Structure of the graph homomorphisms II.

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Abstract

We consider three aspects of homomorphisms of graphs and hypergraphs which are related to the structure of color classes: 1. density, 2. fractal property and 3. generating color classes. In particular we prove the density theorem for hypergraphs and we show that for connected oriented graphs all jumps are balanced (and give an example that the connectivity is needed here). We also show that the color classes are the only homomorphism-defined classes of graphs which are finitely generated.

1 Introduction and statement of results

Graph theory receives its mathematical motivation connection from the two main areas of mathematics: algebra and geometry (topology and the graph notions stood at the birth of algebraic topology). Consequently various operations and comparisions for graphs stress either its algebraical part (e.g. various products) or geometrical part (e.g. contraction, subdivision). It is only natural that the key place in the modern graph theory is played by (fortunate) mixtures of both approaches as exhibited best by the various modifications of the notion of graph minor. However from the algebraical point of view perhaps the most natural graph notion is the notion of a homomorphism:

Given two graphs $G = (V, E)$ and $G' = (V', E')$ a homomorphism f of G into G' is any mapping $f : V \longrightarrow V'$ which satisfies the following

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condition: (1) $[x,y] \in E$ implies $[f(x), f(y)] \in E'$ The condition (1) should

be understood as follows: on both sides of the implication one considers the same type of edges (undirected, directed). The analogous definitions give notions of the homomorphism for hypergraphs (set systems) and relational systems.

Homomorphism is an algebraical notion which in graph theory found its way to problems related to products, reconstruction and chromatic polynomial , to name just a few. Our approach here is motivated by the chromatic number connection expressed by the following observation which holds for every undirected graph G: $G \longrightarrow K_k$ iff $\chi(G) \leq k$.

Motivated by this we call a homomorphism $G \longrightarrow H$ a *H-coloring* of G and given H we call the class of all graphs G which are H-colorable the *color class determined by H*. The color class determined by H is denoted by $\longrightarrow H$ or by $\mathcal{C}_H$. Thus $\longrightarrow H$ is the class of all H-colorable graphs G. The class of all color classes determines the partially ordered class $\mathcal{C}$ ordered by the inclusion. The structure of $\mathcal{C}$ is one of the subjects of this paper.

Note first that the inclusion of the color classes $\longrightarrow H \subset\longrightarrow H'$ is equivalent to $H \longrightarrow H'$. Thus the graphs H and H' determine the same color class iff H and H' are homomorphism equivalent (by this we mean that both $H \longrightarrow H'$ and $H' \longrightarrow H$). This is (for finite graphs) best to express by means of the notion of the *core* (core of H is the minimal subgraph of H which is a homomorphic image of H;[HN1]) by saying that H and H' determine the same color class iff H and H' have the same core.

The color classes corresponding to chromatic number are the color classes determined by complete graphs. They form a chain isomorphic to N. This simplistic illusion is quickly destroyed by the moment of thought and it appears that to the contrary of the first evidence the class $\mathcal{C}$ is a very rich class. The following are extremal results in this direction, see [PT] and [W]:

Theorem 1.1 *The class $\mathcal{C}$ is universal for all partially ordered classes. Explicitely, every partially ordered class is isomorphic to an induced subclass of $\mathcal{C}$. Moreover the class of all color classes determined by finite graphs is universal for all countable partially ordered sets.*

Theorem 1.2 *The class of all color classes of finite undirected graphs is (order) dense with the the unique exception of the pair (K_1, K_2). Explicitely*

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for every pair of graphs G_1, G_2 which is not homomorphicaly equivalent to the pair K_1, K_2, there exists a graph G such that G is not homomorphically equivalent to neither G_1 nor G_2 but $G_1 \longrightarrow G$ and $G \longrightarrow G_2$.

The original proof of Welzl [W] is a difficult ad hoc argument. M.Perles and J.N. found independently (at about 1990) a much simpler and perhaps more natural proof. As neither of us published it (but several times lectured about it) the proof eventually got to a forthcomming survey by G. Hahn and C. Tardiff. Thus it is perhaps fitting to include the original proof here as that proof allows us to prove several stronger statements. They may be formulated as follows

Theorem 1.3 *The class of all color classes of undirected graphs is (order) dense with the unique exception of the pair K_1, K_2.*

Theorem 1.4 *The class of all color classes of hypergraphs is dense.*

Thus for hyperhraphs there are no exceptional pairs which we may call (order) jumps. Explicitely, given a partially ordered class $\mathcal{K}$ we call a pair A, B of its elements a *jump* if $A < B$ but there is no C with $A < C < B$.

In another direction we generalize Theorem 2 as follows:

Theorem 1.5 *Every jump in the class of all connected directed graphs is a pair of balanced graphs.*

(Recall that a directed graph is called *balanced* if every its cycle contains equall number of forwarding and backwarding arcs. Alternatively, a graph G is balanced if there s a homomorphism of G to a monotonne path.)

The problem to characterize all the jumps in the class of directed graphs seems to be very difficult. It is known that jumps exist. Except of the trivial pair single vertex and single arc and another easy pair (single arc, monotonne path of length 2) there are infinitely many jumps of the form P_{k+1}, P_k where the graph P_k has $4 + 2(k-1)$ vertices and it is depicted on Fig.1. Let us call all these jumps *standard jumps.*

This has been proved by author and X. Zhu in [NZ] where the following (presently difficult) result has been proved

Theorem 1.6 *The standard jumps are the only jumps in the class of all finite oriented paths.*

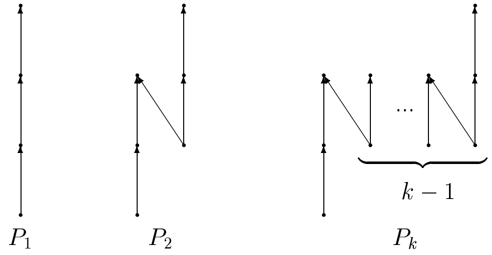

Figure 1: Graphs P_k

Even for the class of all finite directed trees the characterization of all jumps is presently unknown. However, viewing the richness of the homomorphisms between the trees (see e.g.[HNZ1] and [HNZ2], this is perhaps hardly surprising.

But perhaps there is more and more evidence that the standard jumps are the only jumps even for the class of all connected oriented graphs.

It is interesting to note that for not connected graphs the above Theorem 5 does not hold:

Proposition 1.7 *Consider the unbalanced cycle C_{2k+3} formed by the alternating path of length $2k+1$ together with the terminal vertex joined with the first and the last vertex of this path. Then the disconnected graphs G_1 with components C_{2k+3} and P_{k+1} and G_2 with components C_{2k+3} and P_k form a jump in the class of all oriented graphs. There are infinitely many such examples even for a fixed pair of paths.*

Let us state yet another result related to both Density Theorem 2 and Universality Theorem 1. We call it *Fractal Property* of the class $\mathcal{C}$. Before stating it let us introduce the following: A mapping $F : \mathcal{C} \longrightarrow \mathcal{C}$ is said to be *embedding* if F is 1-1 and for any pair of color classes A and B holds: $A \leq B$ iff $F(A) \leq F(B)$. Given a pair A, B of color classes we denote by $[A, B]$ the

3 Generating color classes

Here we prove Theorem 9.
Proof. Suppose that the graphs A and H and the classes $\mathcal{K}$ and $\longrightarrow H$ are given with the properties of the Theorem. Obviously we may suppose that the graph H is a core. Assume for the contradiction that the class $\mathcal{K}$ is finitely generated by the set $\mathcal{H}$. Put $\mathcal{H} = H_1, H_2, ..., H_t$. Denote by H' the disjoint union of the graphs H_i. Obviously for each i $H_i \longrightarrow H$ and also $H \not\longrightarrow H_i$ (for otherwise for some i $A \longrightarrow H_i$ which is a contradiction with the fact that $\mathcal{H}$ generates $\mathcal{K}$). Now let G be a connected graph with the following properties claimed by the incomparability lemma: $G \longrightarrow H$, $G \not\longrightarrow H'$ and $A \not\longrightarrow G$ (it follows from the connectivity that if $G \longrightarrow H'$ then $G \longrightarrow H_i$ for some i, a contradiction). Thus G is A-free, and obviously $G \notin \mathcal{K}$, a contradiction. □

Now we prove Theorem 10.
Proof. Let the graphs $A, B, C,$ be given with properties of Theorem 10. It follows that these graphs fail to be bipartite. Now let g be chosen such that in each of the graphs A, B, C there exists an odd cycle of length $\leq g$ and let G' be the graph with chromatic number $> n^{n'}$ where n and n' denotes the number of vertices of the graphs B and C. Similarily as in the proof of Theorem 2 we can deduce that the graph $G' \times C$ has all the desired properties.
□

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□

Proof. Theorem 5

Assume that G_1, G_2 are directed graphs with a homomorphism $G_1 \longrightarrow G_2$ but no homomorphism $G_2 \longrightarrow G_1$. We use the following observation: If G is not balanced and $G \longrightarrow G'$ then also G' fails to be balanced (this is best to see by the characterization of balanced graphs as those graphs which can be colored by monotonne paths; see above). It follows that if one of the graphs G_i is unbalanced then also G_2 is unbalanced. Let G_2 be connected and k denotes the length of the shortest unbalanced cycle in G_2. Now we can similarily to the above proof of Theorem 2: Let H be a directed graph with chromatic number (of its symmetrization) $> n^{n'}$ while every unbalanced cycle in H has length $> k$ (here we can use again shift like graphs). Then $G = G_1 \cup (HxG_2)$ is the desired graph. The arrows $G_1 \longrightarrow G$ and $G \longrightarrow G_2$ we get similarly as above. Also the non-existence of a homomorphism $G \longrightarrow G_1$ can be proved analogously to the above. Finally, the non-existence of the mapping $G_2 \longrightarrow G$ follows from the connectivity of G_2 together with the above observation about non-balanced graph-homomorphism.

□

Proof. Theorem 4:

In the proof we use the following hypergraph product: Given hypergraphs $H = (X, \mathcal{M})$ and $H' = (X', \mathcal{M}')$ we define their product $H \times H' = (X \times X', M \times M'; M \in \mathcal{M}, M' \in \mathcal{M}')$. It is easy to check that the projections are homomorphisms. Now given two hypergraphs H_1 and H_2 with homomorphism $H_1 \longrightarrow H_2$ and no homomorphism $H_2 \longrightarrow H_1$ we define the desired hypergraph H as follows: First we find a hypergraph $H_0 = (Y, \mathcal{N})$ with the following properties: the chromatic number of H_0 is $> n^{n'}$ where n and n' is the number of points of hypergraphs H_1 and H_2 ; ii. every hyperedge $N \in \mathcal{N}$ has size $> n'$ (any sufficiently large $n' + 1$ uniform hypergraph will do). The desired hypergraph H will be constructed as the disjoint union of H_1 and $H_0 \times H_2$. Obviously $H_1 \longrightarrow H \longrightarrow H_2$. There is no homomorphism $H_2 \longrightarrow H$ as the set system H_2 contains a hyperedge of size $\leq n'$ while the set system $H_0 \times H_2$ has no such edge. The fact that there is no homomorphisms $H \longrightarrow H_1$ proceeds in a complete analogy with the above proof of Theorem 2. (We only have to use the different product but all the homomorphism properties are preserved.)

□

class of all color classes C satisfying $A \leq C \leq B$. We call $[A, B]$ the *interval* in $\mathcal{C}$. Now we can state the fractal property of the color classes $\mathcal{C}$:

Theorem 1.8 *Let $[A, B]$ be an interval of $\mathcal{C}$ where at least one of A, B is not balanced (consequently B is not balanced). Then there is an embedding of $\mathcal{C}$ into the interval $[A, B]$.*

Thus each non balanced interval of $\mathcal{C}$ contains a copy of the whole class $\mathcal{C}$. We shall prove this theorem elsewhere. Let us note that the fractal property is not known to be true in some cases where the density theorem is valid. This is the case e.g. for the class of all finite paths.

It has been shown in [13] that despite of its complexity the color classes may be generated in a simple way by means of subgraphs and products. This may be formalized as follows: Given a set $\mathcal{H}$ of graphs we denote by $SP(\mathcal{H})$ the class of all graph G which are isomorphic to an induced subgraph of a product of graphs belonging to $\mathcal{H}$. We also say that $SP(\mathcal{H})$ is generated by the set (of generators) $\mathcal{H}$. It has been proved in [13] that each color class $\longrightarrow H$ is generated by a finite set of graphs $\mathcal{H}$ (and because of the symmetry, the class of all k-colorable graphs is generated by exactly one graph). Here we complement this by proving the following result which shows that color classes are essentially the only finitely generated classes. Towards this end let us define the following:

Let A be a fixed graph. we say that a graph is A-free (or A-homfree) if there exists no homomorphism $A \longrightarrow G$. Somewhat more precise term would be A-homfree. In [3] Häggkvist and Hell use the term A-mote.

Denote by $Forb(A)$ the class of all A-free graphs. Using the techniques of [13] the main result of [3] may be formulated as follows:

Theorem 1.9 *For every A and for every positive integer k there exists a finite set $\mathcal{A}_k$ of A-free graphs such that the class of all A- free graphs with all its vertices of degree bounded by k is a subsets of $SP(\mathcal{A}_k)$.*

In other words the class of all bounded degree A-free graphs is finitely generated by a finite set of A-free graphs.

Particularly, the class of all A- free bounded degree graphs is a subset of a color class determined by an A- free graph (which is the main result of [3]. Now we show that the set of all A- free graphs in a fixed color classes fail to be finitely generated. More precisely we prove the following:

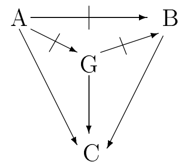

Figure 2: Schema for the theorem 11

Theorem 1.10 *Let A and H be fixed graphs and denote by $\mathcal{K}$ the class of all A- free graphs in the color class $\longrightarrow H$. Further assume that the class $\mathcal{K}$ is non-trivial in the sense that at least one of its members contains an edge and that it does not coincide with $\longrightarrow H$. Then $\mathcal{K}$ fails to be finitely generated.*

This theorem will be deduced from the following result sometimes called Incomparability Lemma:

Theorem 1.11 *Let A, B, C be graphs such that there exists a homomorphism $A \longrightarrow C$, $B \longrightarrow C$ and no homomorphisms , $C \longrightarrow A$, $C \longrightarrow B$ and $A \longrightarrow B$. Then there exists a connected graph G with the following properties: $G \longrightarrow C$, while there are no homomorphisms $A \longrightarrow G$, $G \longrightarrow B$ (see the following schema).*

This result was proved in [14] by nonconstructive means. A direct construction was given by N. Fiklíková (diploma thesis, Charles University). Here we give a surprisingly simple construction. This construction is in the spirit of the above short proof of the Density Theorem and thus it is perhaps a fitting conclusion of this paper (in Section 3).

Let us remark that the proof of the above result has the following corollary:

The class of all graphs G in $\not\longrightarrow H$ with chromatic number $< k$ fails to be a subset of a finitely generated class for every H and k. Despite of this fact these classes can be "finitely" generated by Hajos-type construction. See [12]of which the present paper is a sequel.

The paper is organized as follows: In section 2 we give a short proof of the density Theorem 2 together with Theorems 3.4.5. In Section 3 we prove Theorems 9 and 10.

2 Density theorems - Proofs

We begin with an easy proof of Density Theorem 2.

Proof. (M.Perles and J.N.): Let G_1 and G_2 be given undirected graphs, let $f : G_1 \longrightarrow G_2$ be a homomorphism , let there be no homomorphism $G_2 \longrightarrow G_1$. As this pair is not equivalent to the jump K_1, K_2 every component of the graph G_2 has the chromatic number > 2. At least one of these components fails to be G_1 colorable and let it contain an odd cycle of length k. Now choose a graph H with the following properties: H contains no odd cycle of length lek and the chromatic number of H is $> n^{n'}$ where n and n' denotes the number of vertices of the graphs G_1 and G_2. This graph exists by the celebrated Erdős Theorem [1]. Now let $G = G_1 \cup (H \times G_2)$. Here $\times$ denotes the direct product of two graphs and union sign means the disjoint union. We shall prove that G is the desired graph. Obviously $G_1 \longrightarrow G$ and $G \longrightarrow G_2$ follows as the second projection of $H \times G_2$ is a homomorphism into G_2. On the other hand there is no homomorphism G_2 into G as homomorphisms preserve odd cycles and they cannot increase the length of the shortest of them. Thus it suffices to prove that there is no homomorphism $G \longrightarrow G_1$. Let us suppose for the contradiction that there is a homomorphism $f : H \times G_2 \longrightarrow G_1$. Thus for any vertex x of H we have an induced mapping $f_x : V(G_2) \longrightarrow V(G_1)$ defined by $f_x(y) = f(x, y)$. (This mapping need not be a homomorphism.) As there are at most $n^{n'}$ of such mappings there are vertices x and x' forming an edge of H such that the mappings f_x and f'_x are identical equall , say, to g. However in this case g is a homomorphism of G_2 into G_1, contrary to our assumption. $\square$

Proof. Theorem 3

We proceede exactly as in the above proof for finite graphs. The only difference is that we need a graph H without odd cycles of length lek with chromatic number larger than a given cardinal number. This is another folkloristic result of Erdős and Hajnal which is easy to prove by considering so called iterated shift graphs together with the transfinite Ramsey theorem, see [2].