The possible radius is largest when r is smallest, that is, $r = \frac{1}{8}$. Fig. 11 shows how to upper-bound the radius of C'; the radius of the disc

D drawn there, which is well below $\frac{1}{20}$, is an upper bound for the radius of C'. Now we repeat the considerations made above with the region I' instead

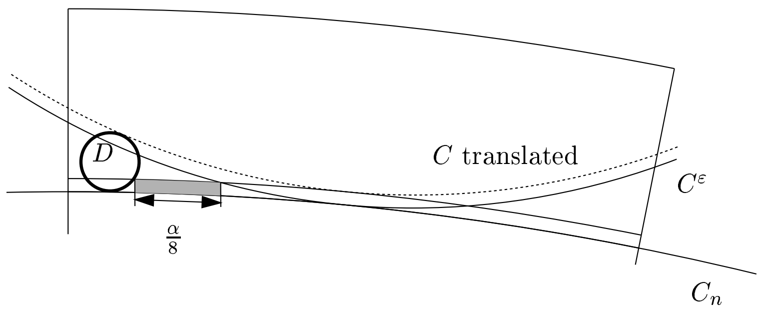


Figure 11: Estimating the radius of C'.

of I, that is, we choose the largest disc $C' \neq C_n$ biting into I', we let r' be its radius, and discuss the cases depending on the range of r'. The first two cases ($r' \leq \frac{\beta}{80}$ and $\frac{\beta}{80} < r' \leq \frac{1}{8}$) work in the same way as above; the only small change is that I' is shorter than I, so one has to check that there's always enough room to accommodate the corner w of the region R_{n+1} (as in Fig. 9) in the region R_n. But this works because r' is small enough. And, because of the restriction $r' \leq \frac{1}{20}$, the third case discussed for the region I cannot occur for I'. Theorem 2 is proved.

References

[1] I. Bárány, Problem 1, Auburn Geometry Mini-workshop, held at Auburn University, Alabama, October 13-16, 1997.