[10] J. Beck and V. Sós. Discrepancy theory. In *Handbook of Combinatorics*, pages 1405–1446. North-Holland, Amsterdam, 1995.

[11] J. L. Bentley. Decomposable searching problems. *Inform. Process. Lett.*, 8:244–251, 1979.

[12] J. L. Bentley and M. I. Shamos. A problem in multivariate statistics: algorithm, data structure, and applications. In *Proc. 15th Allerton Conf. Commun. Control Comput.*, pages 193–201, 1977.

[13] G. Bohus. On the discrepancy of 3 permutations. *Random Struct. Algo.*, 1:215–220, 1990.

[14] M. Drmota and R. F. Tichy. *Sequences, discrepancies and applications (Lecture Notes in Mathematics 1651).* Springer-Verlag, 1997.

[15] J.H. Halton. On the efficiency of certain quasi-random sequences of points in evaluating multi-dimensional integrals. *Numer. Math.*, 2:84–90, 1960.

[16] J. M. Hammersley. Monte Carlo methods for solving multivariable problems. *Ann. New York Acad. Sci.*, 86:844–874, 1960.

[17] Gy. Károlyi. Geometric discrepancy theorems in higher dimensions. *Studia Sci. Math. Hungarica*, 30:59–94, 1995.

[18] Gy. Károlyi. Irregularities of point distributions relative to homothetic convex bodies I. *Monatsh. Math.*, 120:247–279, 1995.

[19] L. Lovász, J. Spencer, and K. Vesztergombi. Discrepancy of set-systems and matrices. *European J. Combin.*, 7:151–160, 1986.

[20] J. Matoušek and J. Spencer. Discrepancy in arithmetic progressions. *J. Amer. Math. Soc.*, 9:195–204, 1996.

[21] F. P. Preparata and M. I. Shamos. *Computational Geometry: an Introduction.* Springer-Verlag, New York, NY, 1985.

[22] K. F. Roth. On irregularities of distribution. *Mathematika*, 1:73–79, 1954.

[23] W. M. Schmidt. On irregularities of distribution VII. *Acta Arith.*, 21:45–50, 1972.

[24] J. Spencer. *Ten lectures on the probabilistic method.* CBMS-NSF. SIAM, 1987.

[25] A. Srinivasan. Improving the discrepancy bound for sparse matrices: better approximations for sparse lattice approximation problems. In *Proc. 8th ACM-SIAM Symposium on Discrete Algorithms*, pages 692–701, 1997.

[26] J. G. van der Corput. Verteilungsfunktionen I. *Akad. Wetensch. Amsterdam, Proc.*, 38:813–821, 1935.

[27] G. M. Ziegler. *Lectures on Polytopes*, volume 152 of *Graduate Texts in Mathematics.* Springer-Verlag, 1994.

On the discrepancy for boxes and polytopes

Jiří Matoušek*

Department of Applied Mathematics
Charles University
Malostranské nám. 25, 118 00 Praha 1
Czech Republic

Abstract

We consider so-called Tusnády's problem in dimension d: Given an n-point set P in $\mathbf{R^d}$, color the points of P red or blue in such a way that for any d-dimensional interval B, the number of red points in $P \cap B$ differs from the number of blue points in $P \cap B$ by at most Δ, where $\Delta = \Delta(d,n)$ should be as small as possible. We slightly improve previous results of Beck, Bohus, and Srinivasan by showing that $\Delta(d,n) = O_d(\log^{d+1/2} n\sqrt{\log\log n})$, with a simple proof. The same asymptotic bound is shown for an analogous problem where B is allowed to be any translated and scaled copy of a fixed convex polytope A in $\mathbf{R^d}$. Here the constant of proportionality depends on A and we give an explicit estimate. The same asymptotic bounds follow also for the Lebesgue-measure discrepancy, which improves and simplifies results of Beck and of Károlyi.

1 Introduction

Definitions of discrepancy. Let $\mathcal{A}$ be a family of Lebesgue-measurable subsets of the d-dimensional Euclidean space $\mathbf{R^d}$, and let P be an n-point subset of the unit cube $[0,1]^d$. The *discrepancy of* P for the family $\mathcal{A}$ is defined as

$$D(P,\mathcal{A}) = \sup_{A\in\mathcal{A}} |D(P,A)|\,,$$

where $D(P,A) = n\lambda^d(A\cap[0,1]^d) - |P\cap A|$, with λ^d denoting the d-dimensional Lebesgue measure. The discrepancy measures the irregularity of distribution of P with respect to the sets of the family $\mathcal{A}$ (note that $n\lambda^d(A\cap[0,1]^d)$ is the expected number of points falling into A of a random n-point set in the unit cube, for instance). The discrepancy function of $\mathcal{A}$ is given by

$$D(n,\mathcal{A}) = \inf_{P\subset[0,1]^d,\,|P|=n} D(P,\mathcal{A})\,.$$

The family for which the discrepancy was defined first and studied most intensively is the family $\mathcal{R}_d$ of axis-parallel boxes in $\mathbf{R^d}$ (or d-dimensional intervals), but more recently, lot of work has also been devoted to investigating the discrepancy for other families, such as the

*Research supported by Czech Republic Grant GAČR 0194 and by Charles University grants No. 193,194.

balls or the halfspaces in $\mathbf{R}^\mathbf{d}$, and so on. The history and overview of discrepancy theory are presented, among others, in the survey Beck and Sós [10], in the book Beck and Chen [7] or, more recently, in Drmota and Tichy [14].

The above-defined notion of discrepancy will be called the *Lebesgue-measure discrepancy*, in order to distinguish it from a seemingly different but closely related notion of *combinatorial discrepancy*, to be defined next. Let X be a finite set and let $\mathcal{S} \subseteq 2^X$ be a family of subsets of X. By a *coloring* we mean a mapping $\chi : X \to \{-1, +1\}$. The *discrepancy* of $\mathcal{S}$, denoted by $\mathrm{disc}(\mathcal{S})$, is the minimum, over all colorings χ, of

$$\mathrm{disc}(\chi, \mathcal{S}) = \max_{S \in \mathcal{S}} |\chi(S)| \ ,$$

where we use the shorthand $\chi(S)$ for $\sum_{x\in S} \chi(x)$. For a survey of basic results and techniques in combinatorial discrepancy theory see Beck and Sós [10], or also Spencer [24] or Alon and Spencer [1].

If $\mathcal{A}$ is a family of subsets of $\mathbf{R}^\mathbf{d}$ and P is a finite point set in $\mathbf{R}^\mathbf{d}$, we let $\mathrm{disc}(P, \mathcal{A})$ denote the discrepancy of the (finite) set system induced by $\mathcal{A}$ on P, that is, of the set system $\{A \cap P : A \in \mathcal{A}\}$. The quantity $\mathrm{disc}(P, \mathcal{A})$ is called the *combinatorial discrepancy* of P for $\mathcal{A}$, and the corresponding discrepancy function of $\mathcal{A}$ is

$$\mathrm{disc}(n, \mathcal{A}) = \max_{P\subset \mathbf{R}^\mathbf{d}, |P|=n} \mathrm{disc}(P, \mathcal{A})\,.$$

The combinatorial and Lebesgue-measure discrepancy are related: Roughly speaking, the combinatorial discrepancy function is an upper bound for the Lebesgue-measure discrepancy function. This relation has been used in numerous papers. Currently, lower bounds for geometric discrepancy are usually easier to prove in the Lebesgue-measure setting, while for many interesting families, the best known upper bounds for the Lebesgue-measure discrepancy are obtained via the combinatorial setting (one example is the family of halfspaces in $\mathbf{R}^\mathbf{d}$, but as we will see later, the classical discrepancy for axis-parallel boxes is a notable exception). The following proposition is one possible precise formulation of the relationship of the combinatorial discrepancy and the Lebesgue-measure discrepancy:

Proposition 1.1 *Let $\mathcal{A}$ be a class of Lebesgue-measurable sets in $\mathbf{R}^\mathbf{d}$ containing a set A_0 with $[0,1]^d \subseteq A_0$. Suppose that $D(n, \mathcal{A}) = o(n)$ for $n \to \infty$, and that $\mathrm{disc}(n, \mathcal{A}) \le f(n)$ for all n, where $f(n)$ is a function satisfying $f(2n) \le (2-\delta)f(n)$ for all n and some fixed $\delta > 0$. Then we have*

$$D(n, \mathcal{A}) = O(f(n)).$$

The main idea of this proposition was formulated by Beck [2], and a general formulation for classes of planar convex sets was given by Lovász, Spencer, and Vesztergombi [19] (the proof for our more general formulation remains almost the same, though). Let us also remark that an upper bound on the combinatorial discrepancy function implies a corresponding upper bound not only on the Lebesgue-measure discrepancy, but also on the discrepancy with respect to an essentially arbitrary probability measure on $\mathbf{R}^\mathbf{d}$ (under some mild assumptions on the family $\mathcal{A}$).

Tusnády's problem in dimension d. For the family $\mathcal{R}_d$ of axis-parallel boxes, the best known asymptotic upper bound for the Lebesgue-measure discrepancy is

$$D(n, \mathcal{R}_d) = O_d(\log^{d-1} n)$$

It remains to show that if we make such decompositions for all possible $B \in \mathrm{POL}(H)$, altogether we obtain at most $k = O_d(\ell^{d-1})$ distinct families of the form $\mathrm{POL}(G)$ participating in these decompositions, where G is a d-tuple of hyperplanes. Let G be one such family of d hyperplanes. By inspecting the construction of the decomposition, we find that this G is determined by a sequence $F_{d-1} \supset F_{d-2} \supset \ldots \supset F_1$ of faces of some polytope $B \in \mathrm{POL}(H)$, $\dim(F_j) = j$ (see Fig. 3). We show that G is also determined, up to a parallel translation, by a $(d-1)$-tuple of hyperplanes from H.

We put $F^{(1)} = F_{d-1}$. Then we choose a facet $F^{(2)}$ of B with $F_{d-2} = F^{(1)} \cap F^{(2)}$, and in general we let $F^{(j)}$ be a facet of B such that $F_{d-j} = F^{(1)} \cap F^{(2)} \cap \ldots \cap F^{(j)}$, $j = 1, 2, \ldots, d-1$ (such an $F^{(j)}$ need not be unique in general but we fix some arbitrarily). Let h_j be the hyperplane containing the facet $F^{(j)}$ (this hyperplane is parallel to some hyperplane of H). The family G can be reconstructed from the $(d-1)$-tuple $(h_1, h_2, \ldots, h_{d-1})$ as follows: $G = \{g_1, g_2, \ldots, g_d\}$, where

$$g_j = \{(x_1, \ldots, x_d) \in \mathbf{R}^\mathbf{d} : (x_1, \ldots, x_{d-j+1}, t_1, \ldots, t_{j-1}) \in h_1 \cap \ldots \cap h_j \text{ for some } t_1, t_2, \ldots, t_{j-1} \in \mathbf{R}\}$$

(in particular, $g_1 = h_1$ and g_d is a parallel translate of the hyperplane $\{x_1 = 0\}$). Therefore, the number of possible families G, up to a parallel translation, is no larger than the number of ordered $(d-1)$-tuples of hyperplanes of H, which is $O_d(\ell^{d-1})$. This concludes the proof of Lemma 4.1. □

References

[1] N. Alon and J. Spencer. *The probabilistic method*. J. Wiley & Sons, 1993.

[2] J. Beck. Roth's estimate on the discrepancy of integer sequences is nearly sharp. *Combinatorica*, 1(4):319–325, 1981.

[3] J. Beck. Some upper bounds in the theory of irregularities of distribution. *Acta Arith.*, 43:115–130, 1984.

[4] J. Beck. Irregularities of distribution II. *Proc. London Math. Soc. (3)*, 56:1–50, 1988.

[5] J. Beck. A two-dimensional van Aardenne-Ehrenfest theorem in irregularities of distribution. *Compositio Math.*, 72:269–339, 1989.

[6] J. Beck. Balanced two-colorings of finite sets in the cube. *Discrete Math.* 73,1/2:13-25, 1989.

[7] J. Beck and W. Chen. *Irregularities of Distribution*. Cambridge University Press, 1987.

[8] J. Beck and W. Chen. Irregularities of point distribution relative to convex polygons. In *Irregularities of partitions (Fertőd 1986)*, pages 1–22. Springer-Verlag, 1989.

[9] J. Beck and W. Chen. Irregularities of point distribution relative to convex polygons II. *Mathematika*, 40:127–136, 1993.

χ. For any $B \in \mathrm{POL}(H)$, we get $|\chi(P \cap B)| \leq q\,\mathrm{disc}(\chi, \mathcal{M}) = O_d(q\sqrt{s \log n})$. Iteration of the partial coloring step finishes the argument, leading to a total discrepancy bound of $O_d(q\sqrt{k} \cdot \log^{d+1/2} n\sqrt{\log(\ell + \log n)})$.

Proof of Lemma 4.1 (i). We decompose a given convex polytope $B \in \mathcal{T}_A$ by the following recursive procedure (generalizing the approach used for convex polygons in Section 3). Without loss of generality, we assume that no face of B contains a segment parallel to one of the coordinate axes. Regarding the x_d-axis vertical, we write $B = U \setminus L$, where L is the set of points lying vertically below B and $U = L \dot{\cup} B$. We further decompose both L and U into semiinfinite vertical prisms, each prism being bounded from above by one of the facets of B. If $d = 2$, this is the whole decomposition, but for larger dimensions, we have to decompose these prisms further.

Consider one such vertical prism T_F bounded from above by a facet F (see Fig. 3). We project F vertically into the $\{x_d = 0\}$ plane, obtaining a $(d-1)$-dimensional convex

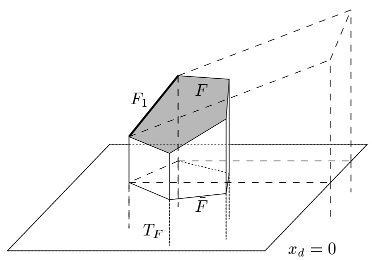

Figure 3: The decomposition of a vertical prism T_F; one set in the resulting decomposition, corresponding to the sequence $F \supset F_1$ of faces, is drawn by long dashes.

polytope $\bar{F}$, whose faces are in one-to-one correspondence with the faces of B contained in F. Identifying $\mathbf{R^{d-1}}$ with the $\{x_d = 0\}$ hyperplane in $\mathbf{R^d}$, we recursively express $\bar{F}$ as a member of $\mathrm{AC}_{\bar{k}}(\mathrm{POL}(\bar{H}_1) \cup \ldots \cup \mathrm{POL}(\bar{H}_{\bar{k}}))$, where $\bar{H}_1, \ldots, \bar{H}_{\bar{k}}$ are suitable collections of $d-1$ hyperplanes in $\mathbf{R^{d-1}}$ and $\bar{k} \leq \mathrm{flc}(F)$. It remains to extend each of the sets in the decomposition of $\bar{F}$ vertically and intersect it with the halfspace lying below the hyperplane h_F containing the facet F. This yields an admissible $\bar{k}$-decomposition of T_F using sets from $\mathrm{POL}(H_1) \cup \ldots \cup \mathrm{POL}(H_{\bar{k}})$, where $H_i = \{h \times \mathbf{R} : \mathbf{h} \in \bar{\mathbf{H}}_\mathbf{i}\} \cup \{\mathbf{h_F}\}$. Combining these decompositions appropriately for all facets F, we get an admissible k-decomposition of B, where $k \leq \sum_F \mathrm{flc}(F) = \mathrm{flc}(B) = \mathrm{flc}(A)$.

Obviously, if we do this decomposition procedure for another copy $B' \in \mathcal{T}_A$ of the polytope A, the resulting families $\mathrm{POL}(H_i)$ used in the decomposition are the same. This finishes the proof of part (i).

Proof of Lemma 4.1 (ii). Given a polytope $B \in \mathrm{POL}(H)$, we apply exactly the same decomposition procedure as in the proof of part (i); we may again assume that H is in a suitably general position with respect to the coordinate axes. This yields an admissible decomposition of B using sets from $\mathrm{POL}(H_1) \cup \ldots \cup \mathrm{POL}(H_q)$, where $q \leq \mathrm{flc}(B) \leq \mathrm{flc}(H)$ and each H_i consists of d hyperplanes.

[26, 15, 16] (here and in the sequel, the subscripts of the $O()$ or $\Omega()$ notation denote that the constant of proportionality may depend on the quantities in the subscript). This bound is known to be tight in dimension 2 [23], but for all higher dimensions, the known lower bound is considerably smaller, only $\Omega_d(\log^{(d-1)/2} n)$ [22], with a small improvement for $d = 3$ [5].

The lower bounds for the combinatorial discrepancy for axis-parallel boxes are derived from the Lebesgue-measure discrepancy and thus asymptotically the same, but the known upper bounds are somewhat worse than those for the Lebesgue-measure case. Mainly the upper bounds in the two-dimensional case have been studied; this is the so-called *Tusnády's problem*. An upper bound of $O(\log^4 n)$ for the (2-dimensional) Tusnády's problem was obtained by Beck [3], and later improved to $O((\log n)^{3.5+\varepsilon})$ [6]. This was further strengthened to $O(\log^3 n)$ by Bohus [13] via a bound for the so-called "k-permutations problem". An $O(\log^{5/2} n)$ bound was recently proved by Srinivasan [25], also via the k-permutations problem and using the so-called *entropy method* (see, e.g., [20] or [1]). Independently, I noted an $O(\log^{5/2} n\sqrt{\log\log n})$ bound several years ago in unpublished lecture notes on geometric discrepancy (currently a book manuscript). As for higher dimensions, for $d \leq 4$, the best published bound is $O((\log n)^{(3d-1)/2})$ [25], and for $d \geq 5$, it is $O_{d,\varepsilon}((\log n)^{d+3/2+\varepsilon})$ [6]. (Here and in the sequel, $\varepsilon > 0$ denotes a constant that can be chosen arbitrarily small.) Here we prove

Theorem 1.2 *For any $d \geq 2$, we have*

$$\mathrm{disc}(n, \mathcal{R}_d) = O_d\left(\log^{d+1/2} n\sqrt{\log\log n}\right).$$

This bound is a bit worse for $d = 2$ than Srinivasan's result but slightly better than the previously known results for all dimensions $d > 2$. Our proof is quite simple, using only the simplest version of Beck's "partial coloring" technique, and with only a very small modification, it will be used also for proving results concerning convex polytopes formulated below. By employing the entropy method and more complicated calculations, it is probably possible to remove the $\sqrt{\log\log n}$ factor in the bound in Theorem 1.2 (certainly so for $d = 2$), but since I have no reason to suspect that the resulting $O_d(\log^{d+1/2} n)$ upper bound is tight, I prefer to present the little worse bound with the considerably simpler proof. The proof is given in Section 2.

Convex polygons and polytopes. For a fixed convex set A in $\mathbf{R^d}$, let $\mathcal{T}_A = \{aA + x : a \in [0, \infty),\ x \in \mathbf{R^d}\}$ denote the family of all translated and scaled copies of A (no rotation allowed). Here we are going to investigate the combinatorial discrepancy $\mathrm{disc}(n, \mathcal{T}_A)$ for A being a given convex ℓ-gon in the plane or a convex polytope with at most ℓ facets in $\mathbf{R^d}$.

The Lebesgue-measure discrepancy $D(n, \mathcal{T}_A)$ for a fixed convex ℓ-gon in the plane has been considered by Beck [4] and by Beck and Chen [8] (see also [9] for a related work). The best published asymptotic bound for this case is due to Beck [4]:

$$D(n, \mathcal{T}_A) = O_{A,\varepsilon}((\log n)^{4+\varepsilon}). \tag{1}$$

Let us remark that this result and some of the upper bounds of Beck and of Károlyi mentioned below are actually proved in a somewhat different and slightly stronger setting (they consider the discrepancy of an infinite point set spread out in the whole $\mathbf{R^d}$, the scaling factor a in the definition of $\mathcal{T}_A$ is restricted to $[0, 1]$, and the bounds depend on the diameter of A).

Beck (private communication) noted that the argument in his paper [6] can be combined with the approach of [4] to yield an $O_{A,\varepsilon}((\log n)^{3.5+\varepsilon})$ bound for $D(n, \mathcal{T}_A)$ and also for $\mathrm{disc}(n, \mathcal{T}_A)$. Here we prove

Theorem 1.3 *Let A be a convex ℓ-gon in the plane. Then*

$$\mathrm{disc}(n, \mathcal{T}_A) = O\left(\ell\,(\log n)^{2.5}\sqrt{\log(\ell + \log n)}\right).$$

This theorem is proved in Section 3.

Beck's upper bound (1) was actually proved for the discrepancy for a family larger than $\mathcal{T}_A$, namely for the family of all convex polygons whose all sides are parallel to the sides of A. Károlyi [17] continued investigations in a similar spirit in higher dimensions. To formulate the results, we need the following definition: For a finite set H of hyperplanes in $\mathbf{R^d}$, let $\mathrm{POL}(H)$ denote the set of all polytopes $\bigcap_{i=1}^{\ell}\gamma_i$, where each γ_i is a halfspace with boundary parallel to some $h \in H$ (obviously, for each $h \in H$, it suffices to consider at most two γ_i's in the intersection). So in the plane, Beck's results provide the strengthening of (1) mentioned above, i.e. $D(n, \mathrm{POL}(H)) = O_{H,\varepsilon}((\log n)^{4+\varepsilon})$. Károlyi [17] proved the estimate

$$D(n, \mathrm{POL}(H)) = O_{H,\varepsilon}\left((\log n)^{\max(3d/2+1+\varepsilon, 2d-1)}\right)$$

for any fixed finite set H of hyperplanes in $\mathbf{R^d}$. Here we establish a somewhat better bound with a considerably simpler proof. To formulate the results, we need some terminology and facts concerning convex polytopes; we refer to Ziegler [27] for background, references and proofs. A convex polyhedron in $\mathbf{R^d}$ is defined as an intersection of finitely many closed halfspaces, and a convex polytope is a bounded convex polyhedron. Since we are interested in discrepancy of finite point sets, we may assume that all the polyhedra we are dealing with are bounded (this saves us some terminological and technical complications). A d-dimensional convex polytope A has faces of dimensions 0 through d; the 0-faces are the vertices, the 1-faces are the edges, the $(d-1)$-faces are called the facets of A, and the unique d-face is A itself (the empty set is sometimes regarded as a face of dimension -1). For our purposes, we define a *flag* of A as a sequence $F_0, F_1, \ldots, F_{d-1}, F_d = A$, where F_i is an i-face of A, $i = 0, 1, \ldots, d$, and where $F_0 \subset F_1 \subset \ldots \subset F_d$. The *flag complexity* of A, denoted by $\mathrm{flc}(A)$, is the number of flags of A. Note that for a convex ℓ-gon A in the plane, we have $\mathrm{flc}(A) = O(\ell)$. For an arbitrary polyhedron in $\mathbf{R^d}$ that is an intersection of ℓ halfspaces we have, by an extension of the Upper bound theorem, $\mathrm{flc}(A) = O_d(\ell^{\lfloor d/2\rfloor})$. For a finite set H of hyperplanes in $\mathbf{R^d}$, we put $\mathrm{flc}(\mathrm{POL}(H)) = \max\{\mathrm{flc}(A) : A \in \mathrm{POL}(H)\}$.

Theorem 1.4

(i) *Let A be a convex polytope in $\mathbf{R^d}$, $d \geq 3$, with at most $\ell \geq d+1$ facets. Then*

$$\mathrm{disc}(n, \mathcal{T}_A) = O\left(\mathrm{flc}(A)^{3/2}(\log n)^{d+1/2}\sqrt{\log(\ell + \log n)}\right) =$$

$$O\left(\ell^{\frac{3}{2}\lfloor d/2\rfloor}(\log n)^{d+1/2}\sqrt{\log(\ell + \log n)}\right).$$

(ii) *Let H be a family of $\ell \geq d$ hyperplanes in $\mathbf{R^d}$, $d \geq 2$. Then*

$$\mathrm{disc}(n, \mathrm{POL}(H)) = O\left(\mathrm{flc}(H)\ell^{(d-1)/2}\log^{d+1/2} n\sqrt{\log(\ell + \log n)}\right) =$$

$$O\left(\ell^{(d-1)/2+\lfloor d/2\rfloor}\log^{d+1/2} n\sqrt{\log(\ell + \log n)}\right).$$

be written as a disjoint union of some number of sets from $\mathcal{F}_i$ plus a set of at most $O(t\log n)$ extra points. Hence, the intersection $P \cap U$ can be written as a disjoint union of sets from $\mathcal{F}$ plus a set $M_{U(B)}$ of at most $s = O(\ell t \log n)$ points, and similarly for L and $M_{L(B)}$.

We put $\mathcal{M} = \{M_{U(B)}, M_{L(B)} : B \in \mathcal{T}_A\}$. We claim that $|\mathcal{M}| = O(\ell^3 n^3)$ (this is a weak bound but it is easy to see and sufficient for our purposes). Since $M_{L(B)}$ and $M_{U(B)}$ only depend on $P \cap B$, it suffices to show that the set system induced by $\mathcal{T}_A$ on P has $O(\ell^3 n^3)$ distinct sets. To see this, represent each polygon $B = aA + (x, y) \in \mathcal{T}_A$ by the point $(x, y, a) \in \mathbf{R^3}$. For each point $p \in P$, the set of polygons $B \in \mathcal{T}_A$ containing p is a polyhedral cone in $\mathbf{R^3}$ bounded by ℓ facets (the slice by a horizontal plane $\{z = a\}$ is a congruent copy of the polygon aA). The number of distinct subsets of P definable by sets from $\mathcal{T}_A$ is no larger than the number of cells in the arrangement of n such polyhedral cones in $\mathbf{R^3}$. This in turn is no larger than the number of cells in an arrangement of ℓn planes in $\mathbf{R^3}$, i.e. $O(\ell^3 n^3)$.

The Partial coloring lemma 2.1 yields the existence of a substantial partial coloring χ with $\chi(F) = 0$ for all $F \in \mathcal{F}$ and $|\chi(M)| = O(\sqrt{s \log|\mathcal{M}|}) = O(\ell\,(\log n)^{3/2}\sqrt{\log(\ell + \log n)})$ for all $M \in \mathcal{M}$ (note that we may assume $n \geq \ell$ for otherwise the bound in Theorem 1.3 is trivial). For any $B \in \mathcal{T}_A$, we have $|\chi(P \cap B)| = |\chi(P \cap U(B)) - \chi(P \cap L(B))| = |\chi(M_{U(B)}) - \chi(M_{L(B)})| = O(\ell\,(\log n)^{1.5}\sqrt{\log(\ell + \log n)})$. Theorem 1.3 now follows by iterating the partial coloring step as in the previous section.

4 Convex polytopes

Here we prove Theorem 1.4. We begin with some preparatory definitions and lemmas. Let $\mathcal{S}$ be a set system. For $q \geq 0$, we define the system of $\mathrm{AC}_q(\mathcal{S})$ of *admissible q-combinations* of sets from $\mathcal{S}$ as follows. We put $\mathrm{AC}_0(\mathcal{S}) = \{\emptyset\}$, and

$$\mathrm{AC}_{q+1}(\mathcal{S}) = \{A\dot{\cup}B : A \in \mathrm{AC}_{q_1}(\mathcal{S}),\ B \in \mathrm{AC}_{q_2},\ A \cap B = \emptyset,\ q_1 + q_2 = q\} \cup$$

$$\{A \setminus B : A \in \mathrm{AC}_{q_1}(\mathcal{S}),\ B \in \mathrm{AC}_{q_2},\ B \subseteq A,\ q_1 + q_2 = q\}.$$

By induction on q, it is easy to see that if χ is a (partial) coloring of the ground set of $\mathcal{S}$ with $\mathrm{disc}(\chi, \mathcal{S}) \leq \Delta$ then $\mathrm{disc}(\chi, \mathrm{AC}_q(\mathcal{S})) \leq q\Delta$.

Lemma 4.1

(i) *Let $A \subset \mathbf{R^d}$ be a convex polytope. There exist $k \leq \mathrm{flc}(A)$ families $H_1, H_2, \ldots, H_k$ consisting of d hyperplanes each, such that any $B \in \mathcal{T}_A$ belongs to $\mathrm{AC}_k(\mathrm{POL}(H_1) \cup \mathrm{POL}(H_2) \cup \ldots \cup \mathrm{POL}(H_k))$.*

(ii) *Let H be a family of ℓ hyperplanes in $\mathbf{R^d}$. There exist $k = O_d(\ell^{d-1})$ families $H_1, H_2, \ldots, H_k$ consisting of d hyperplanes each, such that any $B \in \mathrm{POL}(H)$ belongs to $\mathrm{AC}_q(\mathrm{POL}(H_1) \cup \mathrm{POL}(H_2) \cup \ldots \cup \mathrm{POL}(H_k))$, where $q \leq \mathrm{flc}(H)$.*

First, we derive Theorem 1.4 from Lemma 4.1. This is quite similar to the proofs in the previous sections and we only write out part (ii) here. For a given n-point set $P \subset \mathbf{R^d}$, we let $\mathcal{F} = \bigcup_{i=1}^{k}\mathcal{F}_i$, where $\mathcal{F}_i = \mathrm{CI}^t_{V_i}(P)$, with V_i being the set of normal vectors for the hyperplanes in the family H_i as in Lemma 4.1(ii). The parameter t is set to $Kk\log^{d-1} n \log(\ell + \log n)$. For each $T \in \mathrm{POL}(H_i)$, $P \cap T$ can be written as a disjoint union of some sets of $\mathcal{F}_i$ plus a set M_T of $s = O_d(t\log^{d-1} n)$ extra points. We put $\mathcal{M} = \{M_T : T \in \mathrm{POL}(H_1) \cup \ldots \cup \mathrm{POL}(H_k)\}$ and apply the Partial coloring lemma, obtaining a substantial partial coloring number of points is reduced at least by the factor $\frac{9}{10}$, the number of iterations is $O(\log n)$. Hence, for any $B \in \mathcal{R}$, we have

$$|\chi(P \cap B)| \leq |\chi_1(P \cap B)| + |\chi_2(P \cap B)| + \cdots \leq O(\Delta \log n) =$$

$$O_d((\log n)^{d+1/2} \sqrt{\log \log n}) .$$

This concludes the proof of Theorem 1.2. □

3 Convex polygons

Here we prove Theorem 1.3. Let A be the given convex ℓ-gon, and let $h_1, h_2, \ldots, h_\ell$ be the lines extending the sides of A. Without loss of generality, we may assume that none of the h_i's is vertical. For $i = 1, 2, \ldots, \ell$, we define a two-element set H_i consisting of the line h_i and of the vertical line $\{x = 0\}$.

Consider a convex polygon $B \in \mathcal{T}_A$. Let $L = L(B)$ denote the set of points lying vertically below B, and let $U = U(B) = L \dot\cup B$, as in Fig. 2. We have $B = U \setminus L$ with $L \subseteq U$, and both L and U can be further written as a disjoint union of vertical semiinfinite trapezoids[1]. The total number of these trapezoids used in both L and U is ℓ, and each of them belongs to some of the families $\mathrm{POL}(H_i)$.

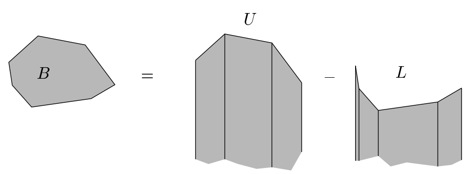

Figure 2: Decomposing a convex ℓ-gon.

Let P be a given n-point set in $\mathbf{R^2}$. To construct a substantial partial coloring for P, we proceed similarly as in the preceding section. Let v_i be a normal vector of h_i, let e_x be the horizontal unit vector, let $V_i = (v_i, e_x)$, and let $\mathcal{F}_i = \mathrm{CI}^t_{V_i}(P)$, where t is a suitable parameter. Putting $\mathcal{F} = \bigcup_{i=1}^{\ell} \mathcal{F}_i$, we derive from Lemma 2.2 that if the condition (2) should be satisfied for $\mathcal{F}$ we must choose

$$t \geq K \ell \log n \log(\ell + \log n)$$

for a large enough constant K.

Let $B \in \mathcal{T}_A$ and let $L = L(B)$ and $U = U(B)$ be as above. If $T \in \mathrm{POL}(H_i)$ is one of the vertical semiinfinite trapezoids from the decomposition of U or L, the intersection $P \cap T$ can

[1] Here we are not quite precise concerning the boundaries of U and L and of the semiinfinite trapezoids. But since we are dealing with the discrepancy of a finite set P, by a small perturbation of B preserving the set $P \cap B$ we may assume that no points of P are on the boundary of B or on the boundaries of the semiinfinite trapezoids.

This theorem is proved in Section 4. Note that both part (i) and part (ii) imply Theorem 1.2, and so in order to improve the dependence on n for A or H fixed one would have to improve the bounds for Tusnády's problem as well.

In conclusion, let us remark that Beck [4] (see also [7]) used bounds on the discrepancy for convex ℓ-gons in an investigation of the Lebesgue-measure discrepancy $D(n, \mathcal{T}_A)$ for an arbitrary convex set A in the plane. He discovered that the behavior of this quantity depends mainly on the smoothness of the boundary of A, and that the smoothness can be quantified here by the approximability of A by convex polygons. Károlyi [18] obtained bounds in a similar spirit in higher dimensions. It seems possible to simplify and slightly improve these results using the basic approach as in the proofs of Theorem 1.3 and 1.4 and some approximation arguments similar to those in [4], [18]. We will not elaborate on this in the present paper.

2 Axis-parallel boxes

The Partial coloring lemma. We first recall a method for obtaining upper bounds in combinatorial discrepancy due to Beck [2]. For a finite set X, we define a *partial coloring* to be a mapping $\chi : X \to \{0, +1, -1\}$, and a let *substantial* partial coloring be a partial coloring χ with $\chi(x) \neq 0$ for at least $\frac{1}{10}|X|$ points $x \in X$. The points x with $\chi(x) = 0$ will be called *uncolored* by χ.

Lemma 2.1 (Beck's Partial coloring lemma) *Let $\mathcal{F}$ and $\mathcal{M}$ be set systems on an n-point set X, $|\mathcal{M}| > 1$, such that $|M| \leq s$ for every $M \in \mathcal{M}$ and*

$$\prod_{F \in \mathcal{F}} (|F| + 1) \leq 2^{(n-1)/5}. \tag{2}$$

Then there exists a partial coloring $\chi : X \to \{-1, 0, +1\}$, such that at least $\frac{n}{10}$ elements of X are colored, $\chi(F) = 0$ for every $F \in \mathcal{F}$, and $|\chi(M)| \leq \sqrt{2s \ln(4|\mathcal{M}|)}$ for every $M \in \mathcal{M}$.

We remark that the constants in the lemma can be improved somewhat, for example the constant $\frac{1}{10}$ can be replaced by 0.48.

The Partial coloring lemma is usually employed as follows. When we need to find a low-discrepancy coloring for some set system $\mathcal{S}$ on a set X, we define an auxiliary set system $\mathcal{F}$ such that

- $\mathcal{F}$ has sufficiently few sets. More exactly, it satisfies the condition (2), and
- each set $S \in \mathcal{S}$ can be written as a disjoint union of some sets from $\mathcal{F}$, plus some extra set M_S which is small (smaller than some parameter s, for all $S \in \mathcal{S}$).

We then define $\mathcal{M} = \{M_S : S \in \mathcal{S}\}$. The Partial coloring lemma yields a partial coloring χ that has zero discrepancy on all sets of $\mathcal{F}$, and so $|\chi(S)| = |\chi(M_S)| = O(\sqrt{s \log |\mathcal{S}|})$. In this way, some 10% of points of X are colored. We then look at the set of yet uncolored points, restrict the system $\mathcal{S}$ on these points, and repeat the construction of a partial coloring. In $O(\log n)$ stages, everything will be colored.

Canonical intervals and canonical boxes. Next, we are going to define appropriate decompositions for the set system induced by axis-parallel boxes on a given finite set $P \subset \mathbf{R^d}$.

We use decompositions as in the so-called *range trees* in computational geometry (Bentley and Shamos [12], Bentley [11]; or see Preparata and Shamos [21]). These decompositions are also similar to those used in the previous results on Tusnády's problem and no doubt in other instances in mathematics as well.

For a future use, our definition is a bit more general than needed in this section. Let v be a vector in $\mathbf{R^d}$, and let P be an n-point set in $\mathbf{R^d}$. Let $p_1, p_2, \ldots, p_n$ be the points of P listed in a nondecreasing order with respect to their projections on v, that is, such that $\langle v, p_1\rangle \le \langle v, p_2\rangle \le \ldots \le \langle v, p_n\rangle$. This ordering need not be unique, but we choose one such ordering arbitrarily and consider it fixed once and for all. We define a *canonical interval of P in the direction v* as a subset of P of the form $\{p_{q2^k+1}, p_{q2^k+2}, \ldots, p_{(q+1)2^k}\}$, $k, q = 0, 1, 2, \ldots$ (see Fig. 1). We let $\mathrm{CI}_v(P)$ be the family of all canonical intervals of P in the direction v;

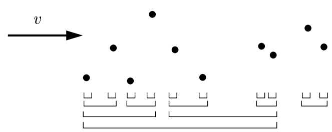

Figure 1: A schematic illustration of $\mathrm{CI}_v(P)$.

note that this is a set system on the ground set P. If $\mathcal{S}$ is a system of finite subsets of $\mathbf{R^d}$, we write

$$\mathrm{CI}_v(\mathcal{S}) = \bigcup_{S\in\mathcal{S}} \mathrm{CI}_v(S)\,.$$

Further we let, for an integer parameter $t \ge 1$,

$$\mathrm{CI}^t_v(P) = \{C \in \mathrm{CI}_v(P) :\ |C| \ge t\}$$

and analogously for $\mathrm{CI}^t_v(\mathcal{S})$. Finally, if $V = (v_1, v_2, \ldots, v_k)$ is a finite sequence of vectors in $\mathbf{R^d}$, we put

$$\mathrm{CI}_V(P) = \mathrm{CI}_{v_1}(\mathrm{CI}_{v_2}(\ldots \mathrm{CI}_{v_k}(P)\ldots))\,,$$

and similarly for $\mathrm{CI}^t_V(P)$.

It is well known, and easy to see, that if B is a slab in $\mathbf{R^d}$ bounded by two hyperplanes perpendicular to a vector v (or a halfspace with boundary perpendicular to v), then the intersection $P \cap B$ can be expressed as a disjoint union of at most $2\lfloor\log_2 n + 1\rfloor$ sets from $\mathrm{CI}_v(P)$. If we consider only canonical intervals of size at least t, i.e. ones from CI^t_v, we get the following statement: The intersection $P \cap B$ can be expressed as a disjoint union of at most $2\lfloor\log_2 n + 1\rfloor - 2\lceil\log_2 t\rceil \le 2\log_2\frac{2n}{t}$ sets from $\mathrm{CI}^t_v(P)$ plus a set of size at most $4t$. Using these observations and proceeding by induction on k, one gets

Lemma 2.2 *Let $V = (v_1, v_2, \ldots, v_k)$ be a sequence of k vectors in $\mathbf{R^d}$, let $t \ge 1$ be an integer, let P be an n-point set in $\mathbf{R^d}$, and let $B \in \mathrm{POL}(H)$, where H consists of hyperplanes with normal vectors $v_1, v_2, \ldots, v_k$. Put $\mathcal{C} = \mathrm{CI}^t_V(P)$. Then we have*

(i) *$P \cap B$ can be written as a disjoint union of at most $O_k((\log\frac{2n}{t})^k)$ sets from $\mathcal{C}$ plus a set of size at most $O_k(t(\log\frac{2n}{t})^{k-1})$.*

(ii)
$$\log\prod_{C\in\mathcal{C}}(|C|+1) = O_k\left(\frac{n}{t}(\log\frac{2n}{t})^{k-1}\log 2t\right)\,.$$

Proof sketch. As was remarked above, the proof goes by induction on k. As for part (i), we can write an $B \in \mathrm{POL}(H)$ as an intersection $B_1 \cap B_2 \cap \ldots \cap B_k$, where each B_i is a slab (or halfspace) bounded by hyperplanes perpendicular to the vector v_i. The intersection $P \cap B_k$ can be written as $C_1 \dot\cup C_2 \dot\cup \ldots \dot\cup C_q \dot\cup R_1$, where $C_1, \ldots, C_q$ are sets from $\mathrm{CI}^t_{v_k}(P)$, $q \le 2\log_2\frac{2n}{t}$, and R_1 is a set of size at most $4t$. By the inductive assumption, each C_i can be expressed as a disjoint union of sets from $\mathrm{CI}^t_{(v_1,\ldots,v_{k-1})}(C_i)$ plus a set of size $O_k(t(\log\frac{2n}{t})^{k-2})$, and part (i) follows easily.

As for part (ii), we observe that the set system $\mathrm{CI}^t_{v_k}(P)$ has $\lfloor n/2^j\rfloor$ sets of size 2^j, $t \le 2^j \le n$. Thus, if $p_k(n,t)$ denotes the maximum possible value of $\log_2\prod_{C\in\mathcal{C}}(|C|+1)$ for $\mathcal{C} = \mathrm{CI}^t_V(P)$ with $|P| \le n$ and V having k vectors, we get the recurrence

$$p_k(n,t) \le \sum_{t\le 2^j\le n}\frac{n}{2^j}p_{k-1}(2^j,t)\,.$$

with the initial condition $p_0(n,t) \le \log_2(n+1)$, and the claimed bound in part (ii) follows by a simple calculation. $\Box$

Proof of Theorem 1.2. Let P be a given n-point set in $\mathbf{R^d}$. We use the Partial coloring lemma 2.1 with $\mathcal{F} = \mathrm{CI}^t_E(P)$, where $E = (e_1, e_2, \ldots, e_d)$ is the orthonormal basis in $\mathbf{R^d}$ (e_i is parallel to the x_i-axis) and t is a suitable parameter, to be determined by calculation. From Lemma 2.2(ii), we get that in order that the condition (2) in the Partial coloring lemma be satisfied, we need that

$$K\,\frac{n}{t}(\log\frac{2n}{t})^{d-1}\log 2t \le \frac{n-1}{5}\,,$$

where K is some constant (depending on d). Therefore, we must choose

$$t \ge K'\log^{d-1} n\log\log n$$

for another large enough constant K'.

Consider an axis-parallel box $B \in \mathcal{R}_d$. According to Lemma 2.2(i), the intersection $P \cap B$ can be written as a disjoint union of some sets from $\mathcal{F}$ plus a set M_B of size at most $s = O_d(t(\log\frac{2n}{t})^{d-1}) = O_d((\log n)^{2d-2}\log\log n)$. Clearly, the set system induced by axis-parallel boxes on P has no more than n^{2d} distinct sets, and hence also the set system $\mathcal{M} = \{M_B :\ B \in \mathcal{R}_d\}$ has at most n^{2d} sets. The Partial coloring lemma applied on the set systems $\mathcal{F}$ and $\mathcal{M}$ yields the existence of a substantial partial coloring χ with $\chi(F) = 0$ for all $F \in \mathcal{F}$ and $|\chi(M)| = O_d(\sqrt{s\log|\mathcal{M}|}) = O_d(\log^{d-1/2} n\sqrt{\log\log n})$ for all $M \in \mathcal{M}$.

To get a full coloring, it remains to iterate this coloring procedure. Namely, we set $\chi_1 = \chi$, we let $P_1 \subset P$ be the set of points that remain uncolored under χ_1, and we use P_1 in the role of P in the above considerations, obtaining a substantial partial coloring χ_2 (note that the auxiliary set systems $\mathcal{F}$ and $\mathcal{M}$ are defined from scratch for P_1; we do not re-use the ones defined for P), and so on. We finish the iterations as soon as only few points remain uncolored (say Δ points, where $\Delta = O_d(\log^{d-1/2} n\sqrt{\log\log n})$ is the discrepancy bound for the first partial coloring χ_1), and we color these points arbitrarily. A full coloring χ of P is obtained by combining all the partial colorings $\chi_1, \chi_2, \ldots$. Since in each iteration, the