# Colored Homomorphisms of Colored Mixed Graphs

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**Abstract**

The homomorphisms of oriented or undirected graphs, the oriented chromatic number, the relationship between acyclic coloring number and oriented chromatic number, have been recently studied in [1, 3, 5, 6, 8, 10, 11]. Improving and combining earlier techniques of Alon and Marshall [1] and Raspaud and Sopena [10], we prove here a general result about homomorphisms of colored mixed graphs which implies all these earlier results about planar graphs. We also determine the exact chromatic number of colored mixed trees. For this, we introduce the notion of colored homomorphism for mixed graphs containing both colored arcs and colored edges.

## 1 Introduction

We will denote by $G = (V, A \cup E)$ a graph with vertices linked by arcs and edges, such that the underlying undirected graph is simple. $G$ will be called a mixed graph. $V$ is the set of its vertices, $A$ is the set of its arcs and $E$ is the set of its edges. In many aspects the mixed graphs present surprising difficulty. A striking example in given in [9]: the question whether an undirected (or directed) graph is Eulerian is notoriously known and easy, for mixed graphs the problem becomes NP-hard. Here we show that for colorings of mixed graph one can obtain results analogous to those of [1] and [10] (for colored undirected and oriented graphs). We assume that the arcs are colored with $n$ colors and the edges with $m$ colors. Henceforth we will say that the mixed graph $G$ is $(n,m)$-colored and adopt the following convention:
A $(0,m)$-colored mixed graph is an undirected graph with $m$-colored edges.
A $(n,0)$-colored mixed graph is an oriented graph with $n$-colored arcs.

Now let $G_1 = (V_1, A_1 \cup E_1)$ and $G_2 = (V_2, A_2 \cup E_2)$ be two $(n,m)$-colored mixed graphs. A colored homomorphism from $G_1$ to $G_2$ is any mapping $f : V_1 \to V_2$ satisfying :

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$$(x,y) \in A_1 \Longrightarrow (f(x), f(y)) \in A_2.$$
$$\{x,y\} \in E_1 \Longrightarrow \{f(x), f(y)\} \in E_2.$$

and the color of the arc or edge linking $f(x)$ and $f(y)$ is the same as that of the arc or edge linking $x$ and $y$.
This definition is a generalized version of the one given for undirected graphs in [1].
The existence of a colored homomorphism from $G_1$ to $G_2$ will be denoted $G_1 \rightarrow G_2$.

Given a $(n,m)$-colored mixed graph $G = (V, A \cup E)$, the problem is to find the smallest number of vertices of a $(n,m)$-colored mixed graph $G' = (V', A' \cup E')$ for which $G \rightarrow G'$. This number will be denoted here $\vec{\chi}(G,n,m)$ and called the *(n,m)-colored oriented chromatic number* of $G$.
Let $\mathcal{G}$ be a family of undirected graphs. We will denote by $\vec{\chi}(\mathcal{G},n,m)$ the maximum of $\vec{\chi}(G,n,m)$ taken over all the possible mixed $(n,m)$-colored graphs having as underlying undirected graph an element of $\mathcal{G}$. In [10] it is proved that $\vec{\chi}(\mathcal{P},1,0)$ is bounded by a constant where $\mathcal{P}$ is the family of the planar graphs. In fact it is proved that $\vec{\chi}(\mathcal{P},1,0) \leq 80$ and presently this is the best known result. In [1] it is proved that $\vec{\chi}(\mathcal{P},0,m) \leq 5m^4$. In this note, we will prove that $\vec{\chi}(\mathcal{P},n,m) \leq 5(2n+m)^4$.
As in [10, 1] this result is a consequence of a more general result dealing with the acyclic chromatic number. A proper *$k-$coloring* of the vertices of an undirected graph is said to be *acyclic* if the subgraph induced by the vertices with any two colors has no cycle. The acyclic chromatic number of a graph $G$ is the smallest $m$ such that $G$ has an acyclic $m$-coloring. It will be denoted by $a(G)$. We shall make use of the famous result of Borodin result [2] who proved $a(G) \leq 5$ for every planar graph $G$. We will prove in this note the following theorems:

**Theorem 1** *Let $\mathcal{T}$ be the class of $(n,m)$-colored mixed forests:*

*1)* $\vec{\chi}(\mathcal{T},n,0) = 2n+1$

*2)* $\vec{\chi}(\mathcal{T},n,m) = 2(n + \lfloor \frac{m}{2} \rfloor + 1)$ *for* $m \neq 0$.

**Theorem 2** *If $G$ is a $(n,m)$-colored mixed graph for which the acyclic chromatic number of the underlying undirected graph is at most $k$ then $\vec{\chi}(G,n,m) \leq k(2n+m)^{k-1}$.*

The proof of Theorem 2 is a refinement of the technique used in [10] for oriented graphs without any coloration of the arcs and the technique used in [1] for edge colored graphs without any orientation.
Notice that if $n = 0$ we obtain the result of Alon and Marshall and if $m = 0$ we obtain a generalization of the result of Raspaud and Sopena.
Similarly as in [1], we can show that the main term in the upper bound in Theorem 2 is in a sense the best possible.

**Observation 1** *Graphs of acyclic chromatic number at most $k$ could have $(n,m)$-colored oriented chromatic number greater than $(2n+m)^{k-1} + k - 1$.*



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that a coloring $c$ of $X$ *respects* $P_1, \ldots, P_t$ if the following two conditions hold:

(i) (non triviality) $c(x) \neq c(y)$ whenever $x \neq y$ and $P_i(x, y)$ holds for some $i$;

(ii) (compatibility) If $P_i(x, y)$ holds and $c(x) = c(x')$, $c(y) = c(y')$, then also $P_i(x', y')$ holds.

The minimal number of colors of a coloring of $X$ respecting $P_1, \ldots, P_t$ will be denoted by $\chi(X, P_1, \ldots, P_t)$. On can check that $\chi(X, P_1, \ldots, P_t)$ equals $\vec{\chi}(G, n, m)$ for the $(n, m)$-colored mixed graph $G = (V, A \cup E)$ defined as follows:
$V = X$; an edge $e = \{x, y\}$ belongs to the set $E$ if and only if $P_i(x, y) = P_i(y, x)$ and there exists $i$ with $P_i(x, y) = 1$ for some $i$; an arc $e = (x, y)$ belongs to the set $A$ if and only $P_i(x, y) = 1$ and $P_i(y, x) = 0$ for some $i$. We put $m = 2^t$, $n = 2^{2t}$. An edge $e = \{x, y\}$ gets the color consisting of the set $\{i; P_i(x, y) = P_i(y, x) = 1\}$ (so color of $\{x, y\}$ is a subset of $\{1, \ldots, t\}$). An arc $e = (x, y)$ gets the colors from the set $\{i; P_i(x, y) \neq P_i(y, x)\} \cup \{i'; P_{i'}(x, y) = P_{i'}(y, x)\}$ (so color of $(x, y)$ is a subset of $\{1, \ldots, t\} \cup \{1', \ldots, t'\}$).
Our Theorem 2 can then be formulated as follows:

**Theorem 3** *Let $X$ be a set and let the support of $t$-binary predicates $P_1, \ldots, P_t$ (i.e. the set $\{\{x, y\};\ P_i(x, y)$ holds for some $i$ $\}$ be a planar graph. Then $\chi(X, P_1, \ldots, P_t)$ is bounded by a function of $t$ only. For example $\chi(X, P_1, \ldots, P_t) < 5.2^{8(t+1)}$; we don't try to optimize here.*

**2.** Theorem 1 determines the maximal oriented chromatic of a $(n, m)$-colored mixed tree. Some instances of our problem have been studied earlier. For example the harmonious chromatic number of a graph is the oriented chromatic number of an undirected graph where all the edges get distinct colors (i.e. a $(0, |E|)$-colored mixed graph). Edwards and McDiarmid recently proved [5] that to find the harmonious chromatic number of a tree is NP-complete. This implies that the oriented chromatic number of colored mixed trees is in general a NP-complete problem. Is there a polynomial algorithm for each fixed pair of values $m$,$n$? This holds for $n = 1$, $m = 0$; $n = 0$, $m = 1$; $n = 1$, $m = 1$.

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To see that, let $\vec{\mathcal{K}}_{k-1,(2n+m)^{k-1}}$ be the complete bipartite graph with $k-1$ vertices in one set of the bipartition and $(2n+m)^{k-1}$ vertices in the other bipartition's set. The edges are oriented and colored in such a way that from all the $(2n+m)^{k-1}$ vertices of the second partition's set, we have all the possible configurations for orientation and color of the incident arcs and edges. It is easy to see that the acyclic chromatic number of this graph is at most $k$. Moreover $\vec{\chi}(\vec{\mathcal{K}}_{k-1,(2n+m)^{k-1}}, n, m) = (2n+m)^{k-1} + k - 1$.
We remark that in the case where we consider only oriented graphs (no edges and only monochromatic arcs), this observation improves the lower bound given in Obervation 3 of [6].
Combining Theorem 2 with the theorem of Borodin we obtain:

**Corollary 1** $\vec{\chi}(\mathcal{P}, n, m) \leq 5(2n+m)^4$

One can also generalize a construction given in [1] to obtain a lower bound $\vec{\chi}(\mathcal{P}, n, m) \geq (2n+m)^3 + 3$.

## 2 Proof of the theorems

**Proof of Theorem 1:**

We first prove the lower bounds.
Let $m$ and $n$ be any two integers and $S$ be a star with $2n+m+1$ vertices and $n$ arcs entering the central vertex, the arcs being colored with the $n$ different colors, and $n$ arcs outgoing from the central vertex, the arcs being colored with the $n$ different colors. Moreover $m$ edges are incident to the central vertex, the edges being colored with the $m$ different colors. It is easy to see that $\vec{\chi}(S, n, m) \geq 2n + m + 1$. Thus we have the claimed lower bound for $m = 0$ and $m$ odd.
Now the case $m$ even:
We consider the following graph, constructed from the star $S$. We take $2n+m$ copies of the star $S$. In each copy we choose a vertex different from the central vertex in order to have $2n+m$ vertices different with respect to the color and the orientation (if any) of the incident edge. Then we identify all these $2n+m$ vertices in one vertex. We will denote by $S^*$ this complete $(2m+n)$-ary tree of height equal to two. It is clear by the fact that $S^*$ contains $S$ that the target graph must have at least $2n+m+1$ vertices and contain the star $S$ as a spanning subgraph, the star contained in $S^*$ must be mapped into the star contained in the target graph. It is the easy to see by a simple counting argument that the external vertices (i.e. the leaves of the tree) cannot be mapped onto this target graph having $2n+m+1$ vertices and $S$ as a spanning subgraph. Hence $\vec{\chi}(S_*, n, m) \geq 2n + m + 2$. This gives the lower bound for $m$ even.
Now we prove the upper bounds.

**Case 1:** $m$ is odd:
Let $T$ be a $(n, m)$-colored mixed tree. We now define the target graph, which we will denote by $\vec{\mathcal{K}}_{2n+m+1}$. It is a complete $(n, m)$-colored mixed graph whose the underlying undirected graph is the complete graph $K_{2n+m+1}$.
It well known by a theorem due to Walecki [7] that the set of the edges of $K_{2n+m+1}$ can be decomposed into $n + \frac{m-1}{2}$ Hamiltonian cycles and one perfect matching. To construct the target graph, we orient $n$ Hamiltonian cycles to obtain $n$ Hamiltonian circuits, and we color all the arcs

of each of them with the $n$ different colors, to obtain $n$ monochromatic Hamiltonian circuits. The $\frac{m-1}{2}$ other Hamiltonian cycles remain undirected and for each of them we color the edges alternately with 2 colors of the given $m$ colors. Hence we have $\frac{m-1}{2}$ bicolored Hamiltonian cycles and in total we have used $m-1$ colors. Then we color the edges of the matching by the last remaining color. This completes the definition of $\vec{\mathcal{K}}_{2n+m+1}$.
Notice that each vertex of this complete $(n,m)$-colored mixed graph is incident with one arc of each type (orientation and color) and with one edge of each color.
It easy to see that there is a colored homorphism between $T$ and $\vec{\mathcal{K}}_{2n+m+1}$. For example we can use a depth first search of the tree $T$ from any root and map the root vertex into any vertex of $\vec{\mathcal{K}}_{2n+m+1}$. Then each new vertex $x$ encountered will be mapped into a vertex of $\vec{\mathcal{K}}_{2n+m+1}$ according to the color and the orientation of the edge linking the vertex $x$ to another already visited vertex. Hence $\vec{\chi}(\mathcal{T},n,m) \leq 2n+m+1$. This completes the proof of the case $m$ odd.

**Case 2 :** $m$ is even.
Let $T$ be a $(n,m)$-colored mixed tree. We now define the target graph which we will denote by $\vec{\mathcal{K}}_{2n+m+2}$. It is a complete $(n,m)$-colored mixed graph whose underlying undirected graph is the complete graph $K_{2n+m+2}$. Since $2n+m+2$ is even, we can decompose $K_{2n+m+2}$ into $n+\frac{m}{2}$ Hamiltonian cycles and one matching. As in the previous case, to construct the target graph, we orient $n$ Hamiltonian cycles to obtain $n$ Hamiltonian circuits, and we color all the arcs of each of them with the $n$ different colors, to obtain $n$ monochromatic Hamiltonian circuits. The $\frac{m}{2}$ other Hamiltonian cycles remain undirected and for each of them we color the edges alternately with 2 colors of the given $m$ colors. Hence we have $\frac{m}{2}$ bicolored Hamiltonian cycles and in total we have used $m$ colors. The remaining matching is not used, we can, for example, color it by one of the $m$ colors to obtain a $(n,m)$-colored mixed graph. The definition of $\vec{\mathcal{K}}_{2n+m+2}$ is complete.
Notice again that each vertex of the mixed complete graph is incident with one arc of each type (orientation and color) and with one edge of each color. Then we proceed as in the previous case. Hence $\vec{\chi}(\mathcal{T},n,m) \leq 2n+m+2$.

**Case 3 :** $m = 0$.
If $T$ is an $n$-colored oriented tree (with only arcs and no edges). The target graph will be $\vec{\mathcal{K}}_{2n+1}$, the complete diregular graph with a Hamiltonian circuits decomposition.
Moreover the arcs of a Hamiltonian circuit are monochromatic. Hence we have $n$ Hamiltonian circuits, each of them having a different color. Then we proceed as in the previous cases.
This completes the proof of theorem 1. □

**Proof of theorem 2.**
Let $\vec{\mathcal{K}}_{2n+m,2n+m}$ be a complete bipartite mixed graph with $2n+m$ vertices in each bipartition class, defined as follows. The underlying undirected graph is the complete bipartite graph $K_{2n+m,2n+m}$ with $2n+m$ vertices in each bipartition class. It is well know by a theorem due to Dirac [4] that the set of edges of $K_{2n+m,2n+m}$ can be decomposed into $\frac{2n+m-1}{2}$ Hamiltonian cycles plus a perfect matching if $m$ is odd or $\frac{2n+m}{2}$ Hamiltonian cycles if $m$ is even. To obtain the mixed graph, first, we orient $n$ Hamiltonian cycles in such a way that they become $n$ Hamiltonian circuits, and we color the arcs of each of them by one different color taken from the $n$ given colors in order to obtain $n$ monochromatic Hamiltonian circuits. Then, the $\frac{m}{2}$ if $m$ is even, or $\frac{m-1}{2}$ if $m$ is odd, remaining Hamiltonian cycles remain undirected and for each of them we color the edges alternately with 2 colors of the given $m$ colors. We complete by coloring the matching by the remaining color if necessary.
Notice again that each vertex of the mixed complete graph is incident with one arc of each type (orientation and color) and with one edge of each color.
We will denote:
$V(\vec{\mathcal{K}}_{2n+m,2n+m}) = A \cup B$ with $A = \{1,2,\ldots,2n+m\}$ and $B = \{\mathit{1},\mathit{2},\ldots,\mathit{2n+m}\}$. We shall make use of the following:

**Proposition 1** *Let $T$ be a $(n,m)$-colored mixed tree and $V_1$ and $V_2$ be any bipartion of its set of vertices according to the fact that a tree is a bipartite graph. Then there exists a colored homomorphism of $T$ into $\vec{\mathcal{K}}_{2n+m,2n+m}$ which maps $V_1$ into $A$ and $V_2$ into $B$.*

The proof is similar to that of Theorem 1. We can use a depth first search, by mapping the root into any vertex of the correct color class. □

Now we prove Theorem 2.
Let $G = (V, A \cup E)$ be an $(n,m)$-colored mixed graph for which the acyclic chromatic number of the underlying undirected graph is $k$. The target graph $H$ has the following set of vertices: $V(H) = \{(i; a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k)\}$ where $i \in [1,k]$ and $a_l \in [1, 2n+m]$, $l \in [1, i-1] \cup [i+1, k]$.
For $1 \leq i < j \leq k$, two vertices of $H$ $(i; a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k)$ and $(j; b_1, b_2, \ldots, b_{j-1}, b_{j+1}, \ldots, b_k)$ are linked by an edge or an arc with the direction and the color according to the one linking $\mathit{s} \in A$ and $t \in B$ in $\vec{\mathcal{K}}_{2n+m,2n+m}$, where $\mathit{s} = a_j$ and $t = b_i$.

Now we define the colored homomorhism.
Let $V_1, V_2, \ldots, V_k$ be the color classes of an acyclic coloring of the vertices of $G$ with $k$ colors. For any $i$ and $j$, $i < j$, belonging to $[1,k]$, $G[V_i \cup V_j]$, the induced graph by $V_i \cup V_j$, is a forest. Hence by proposition 1 there exists a colored homorphism $c_{i,j}$ which maps $V_i$ (resp. $V_j$) into $A$ (resp. $B$). We define the colored homomorphism $f$ which maps $G$ into $H$ as follows. Let $v$ be a vertex of $G$; then $v$ belongs to one $V_i$ for some $i \in [1,k]$. The image of $v$, will be : $f(v) = (i; c_{1,i}(v), \ldots, c_{i-1,i}(v), c_{i,i+1}(v), \ldots, c_{i,k}(v))$. The constraints of coloration and orientation are satisfied. Indeed, if $v \in V_i$ and $w \in V_j$ are linked by a colored arc or a colored edge, then the vertices $f(v) = (i; c_{1,i}(v), \ldots, c_{i-1,i}(v), c_{i,i+1}(v), \ldots, c_{i,k}(v))$ and $f(w) = (j; c_{1,j}(w), \ldots, c_{j-1,j}(w), c_{j,j+1}(w), \ldots, c_{j,k}(w))$ are, by the definition of the graph $H$, linked by a colored arc or a colored edge according to the link existing between $c_{i,j}(v)$ in $A$ and $c_{i,j}(w)$ in B. It is then clear that the orientation and the color of the arc, if it is an arc, or the color of the edge, if it is an edge, is that of the existing link between $v$ and $w$, because $c_{i,j}$ is a colored homorphism wich maps $G[V_i \cup V_j]$ into $\vec{\mathcal{K}}_{2n+m,2n+m}$ according to the bipartition. The number of the vertices of $H$ is exactly $k(2n+m)^{k-1}$. This completes the proof. □

## 3 Concluding Remarks

**1.** Our Theorem 2 is general enough to imply the following result about binary predicates on a fixed set $X$:
Let $X$ be a set and $P_1, \ldots, P_t$ be binary predicates on $X$ (i.e. $P_i : X \times X \longrightarrow \{0,1\}$). We say