ON THE COMPUTATIONAL COMPLEXITY OF $(\mathcal{O}, \mathcal{P})$-PARTITION PROBLEMS

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Abstract

We prove that for any additive hereditary property $\mathcal{P} > \mathcal{O}$, it is NP-hard to decide if a given graph G allows a vertex partition $V(G) = A \cup B$ such that $G[A] \in \mathcal{O}$ (i.e., A is independent) and $G[B] \in \mathcal{P}$.

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1. Introduction

We consider finite undirected simple graphs. A *graph property* is any isomorphism closed class of graphs. A graph property is *hereditary* if it is closed under taking subgraphs, and it is *additive* if it is closed under taking disjoint

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unions. The class $\mathcal{O}$ of all edgless graphs is the simplest additive hereditary property.

The *join* $G \oplus H$ of two graphs G and H is the graph consisting of the disjoint union of G and H and all the edges between $V(G)$ and $V(H)$.

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be graph properties. A *vertex* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-*partition* of a graph G is a partition $(V_1, V_2, \ldots, V_n)$ of $V(G)$ such that for each $i = 1, 2, \ldots, n$, the induced subgraph $G[V_i]$ has the property $\mathcal{P}_i$. The composition $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ is defined as the class of all graphs having a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. A graph property $\mathcal{P}$ is *reducible* if $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$ for some nonempty properties $\mathcal{P}_1, \mathcal{P}_2$, and it is called *irreducible* otherwise. Every additive hereditary property is uniquely factorizable into irreducible properties [3].

In this paper we address the complexity of recognizing graphs from reducible properties. This question may be viewed as generalization of graph coloring problems. It is plausible to conjecture that recognition of $\mathcal{P} \circ \mathcal{Q}$ graphs is NP-hard for any two nonempty properties, if at least one of them is nontrivial. In this note we present a general reduction which shows the hardness for $\mathcal{Q} = \mathcal{O}$ and $\mathcal{P} \neq \mathcal{O}$.

2. The NP-Hardness Results

Theorem 1. *If $\mathcal{P} \neq \mathcal{O}$ is an additive hereditary property not divisible by $\mathcal{O}$ then the $(\mathcal{O}, \mathcal{P})$-partition problem is NP-hard.*

Proof. Let k be the maximum size of a complete graph belonging to $\mathcal{P}$, i.e., $K_k \in \mathcal{P}$ and $K_{k+1} \notin \mathcal{P}$. Clearly, $k > 1$. Let S_{k+1} be the graph obtained by taking $k+1$ complete graphs of size k and unifying one vertex from each of them, and let F_k denote the graph obtained from K_k by hanging a copy of K_k on each of its vertices. (Here and later on, to *hang* a copy of K_k onto a vertex x in a graph G means to add a clique of size $k-1$ to G and make all of its $k-1$ vertices adjacent to x.)

We will reduce from **1-in-**$(k+1)$**-SAT** which is known to be NP-complete even when the input formula has no negations and all variables occur in exactly $k+1$ clauses each (this is the exact cover problem for $(k+1)$-regular $(k+1)$-uniform hypergraphs) [1].

Suppose first that S_{k+1} and F_k are both in $\mathcal{P}$. Given a formula Φ as specified above, we construct a graph G as follows: For each clause

$c = (x_{c,1}, x_{c,2}, \ldots, x_{c,k+1})$ we introduce a complete graph on $k+1$ vertices $c_1, c_2, \ldots, c_{k+1}$. For each variable x we regard x as a vertex of G. If x occurs in a clause c, say $x = x_{c,i}$ in c, we construct a so called connector gadget by taking a complete graph on $k+1$ vertices $c_i, x, v_1(c,x), \ldots, v_{k-1}(c,x)$, and forcing the vertices $v_1(c,x), \ldots, v_{k-1}(c,x)$ to "be in P". This forcing is done as follows. Mihók proved in [2] that for any property $\mathcal{P} > \mathcal{O}$ not divisible by $\mathcal{O}$, there exists a graph uniquely partitionable into $\mathcal{O} \circ \mathcal{P}$. Take a copy of such a graph and make one vertex from the $\mathcal{O}$ part of it adjacent to $v_1(c,x), \ldots, v_{k-1}(c,x)$. In any $\mathcal{O} \circ \mathcal{P}$ partition of G, this vertex is in the $\mathcal{O}$ part, and since this part is independent, the vertices $v_1(c,x), \ldots, v_{k-1}(c,x)$ are all in the $\mathcal{P}$ part. We claim that G constructed in this way allows an $(\mathcal{O}, \mathcal{P})$-partition if and only if Φ is satisfiable.

Suppose first that G does allow a partition $V(G) = A \cup B$ such that $G[A] \in \mathcal{O}$ and $G[B] \in \mathcal{P}$. We set $x = \mathsf{true}$ iff $x \in B$. Since $K_{k+1} \notin \mathcal{P}$, at least one vertex of each clause gadget is in A, and since A is independent, such vertex is unique. Say this is a vertex c_i in a clause c. Since A is independent, the corresponding variable vertex $x_{c,i}$ is in B and this variable is true in the clause. For any other variable $x_{c,j}$ in c, $c_j \in B$. Since the connector gadget is another copy of K_{k+1} and $k-1$ of its vertices are forced to be in B, the only vertex which can be in A is the corresponding variable vertex $x_{c,j}$, hence every $x_{c,j}, j \neq i$ is false. Thus Φ is 1-in-$(k+1)$-satisfied.

Suppose on the other hand that Φ is 1-in-$(k+1)$ satisfied by a truth valuation ϕ. We set

$$A = \{x | \phi(x) = \mathsf{false}\} \cup \bigcup_c \{c_i | \phi(x_{c,i}) = \mathsf{true}\}$$

$$B = \{x | \phi(x) = \mathsf{true}\} \cup \bigcup_c \{c_i | \phi(x_{c,i}) = \mathsf{false}\}$$

and we add the vertices whose membership is forced to the particular classes (A representing $\mathcal{O}$ and B representing $\mathcal{P}$). Obviously, A is an independent set. The components of $G[B]$ in the forcing uniquely partitionable graphs which hang on $v_i(c,x)$'s are in $\mathcal{P}$ by construction, the remaining components of $G[B]$ are copies of F_k (around the clause gadgets) and S_{k+1} (around the variable vertices which were valued true). Thus $G \in \mathcal{O} \circ \mathcal{P}$ and we are done.

The situation is slightly more complex if $F_k \notin \mathcal{P}$ or $S_{k+1} \notin \mathcal{P}$. Here we first need to introduce some notation.

Let H be a rooted graph and let $s = (s_1, s_2, \ldots, s_n)$ be a finite sequence of positive integers. We denote by $H[s]$ the graph obtained from H by hanging n complete graphs $K_{s_i}, i = 1, 2, \ldots, n$ on the root of H.

For a sequence of k positive integers $s = (s_1, s_2, \ldots, s_k)$, we denote by $F_k(s)$ the graph obtained by hanging complete graphs $K_{s_i}, i = 1, 2, \ldots, k$ on the vertices of K_k, one on each. Thus

$$F_k = F_k(k, k, \ldots, k) \text{ – } k \text{ terms in the parentheses}$$

$$S_{k+1} = K_1[k, k, \ldots, k] \text{ – } k+1 \text{ terms in the parentheses.}$$

If $F_k(k, k, \ldots, k) \in \mathcal{P}$, we set $m = k$ and $t^+ = (k, k, \ldots, k)$. If $F_k(k, k, \ldots, k) \notin \mathcal{P}$, we let m be the least number such that $F_k(m, m, \ldots, m) \notin \mathcal{P}$ and we let h be the smallest index such that for $t_1 = t_2 = \ldots = t_h = m - 1, t_{h+1} = \ldots = t_k = m$ and $t = (t_1, t_2, \ldots, t_k)$, $F_k(t) \in \mathcal{P}$. We then set $t^+ = (t_1^+, t_2^+, \ldots, t_k^+)$ so that $t_1^+ = t_2^+ = \ldots = t_{h-1}^+ = m - 1$, $t_h^+ = \ldots = t_k^+ = m$. Thus $F_k(t) \in \mathcal{P}$ and $F_k(t^+) \notin \mathcal{P}$. We denote by H the graph obtained from $F_k(t^+)$ by deleting one of the hanging cliques of size m, rooted in the vertex whose clique was deleted (e.g., in the vertex corresponding to t_k^+).

For a sequence $s = (s_1, s_2, \ldots, s_n)$ we denote

$$\psi(s) = (s_2, s_3, \ldots, s_n)$$

i.e., the sequence obtained by deleting the first element, and we denote

$$\phi_k(s) = (s_1 - 1, s_1 - 1, \ldots, s_1 - 1, s_2, s_3, \ldots, s_n)$$

i.e., the sequence obtained by replacing the first element s_1 by k occurrences of $s_1 - 1$. Note that $|\psi(s)| = |s| - 1$ and $|\phi_k(s)| = |s| + k - 1$.

If $F_k \notin \mathcal{P}$, we have $H[m] \notin \mathcal{P}$. If $F_k \in \mathcal{P}$, we must have $S_{k+1} \notin \mathcal{P}$ and hence $H[k, k, \ldots, k] \notin \mathcal{P}$ (but $H = H[1] \in \mathcal{P}$). In either case, Lemma 1 says that there is a sequence $s = (s_1, \ldots, s_n)$ such that

$$2 \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq m,$$

$$H[s] \notin \mathcal{P} \text{ but } H[\phi_k(s)] \in \mathcal{P}.$$

We denote by $\widetilde{H}$ the graph $H[\psi(s)] = H[s_2, s_3, \ldots, s_n]$ and we use this $\widetilde{H}$ for the construction of the graph G.

Given a formula Φ as in the first part of the paper, we again plug into G $(k+1)$-cliques for the clauses of Φ. Each variable x will be replaced by a copy of $\widetilde{H}$ with the root in the vertex x and with all vertices except for x being forced to "be in P". If variable x occurs as the i-th variable of a clause c, the connector of x and c will be a copy of K_{s_1} containing c_i, x and $s_1 - 2$ extra vertices which will be also forced to "be in P".

Now the proof is straightforward. Suppose first that $G \in \mathcal{O} \circ \mathcal{P}$, say $V(G) = A \cup B$ such that A is independent and $G[B] \in \mathcal{P}$. Again we set $x = \mathsf{true}$ iff $x \in B$. Since $K_{k+1} \notin \mathcal{P}$, at least one vertex of each clause gadget is in A, and since A is independent, such vertex is unique. Say this be a vertex c_i in a clause c. Since A is independent, the corresponding variable vertex $x_{c,i}$ is in B and this variable is true in the clause. For any other variable $x_{c,j}$ in c, $c_j \in B$. Since the connector gadget K_{s_1} together with the vertices of the variable gadget which are forced to be in B forms $H[s] \notin \mathcal{P}$, it must be $x_{c,j} \in A$ for every $j \neq i$. Thus Φ is Φ is 1-in-$(k+1)$ satisfied.

Suppose on the other hand that Φ is 1-in-$(k+1)$ satisfied by a truth valuation ϕ. We set

$$A = \{x|\phi(x) = \mathsf{false}\} \cup \bigcup_c \{c_i|\phi(x_{c,i}) = \mathsf{true}\}$$

$$B = \{x|\phi(x) = \mathsf{true}\} \cup \bigcup_c \{c_i|\phi(x_{c,i}) = \mathsf{false}\}$$

and we add the vertices whose membership is forced to the particular classes (A representing $\mathcal{O}$ and B representing $\mathcal{P}$). Obviously, A is an independent set. The components of $G[B]$ in the forcing uniquely partitionable graphs which hang on $v_i(c,x)$'s are in $\mathcal{P}$ by construction. The remaining components of $G[B]$ are copies $F_k(s_1-1,\dots,s_1-1) \subset F_k(m-1,m-1,\dots,m-1) \subset F_k(t) \in \mathcal{P}$ (around the clause gadgets) and $\widetilde{H}[s_1-1,\dots,s_1-1] = H[\phi_k(s)] \in \mathcal{P}$ (around the variable vertices which are valued true). Thus $G \in \mathcal{O} \circ \mathcal{P}$ and we are done. ■

Lemma 1. *Let H be a graph such that $H \in \mathcal{P}$ and $H[w] \notin \mathcal{P}$ for some sequence w. Then there exists a sequence s such that*

$$\max s \leq \max w,$$

$$H[s] \notin \mathcal{P},$$

$$H[\phi_k(s)] \in \mathcal{P}.$$

Proof. Let $m = \max w$. Set

$$A = \{s | 1 < s_1 \leq \ldots \leq s_n \leq m, H[s] \in \mathcal{P}\},$$

$$A' = \{s | s \notin A, \psi(s) \in A\}.$$

Let $s \in A'$ be a sequence with minimum possible $s_1 (> 1)$.

If $s_1 = 2$ then $\phi_k(s) = (1, 1, \ldots, 1, s_2, \ldots, s_n)$ and $H[\phi_k(s)] = H[\psi(s)] \in A$ and s has the desired property.

If $s_1 > 2$ then s would be good for us if $H[\phi_k(s)] \in \mathcal{P}$. So we may assume that $H[\phi_k(s)] = H[s_1 - 1, s_1 - 1, \ldots, s_1 - 1, s_2, \ldots, s_n] \notin \mathcal{P}$. Since $(s_2, \ldots, s_n) \in A$, there is a number $j > 0$ such that $(s_1 - 1, \ldots, s_1 - 1, s_2, \ldots, s_n) \in A'$ (with j occurrences of $s_1 - 1$). But this is a contradiction as $2 \leq s_1 - 1 < s_1$. ■

Theorem 2. *For any property $\mathcal{P} \neq \mathcal{O}$, the $(\mathcal{O} \circ \mathcal{P})$-partition problem is NP-hard.*

Proof. If $\mathcal{P} = \mathcal{O}^n$ for $n \geq 2$, then the $(\mathcal{O}, \mathcal{P})$-partition problem is just the $(n+1)$-colorability of graphs and hence well known NP-complete.

Let $\mathcal{P} = \mathcal{O}^n \circ \mathcal{Q}$, where and $\mathcal{Q}$ is not divisible by $\mathcal{O}$. In view of Theorem 1, we may assume that $n > 0$. We know that $\mathcal{O} \circ \mathcal{Q}$-partition is NP-hard. Suppose G is an input graph for this question. Let G have g vertices and let G' be the join (Zykov sum) of G and n independent sets $I_i, i = 1, 2, \ldots, n$, each of size g. We claim that $G' \in \mathcal{O} \circ \mathcal{P} = \mathcal{O}^{n+1} \circ \mathcal{Q}$ if and only if $G \in \mathcal{O} \circ \mathcal{Q}$.

It is clear that if $G \in \mathcal{O} \circ \mathcal{Q}$ then $G' \in \mathcal{O}^{n+1} \circ \mathcal{Q}$.

On the other hand, suppose that G' allows a partition into a graph from $\mathcal{Q}$ and at most $n+1$ independent sets, say $V(G') = Q \cup \bigcup_{i=1}^{n+1} A_i$, where each A_i is independent and $G'[Q] \in \mathcal{Q}$. Each A_i is either a subset of $V(G) - Q$, or a subset of one of the I_i's. Assume $k+1$ of A_i's being subsets of $V(G) - Q$, and without loss of generality let them be $A_1, \ldots, A_{k+1}$. Then only $n-k$ sets $A_{k+2}, \ldots, A_{n+1}$ lie outside of $V(G)$, and consequently, at least k of the I_i's lie inside Q, say $I_1, \ldots, I_k$. But then $G[(Q \cap V(G)) \cup \bigcup_{i=1}^{k} A_i]$ is isomorphic to a subgraph of $G[V(G) \cap Q] \oplus \sum_{i=1}^{k} I_i \subset G'[Q] \in \mathcal{Q}$ and $G \subset G[(V(G) \cap Q) \cup \bigcup_{i=1}^{k} A_i] \oplus G[A_{k+1}] \in \mathcal{Q} \circ \mathcal{O}$.

Since the construction of G' is linear in the size of G, we have concluded the proof. ■

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