

$D$  drawn there, which is well below  $\frac{1}{20}$ , is an upper bound for the radius of  $C'$ . Now we repeat the considerations made above with the region  $I'$  instead

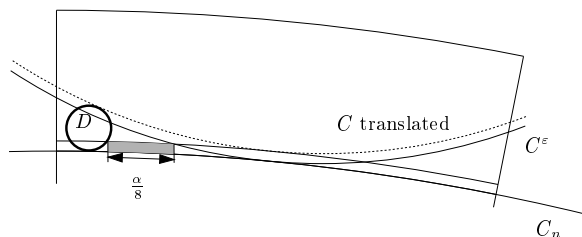


Figure 11: Estimating the radius of  $C'$ .

of  $I$ , that is, we choose the largest disc  $C' \neq C_n$  biting into  $I'$ , we let  $r'$  be its radius, and discuss the cases depending on the range of  $r'$ . The first two cases ( $r' \leq \frac{\beta}{80}$  and  $\frac{\beta}{80} < r' \leq \frac{1}{8}$ ) work in the same way as above; the only small change is that  $I'$  is shorter than  $I$ , so one has to check that there's always enough room to accommodate the corner  $w$  of the region  $R_{n+1}$  (as in Fig. 9) in the region  $R_n$ . But this works because  $r'$  is small enough. And, because of the restriction  $r' \leq \frac{1}{20}$ , the third case discussed for the region  $I$  cannot occur for  $I'$ . Theorem 2 is proved.

## References

- [1] I. Bárány, Problem 1, Auburn Geometry Mini-workshop, held at Auburn University, Alabama, October 13-16, 1997.

## Almost-tiling the plane by ellipses \*

KRYSZYNA KUPERBERG

Department of Mathematics  
Auburn University  
Auburn, AL 36849-5310  
U.S.A.

e-mail: kuperkm@mail.auburn.edu

WLODZIMIERZ KUPERBERG

Department of Mathematics  
Auburn University  
Auburn, AL 36849-5310  
U.S.A.

e-mail: kuperwl@mail.auburn.edu

JIRÍ MATOUŠEK

Department of Applied Mathematics  
Charles University  
Malostranské nám. 25, 118 00 Praha 1  
Czech Republic  
and

Institut für Theoretische Informatik  
ETH Zürich, Switzerland  
e-mail: matousek@kam.mff.cuni.cz

PAVEL VALTR

Department of Applied Mathematics  
Charles University  
Malostranské nám. 25, 118 00 Praha 1  
Czech Republic  
and

DIMACS Center  
Rutgers University, NJ, U.S.A.  
e-mail: valtr@kam.mff.cuni.cz

*Dedicated to our friend Imre Bárány on the occasion of his 50-th birthday.*

### Abstract

For any  $\lambda > 1$  we construct a periodic and locally finite packing of the plane with ellipses whose  $\lambda$ -enlargement covers the whole plane. This answers a question of Imre Bárány. On the other hand, we show that if  $\mathcal{C}$  is a packing in the plane with circular discs of radius at most 1, then its  $(1 + 10^{-5})$ -enlargement covers no square with side length 4.

\*Research of K. Kuperberg was supported by NSF grant No. DMS-9704958. Research of W. Kuperberg was supported by NSF grant No. DMS-9704319. Research of J. Matoušek was supported by ETH Zürich, by Czech Republic Grant GAČR 0194/1996, and by Charles University grants No. 193,194/1996. Research of P. Valtr was supported by DIMACS Center, by Czech Republic Grant GAČR 0194/1996, and by Charles University grants No. 193,194/1996.

# 1 Introduction

Let  $\mathcal{C}$  be a system (finite or infinite) of centrally symmetric convex bodies in  $\mathbb{R}^d$  with disjoint interiors; we call such a  $\mathcal{C}$  a *packing*. For a real number  $\varepsilon > 0$  and for  $C \in \mathcal{C}$ , we let  $C^\varepsilon$  denote  $C$  enlarged by the factor  $1 + \varepsilon$  from its center, that is,  $C^\varepsilon = (1 + \varepsilon)(C - c) + c$ , where  $c$  stands for the center of symmetry  $C$ . Let us call the closure of the set  $C^\varepsilon \setminus C$  the  $\varepsilon$ -ring of  $C$ . We call the system  $\mathcal{C}^\varepsilon = \{C^\varepsilon; C \in \mathcal{C}\}$  the  $(1 + \varepsilon)$ -enlargement of  $\mathcal{C}$ .

For a class  $\mathcal{C}_0$  of centrally symmetric convex bodies in  $\mathbb{R}^d$ , we define the *Bárány number* of  $\mathcal{C}_0$  as the infimum of the numbers  $\varepsilon > 0$  such that there exists a packing  $\mathcal{C} \subseteq \mathcal{C}_0$  whose  $(1 + \varepsilon)$ -enlargement covers the whole  $\mathbb{R}^d$ .

We observe that the system of all circular discs in the plane has Bárány number 0, since we can produce the desired packing for any  $\varepsilon > 0$  by using larger and larger discs (add discs to the packing one by one, and in the  $i$ th step, choose the  $i$ th disc so that its  $\varepsilon$ -ring covers the disc of radius  $i$  around the origin). A different situation may occur if the diameter of the bodies in  $\mathcal{C}_0$  is bounded.

Motivated by a problem concerning convex polytopes, Imre Bárány [1] raised a problem which in our terminology can be rephrased as follows: If  $\mathcal{E}$  stands for the class of all ellipses of diameter at most 1, is the Bárány number of  $\mathcal{E}$  zero?

In this paper we give a positive answer to this question:

**Theorem 1** *For every  $\lambda > 1$  there is a periodic packing of the plane with ellipses whose  $\lambda$ -enlargement is a covering.*

On the other hand, if we allow only discs of bounded radius, then Bárány's question has a negative answer:

**Theorem 2** *Let  $\mathcal{C}$  be a packing of the plane with circular discs of radius at most 1. Then  $(1 + 10^{-5})$ -enlargement of  $\mathcal{C}$  covers no square with side length 4.*

**Remarks.** Our packing in Theorem 1 is locally finite and the details in the construction can be done so that all ellipses in the packing have diameter between  $\varepsilon/10$  and 1, where  $\varepsilon = \lambda - 1$  (however, their width varies from  $\exp[-c/\varepsilon \log^2(1/\varepsilon)]$  to  $c$  and we need  $\exp[c/\varepsilon \log(1/\varepsilon)]$  of them on each unit

$$> 1 + 2r(1 - \cos 2\alpha - \beta) > 1 + \frac{\beta}{40}(1 - \cos 2\alpha - \beta) = 1.000021\dots$$

To show that  $R_{n+1} \subseteq R_n$ , it remains to verify that the point  $w$  cannot go beyond the side boundary of  $R_n$ . Suppose to a contrary that the angle  $cc_nw$  is more than  $\frac{\alpha}{2}$ . Then the distance of the point  $w$  to the line  $cc_n$  is at least  $r_n \sin \frac{\alpha}{2} = \sin \frac{\alpha}{2}$ . On the other hand, the same distance equals  $(1 + \beta)r \sin 3\alpha$ . It follows that  $\sin \frac{\alpha}{2} \leq (1 + \beta)r \sin 3\alpha \leq (1 + \beta)\frac{1}{8} \sin 3\alpha < \frac{1}{6} \sin 3\alpha$ , which contradicts the concavity of the sine function in the interval  $[0, \pi]$ .

To verify the induction hypothesis for  $R_{n+1}$ , it remains to show that no disc  $C'$  of  $\mathcal{C}$  with radius in the interval  $(r, 1]$  may bite into  $R_{n+1}$ . Since  $r \geq \frac{\beta}{80}$  is not too small, any such  $C'$  biting into  $R_{n+1}$  would have to intersect  $C_n$  or  $C$ .  $C_n$  itself doesn't bite into  $R_{n+1}$ , since  $|c_n u|^2 > 1.000021$  and thus  $|c_n u| > 1 + \varepsilon$ .

*The case  $\frac{1}{8} < r \leq 1$ .* Here we have the relatively large disc  $C$  biting into the region  $I$  (Fig. 10). Consider the circular arc  $a$  bounding the region  $I$

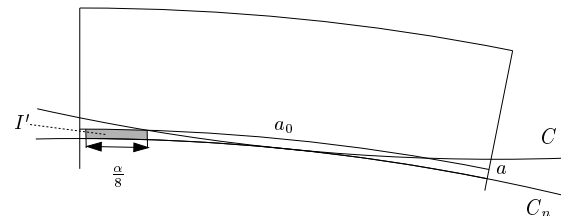


Figure 10: The case  $\frac{1}{8} < r \leq 1$ .

from the outer side, and let  $a_0$  be the portion of this arc contained in the the disc  $C^\varepsilon$ . Calculation shows that even if  $C$  has the largest possible radius 1 and touches  $C_n$  in the middle of the region  $R_n$ , one of the portions of  $a \setminus a_0$  has angular length at least  $\frac{\alpha}{8}$  (this extreme case is shown in Fig. 10). We thus select a portion  $I'$  of the region  $I$  of angular length  $\frac{\alpha}{8}$ , avoiding  $a_0$  but adjacent to it.

What is the largest possible radius of a disc  $C' \neq C_n$  of  $\mathcal{C}$  that may bite into  $I'$ ? The possible radius is largest when  $r$  is smallest, that is,  $r = \frac{1}{8}$ . Fig. 11 shows how to upper-bound the radius of  $C'$ ; the radius of the disc

of side  $\frac{\beta}{20}$  within  $I$  so that  $C_n$  doesn't bite into it, as in Fig. 8(b). This is a valid region of the square type.

The case  $\frac{\beta}{80} < r \leq \frac{1}{8}$ . Let  $c_n$  denote the center of  $C_n$ , and let  $c$  be the center of  $C$ . We set  $C_{n+1} = C$ , we define  $r_{n+1}$  as the radius of  $C_{n+1}$ , and we choose  $R_{n+1}$  as a crescent of  $C_{n+1}$  as follows (Fig. 9). The angle of the axis

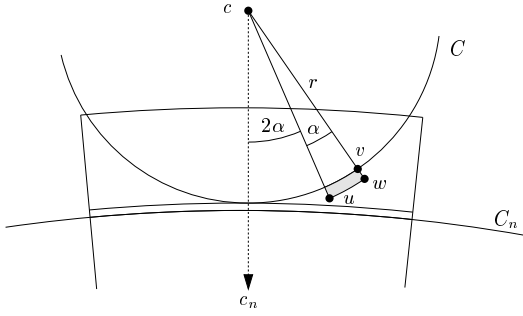


Figure 9: The case  $\frac{\beta}{80} < r \leq \frac{1}{8}$ .

of  $R_{n+1}$  and of the halfline  $cc_n$  is  $\frac{5}{2}\alpha$  and  $R_{n+1}$  lies on the side of the segment  $c_n c$  closer to the axis of  $R_n$ . To verify the inductive hypothesis for  $R_{n+1}$ , we first need to show that  $R_{n+1} \subseteq R_n$ . That is, we need to check that the point  $v$  in Fig. 9 has distance at most  $1 + \beta$  from  $c_n$ , that  $u$  has distance at least 1 from  $c_n$ , and that the point  $w$  cannot go beyond the side boundary of  $R_n$ . Let us check these conditions computationally. As for the first inequality,  $|vc_n| \leq 1 + \beta$ , we use the cosine theorem for the triangle  $c_n cv$ :

$$\begin{aligned} |c_n v|^2 &= |c_n c|^2 + |cv|^2 - 2|c_n c| \cdot |cv| \cos 3\alpha = (|c_n c| - |cv|)^2 + 2|c_n c| \cdot |cv|(1 - \cos 3\alpha) \leq \\ &\leq \left(1 + \frac{\beta}{16} + \frac{\varepsilon}{8}\right)^2 + 2\left(1 + \frac{\beta}{16} + (1 + \varepsilon)\frac{1}{8}\right) \cdot \frac{1}{8}(1 - \cos 3\alpha) = 1.055\dots \end{aligned}$$

Thus,  $|c_n v| < 1.03 < 1 + \beta = 1.0625$ .

Similarly,

$$|c_n u|^2 = (|c_n c| - |cu|)^2 + 2|c_n c| \cdot |cu|(1 - \cos 2\alpha) > (1 - \beta r)^2 + 2 \cdot 1 \cdot r \cdot (1 - \cos 2\alpha) >$$

square). Our methods can be used to prove that Theorem 2 (possibly with a different positive constant instead of  $10^{-5}$ ) holds also in any dimension  $d \geq 2$  and when  $\mathcal{C}_0$  consists of convex bodies in  $\mathbb{R}^d$  with a constant-bounded diameter and curvature. We do not prove these generalizations here, since the idea remains the same but the details become messy. The value  $\varepsilon = 10^{-5}$  in Theorem 2 is certainly not the best possible one could get by our proof method, but it seems that a different method would be needed to determine the Bárány number for discs in the plane exactly or at least to prove a reasonable lower bound for its value.

## 2 Almost-tiling by ellipses

Throughout the construction, a number  $\lambda > 1$  remains fixed. Choose an integer  $n$  such that the regular  $2n$ -polygon  $P = P_{2n}$  circumscribed about a circular disk  $D$  is contained in the interior of the  $\lambda$ -enlargement of  $D$ . Denote two antipodal vertices of  $P$  by  $v^-$  and  $v^+$ . Suppose  $T$  is a triangle with a horizontal base  $B$  and vertex  $v$  above  $B$ . Then there is a (unique) polygon  $P(T)$  satisfying the following conditions (see Fig. 1 which illustrates the case  $n = 4$ ):

- (i)  $P(T) \subset T$ ; (ii) There is an affine transformation  $A$  such that  $A(P) = P(T)$ ; (iii)  $A(v^-)$  is the midpoint of  $B$  and  $A(v^+) = v$ ; (iv) The angles at  $v$  of  $P(T)$  and of  $T$  are equal.

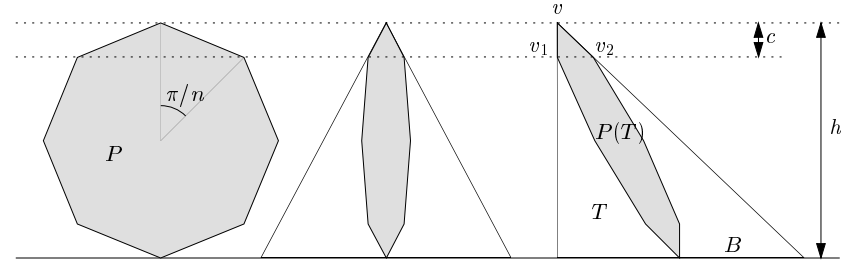


Figure 1: The affine copy of  $P$  properly inscribed in  $T$ .

We say that  $P(T)$  is an *affine copy of  $P$  properly inscribed in  $T$* . The existence and uniqueness of  $P(T)$  follows from the fact that  $P$  itself, oriented so that one of its main diagonals is vertical, is properly inscribed in a triangle, and an affine transformation that sends this triangle onto  $T$  (top vertex onto top vertex and base onto base) determines  $P(T)$  uniquely. Obviously,  $P(T)$  contains an inscribed ellipse, namely  $A(D)$ , whose  $\lambda$ -enlargement contains a neighborhood of  $P(T)$ .

Observe the following property of the polygon  $P(T)$ :

(1) Let  $v_1$  and  $v_2$  be the vertices of  $P(T)$  adjacent to  $v$ . Then the line  $v_1v_2$  is parallel to  $B$  and partitions the height  $h$  of  $T$  at the ratio of

$$c : (h - c) = \left(1 - \cos \frac{\pi}{n}\right) : \left(1 + \cos \frac{\pi}{n}\right),$$

where  $c$  is the portion of  $h$  containing  $v$  (see Fig. 1).

It follows that

(2) For every vertex  $w \neq v$  of  $P(T)$  the distance from  $w$  to the line of  $B$  is less than or equal to  $\mu h$ , where  $\mu < 1$  is a positive constant independent from  $T$ . Specifically,

$$\mu = \frac{1}{2} \left(1 + \cos \frac{\pi}{n}\right).$$

The construction continues with the following lemma:

**Lemma.** *If  $U$  is a neighborhood of a side of a triangle  $T$ , then there is a polygonal region  $W$  containing  $T \setminus U$  and contained in  $T$ , which can be tiled by a finite collection of affine copies of  $P$ .*

**Proof.** Denote the vertices of  $T$  by  $a, b$  and  $c$  so that  $U$  is a neighborhood of  $ab$ . We introduce a rectangular coordinate system so that  $ab$  lies on the  $x$ -axis and the  $y$ -coordinate of  $c$  is positive. The affine copies of  $P$  used for the tiling will be referred to as *tiles*. We construct the tiling by an algorithm describing the successive tiles and their respective proper places. Let  $G_i$  denote the closure of the untiled part of  $T$  at the  $i$ -th stage of the construction ( $i = 0, 1, 2, \dots$ ), at which point  $i$  tiles have been put in place. Obviously, at the beginning, the number of tiles placed is 0 and  $G_0 = T$ . We define the first tile,  $P_1$ , to be an affine copy of  $P$  properly inscribed in  $T$  and we partition  $G_1$  into a collection of triangles  $\mathcal{T}_1 = \{T_1^1, T_1^2, \dots, T_1^{2^{n-2}}\}$  each of which has its base on  $ab$  and top vertex at some vertex of the first tile (see Figure 2).

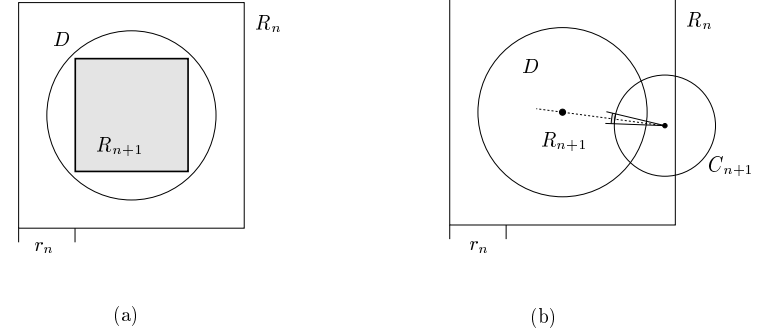


Figure 7: The inductive step for a square-type region  $R_n$ .

we omit, show that thus constructed  $R_{n+1}$  satisfies the inductive hypothesis (i.e. no disc of radius larger than  $r_{n+1}$  bites into  $R_{n+1}$ ).

It remains to discuss the inductive step from  $R_n$  to  $R_{n+1}$  for an  $R_n$  of the crescent type. For a simpler notation, we will measure distances in the units of  $r_n$  from now on, that is, we may assume  $r_n = 1$ . In this case, we let  $I$  denote the intersection of  $R_n$  with the  $\frac{\beta}{16}$ -ring of  $C_n$  (see Fig. 8(a)). Let  $C$

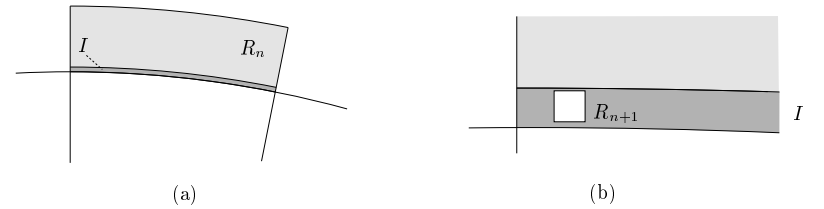


Figure 8: The region  $I$  (a), and the case of a very small  $r$  (b).

be the largest disc of  $\mathcal{C}$  distinct from  $C_n$  biting into  $I$ , and let  $r$  be the radius of  $C$ . Here we distinguish three cases:  $r \leq \frac{\beta}{80}$ ,  $\frac{\beta}{80} < r \leq \frac{1}{8}$ , and  $\frac{1}{8} < r \leq 1$ . *The case  $r \leq \frac{\beta}{80}$ .* Here we set  $r_{n+1} = \frac{\beta}{80}$  and we choose  $R_{n+1}$  as a square

induction, we choose  $r_1 = 1$  and we let  $R_1 = S$ .

A region  $R_n$  of the crescent type is defined using some disc  $C_n \in \mathcal{C}$ , and  $r_n$  is the radius of this  $C_n$  (see Fig. 6). We fix suitable constants<sup>1</sup>  $\alpha = \frac{\pi}{16}$

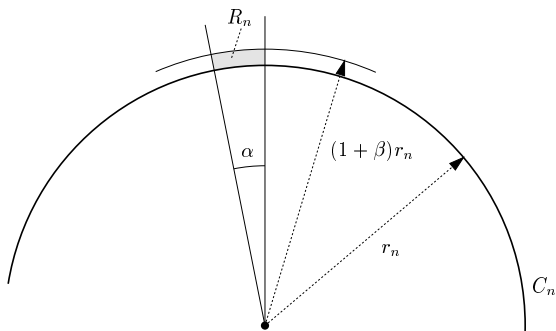


Figure 6: A crescent-type region  $R_n$ .

and  $\beta = \frac{1}{16}$ . Let  $c$  denote the center of  $C_n$ ; then  $R_n$  is the intersection of an angle  $\alpha$  with apex at  $c$  with the  $\beta$ -ring of  $C_n$ . We say that  $R_n$  is a *crescent of  $C_n$* , and we call the halfline originating at  $c$  and dividing  $R_n$  into two equal parts the *axis of  $R_n$* .

We describe how  $R_{n+1}$  is constructed from  $R_n$ . First, we treat the simpler case when  $R_n$  is of the square type. Let  $D$  be the disc of radius  $\frac{3}{2}r_n$  centered at the center of the square  $R_n$  (Fig. 7). Choose  $C_{n+1} \in \mathcal{C}$  as the disc of the largest radius that bites into  $D$ . If the radius of  $C_{n+1}$  is at most  $r_n/2$ , set  $r_{n+1} = r_n/2$  and pick the region  $R_{n+1}$  as a square of side  $4r_{n+1}$  inside the disc  $D$ , as in Fig. 7(a). Otherwise, let  $r_{n+1}$  be the radius of  $C_{n+1}$ . In this case, we pick  $R_{n+1}$  as a crescent of  $C_{n+1}$ . The axis of  $R_{n+1}$  is the halfline originating at the center of  $C_{n+1}$  and passing through the center of  $D$ ; see Fig. 7(b) (if these centers happen to coincide then pick an arbitrary direction of the axis). This finishes the definition of  $R_{n+1}$ . Easy geometric considerations, whose details

<sup>1</sup>The choice of the constants in the proof is somewhat arbitrary. The goal, rather than trying to get the best value of  $\varepsilon$ , was to select them in such a way that realistic pictures can be drawn.

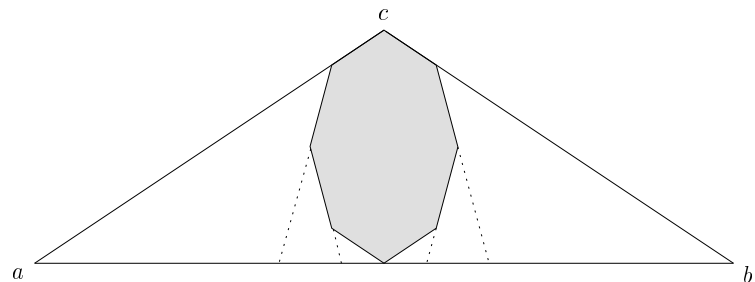


Figure 2: Partitioning  $G_1$  into triangles.

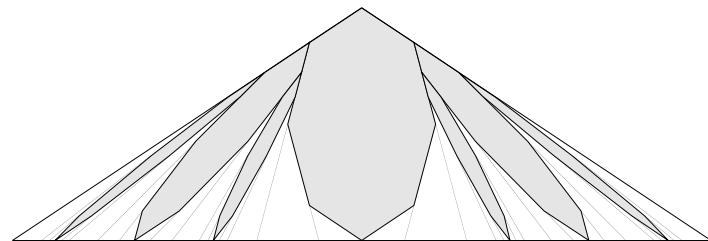


Figure 3: Tiling a triangle minus its base's neighborhood.

The formula for designing and placing the next (*i.e.*, the  $(i+1)$ -st) tile in  $T$  is:

*Next Tile.* Among all triangles of  $\mathcal{T}_i$  choose a tallest one, *i.e.* one whose top vertex  $v$  has a maximum  $y$ -coordinate and call it  $T_i^{\max}$ . Let  $F$  be an affine transformation sending  $T$  onto  $T_i^{\max}$  with  $F(c) = v$  and define  $P_{i+1}$  to be  $F(P_1)$ , which is an affine copy of  $P$  properly inscribed in  $T_i^{\max}$ . Then define the partition  $\mathcal{T}_{i+1}$  of  $G_{i+1}$  by replacing  $T_i^{\max}$  with the images of the triangles in  $\mathcal{T}_1$  under  $F$ . (Fig. 3 shows the tiling stage at  $i = 7$ ).

Let now  $y_i$  be the maximum of the  $y$ -coordinates of points in the closure of  $G_i$ . Of course,  $y_i$  occurs at one of the vertices of  $G_i$ , thus at the top vertex

of one of the triangles of  $\mathcal{T}_i$ . Obviously,  $y_i > 0$ , and  $y_{i+1} \leq y_i$ . Let  $M_i$  be the line  $y = y_i$  and let  $m_i$  be the line  $y = \mu y_i$ , where  $0 < \mu < 1$  is the constant described in (2). As we place the successive tile at the  $(i + 1)$ -st stage of the construction, the top vertex of the tile eliminates one vertex of  $G_i$  lying on  $M_i$ , and, by (2), every non-top vertex of this tile lies below the line  $m_i$ . Thus, between the lines  $m_i$  and  $M_i$ , the set of vertices of  $G_{i+1}$  is obtained from the set of vertices of  $G_i$  by deleting one element. Therefore there exists an integer  $k$  such that no vertex of  $G_{i+k}$  lies above the line  $m_i$ . It follows that for every  $i$  there exists a  $k$  such that  $y_{i+k} \leq \mu y_i$ , and, consequently,  $\lim_{i \rightarrow \infty} y_i = 0$ . This implies that there is an integer  $k_0$  such that all vertices of  $G_{k_0}$  lie in the neighborhood  $U$  of  $ab$ , and the proof of the lemma is complete.

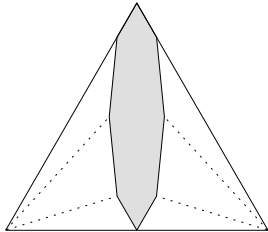


Figure 4: Triangulating the complement of the initial tile.

We now construct a periodic packing of the plane with ellipses. Begin with the familiar regular tiling of the plane with congruent equilateral triangles. Place in one of the triangles a properly inscribed affine copy of  $P$ , and call it the *initial tile*. Partition the remaining portion of the triangle into  $2n - 2$  smaller triangles as shown in Fig. 4. The ellipse inscribed in the initial tile, when homothetically  $\lambda$ -enlarged, covers a neighborhood of the tile, hence it covers a neighborhood of one edge of each of the smaller triangles. Tile each of the smaller triangles minus a neighborhood of the edge already covered, in the manner described in the proof of the Lemma (see Fig. 3). Finally, extend this pattern to all triangles of the regular tiling so that the initial tiles in each of them are translates of each other (see Figure 5).

**Remark (István Talata).** Instead of ellipses, affine images of an arbitrary

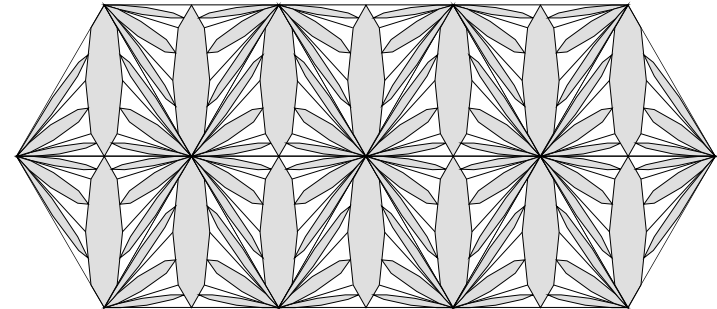


Figure 5: A periodic arrangement.

centrally symmetric convex domain can be used in the above construction, which would require minor modifications only. In other words, the B ar any number of the class of bounded-diameter affine images of a plane centrally symmetric convex domain is zero.

### 3 Packings with discs

Throughout this section, we have  $\varepsilon = 10^{-5}$ . Suppose for contradiction that there exists a packing  $\mathcal{C}$  with discs of radius at most 1 such that its  $(1 + \varepsilon)$ -enlargement covers a square  $S$  with side length 4. Let us say that a disc  $C \in \mathcal{C}$  *bites* into a set  $X \subseteq \mathbb{R}^2$  if  $C^\varepsilon \cap X \neq \emptyset$ . By induction, we are going to construct a sequence of compact sets  $S = R_1 \supset R_2 \supset R_3 \dots$  such that for each  $n = 1, 2, \dots$ , no disc of  $\mathcal{C}$  of radius greater than  $r_n$  bites into  $R_n$ , where  $(r_n)_{n=1}^\infty$  is a decreasing sequence of real numbers tending to 0. Taking a point  $x \in \bigcap_{n=1}^\infty R_n$  leads to a contradiction, since such an  $x$  cannot be covered by any  $C^\varepsilon$  with  $C \in \mathcal{C}$ .

Each of the regions  $R_n$  will be of one of two types, called the *square type* and the *crescent type*. We now describe the shape and the inductive hypothesis for these two types of regions.

A region  $R_n$  of the square type is a square of side  $4r_n$ , and we assume that no disc  $C \in \mathcal{C}$  of radius greater than  $r_n$  bites into  $R_n$ . As a basis of the