Rankings of directed graphs

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Abstract

A ranking of a graph is a coloring of the vertex set with positive integers such that on every path connecting two vertices of the same color there is a vertex of larger color. We consider the directed variant of this problem, where the above condition is imposed only on those paths in which all edges are oriented in the same direction. We show that the ranking number of a directed tree is bounded by that of its longest directed path plus one, and that it can be computed in polynomial time. Unlike the undirected case, however, deciding whether the ranking number of a directed (and even of an acyclic directed) graph is bounded by a constant is NP-complete. In fact, the 3-ranking of planar bipartite acyclic digraphs is already hard.

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1 Introduction

Given an undirected graph G, its *ranking number* $\chi_r(G)$ is the minimum integer k for which there exists a *(vertex) k-ranking*, that is a mapping $f : V(G) \to \{1, 2, \ldots, k\}$ such that every path connecting two vertices u, v of the same rank $f(u) = f(v)$ contains a vertex w with higher rank, $f(w) > f(u)$.

It is well known and easy to see that for the path P_ℓ of length $\ell - 1$ on ℓ vertices,

$$\chi_r(P_\ell) = \lfloor \log \ell \rfloor + 1$$

holds, and that the longest k-rankable path $P_{2^k-1} = x_1 x_2 \ldots x_{2^k-1}$ admits the *unique* optimal ranking f with

$$f(x_i) = \max\{ j : 2^j | i \} + 1$$

for all $1 \leq i < 2^k$. (Throughout, $\log$ means logarithm of base 2.)

This paper is the first approach to the ranking of *directed* graphs. The ranking number of a digraph G is naturally defined as the minimum k such that there exists a mapping $f : V(G) \to \{1, 2, \ldots, k\}$ with the property that every *directed path* (i.e., path in which all edges are oriented consecutively) connecting two vertices u, v of the same rank $f(u) = f(v)$ contains a vertex w with higher rank, $f(w) > f(u)$. We denote the ranking number of a directed graph G again by $\chi_r(G)$.

Obviously, the ranking number of a directed path equals that of the undirected path of the same length. Directed and undirected rankings, however, have a strikingly different behavior already on trees. For instance, an undirected tree containing no path longer than t can have as large ranking number as $\lceil t/2 \rceil + 1$. This is far from being true in the directed case. We shall prove that the ranking number of a directed tree can exceed that of its longest directed path by at most 1 (Corollary 3), hence it grows just with $\log t$.

We also consider rankings from the computational complexity point of view. The problem Ranking takes as input a graph G and a positive integer k, and asks whether $\chi_r(G) \leq k$. It is known that Ranking on undirected graphs is NP-complete in general, but solvable in polynomial time for every fixed k; see [1] for results and further references. For the analogous problem of Directed Ranking, however, we prove in Theorem 8 that it is NP-complete even if the input is restricted to fixed $k = 3$ and to *acyclic* orientations of *planar bipartite* graphs. On the other hand, the 2-rankable directed graphs can be characterized in several different ways, as shown in Section 5. We also prove that the ranking number of directed *trees* can be determined in polynomial time (Section 3).

2 Upper bound for trees

In this section we prove general bounds on the ranking number of oriented trees and also on that of orientations of a path of given length. We begin with some definitions.

Notation. We write $p(\ell) := \lfloor \log \ell \rfloor + 1 = \chi_r(P_\ell)$ for the ranking number of the (directed or undirected) path with ℓ vertices (i.e., $p(\ell) = k$ if and only if $2^{k-1} \le \ell \le 2^k - 1$). Moreover, we define $r_t(\ell)$ and $r_p(\ell)$ as the maximum ranking number of directed trees and that of orientations of undirected paths, respectively, under the condition that *no directed subpath has more than ℓ vertices*.

Our results will show that the above three parameters are very close to each other, in the entire range of ℓ.

Theorem 1 *For every $k \ge 1$ and ℓ such that $2^{k-2} + 1 \le \ell \le 2^{k-1}$,*

$$r_t(\ell) = k\,.$$

Proof. We first show that $\chi_r(T) \le k$ provided that every directed subpath of T has at most 2^{k-1} vertices. Consider an infinite directed path with vertices x_i and edges $x_i x_{i+1}$, $i \in Z$. Define a mapping $\phi : \{x_i : i \in Z\} \to \{1, 2, \ldots, k\}$ by

$$\phi(x_i) = \begin{cases} k & \text{if } i \equiv 0 \bmod 2^{k-1}\,, \\ \max\{j : i \equiv 0 \bmod 2^{j-1}\} & \text{if } i \not\equiv 0 \bmod 2^{k-1}\,. \end{cases}$$

Obviously, any segment of length at most 2^{k-1} is ranked feasibly by ϕ.

Now we consider a directed tree T containing no directed subpath with more than 2^{k-1} vertices. We view such a tree as a Hasse diagram of a partially ordered set, and as such, partition its vertices into levels: we choose an arbitrary vertex and call its level $L(0)$, and then recursively sort the other vertices — a vertex u is placed into level $L(i+1)$ ($L(i-1)$) if there is a vertex v already in level $L(i)$ such that $uv \in E(T)$ ($vu \in E(T)$). A mapping f defined by $f(u) = \phi(x_i)$ for $u \in L(i)$ is then a feasible k-ranking of T. (The above procedure partitions T into levels correctly, since T is a tree.)

We next turn to the lower bound for $r_t(\ell)$, namely $r_t(2^{k-1}+1) > k$. By induction on i we construct a series of trees $T_k(i)$, $i = 0, 1, 2, \ldots, 2^{k-1}$ but in decreasing order, with the following properties:

1. every directed subpath of $T_k(i)$ has at most $2^{k-1} + 1$ vertices,
2. $T_k(i)$ contains a nonextendable directed path P of length $2^{k-1} - 1$ with vertices $x_1, x_2, \ldots, x_{2^{k-1}}$ and arcs $x_h x_{h+1}$, $1 \le h < 2^{k-1}$,

3. for every $j \leq i$, every directed path of $T_k(i)$ passing through x_j has at most 2^{k-1} vertices, and

4. for every feasible k-ranking f of $T_k(i)$ and for every $j > i$, $f(x_j) \neq k$.

The first step of the construction is for $i = 2^{k-1}$, and for $T_k(2^{k-1})$ we simply take the path $P = x_1x_2 \ldots x_{2^{k-1}}$. In the recursive step, we take a copy T' of $T_k(i+1)$ with vertex set disjoint from the vertex set of $T_k(i+1)$ and add the arc $x'_{i+1}x_{i+1}$ to the disjoint union of T' and $T_k(i+1)$ (we assume that the copy of P is denoted by $P' = x'_1x'_2 \ldots x'_{2^{k-1}}$ in T'). This will be our $T_k(i)$, and $P = x_1x_2 \ldots x_{2^{k-1}}$ will keep playing the role of the path P for the property 2.

The properties 1–3 for $T_k(i)$ clearly follow by induction. To prove 4, we revoke the result known from undirected ranking — the longest $(k-1)$-rankable path has $2^{k-1} - 1$ vertices. Hence, in any feasible k-ranking f_{i+1} of $T_k(i+1)$, at least one of the vertices of P is ranked k. If f_i is a k-ranking of $T_k(i)$, by the induction hypothesis none of the vertices x'_j, $j > i+1$ is ranked k, and hence at least one of the vertices x'_j, $1 \leq j \leq i+1$ is ranked k. On the other hand, the directed path $x'_1 \ldots x'_{i+1}x_{i+1} \ldots x_{2^{k-1}}$ contains at most one vertex ranked k, and thus 4 follows for $T_k(i)$.

The tree $T = T_k(0)$ has no directed path with more than $2^{k-1}+1$ vertices and it is not k-rankable. Indeed, if f_0 were a feasible k-ranking, then the property 4 would imply that no vertex of P is ranked k, contradicting the fact that the path with 2^{k-1} vertices is not $(k-1)$-rankable. Thus $r_t(2^{k-1}+1) \geq k+1$. □

Next, we show that the ranking number of directed trees of maximum degree 2 (i.e., orientations of undirected paths) usually equals the ranking number of their longest paths.

Theorem 2 *For every $k \geq 3$ and every ℓ such that $2^{k-1} - 1 \leq \ell \leq 2^k - 2$,*

$$r_p(\ell) = k \,.$$

Proof. We first prove the upper bound, i.e., $r_p(2^k - 2) \leq k$. It is easy to see that every (directed or undirected) path with at most $2^k - 2$ vertices has a feasible k-ranking such that the first vertex is ranked 1 and the last vertex is ranked 2. Thus, if T is an orientation of a path consisting of several segments of length at most $2^k - 3$ (a segment is a maximal directed subpath), we can k-rank each segment separately so that the sources are ranked 1 and the sinks are ranked 2.

On the other hand, to show the lower bound, we take two vertex-disjoint paths of length $2^k - 2$ each, and orient an arc from the first vertex of one of them to the last vertex of the other one. The resulting graph has no feasible k-ranking, because in every k-ranking of a directed path of length $2^k - 2$, both endvertices are ranked 1, thus the added arc would connect two vertices ranked 1, a contradiction. Therefore $r_p(2^k - 1) \geq k + 1$. □

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6 Open problems

There are many interesting related problems arising in the above context in a natural way. Below we mention some of them.

1. Determine the complexity of Directed Ranking on digraphs whose underlying graphs have treewidth at most t for a fixed integer t. (The undirected version is polynomially solvable [1].)
2. Draw a sharper line between the polynomial instances of oriented trees and the NP-complete class of directed acyclic bipartite planar graphs, by describing large subclasses of the latter in which the ranking number still can be determined in polynomial time.
3. What is the complexity of Directed Edge Ranking for a fixed number of colors? (The undirected version is linear [1], but NP-complete if the number of colors is unrestricted [6].)
4. More generally, which classes of directed graphs admit polynomial-time decision algorithms for k-ranking and/or edge k-ranking, for every fixed k?

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Reformulating the results proven above, and relating the ranking number of a directed tree to the ranking number of its longest paths, we obtain:

Corollary 3 *The ranking number of a directed tree is always less than or equal to the ranking number of its longest directed paths plus 1. This bound is best possible, as*

$$r_t(\ell) = \begin{cases} p(\ell) & \text{if } \ell = 2^k\,, \\ p(\ell)+1 & \text{if } \ell \neq 2^k\,. \end{cases}$$

Similarly, for orientations of undirected paths, we have

$$r_p(\ell) = \begin{cases} p(\ell) & \text{if } \ell \neq 2^k-1\,, \\ p(\ell)+1 & \text{if } \ell = 2^k-1\,. \end{cases}$$

We illustrate the functions $p(\ell)$, $r_p(\ell)$, and $r_t(\ell)$ in the schematic figure 1.

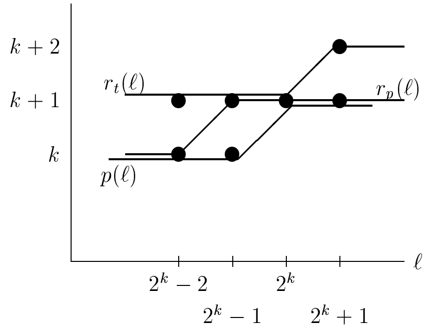

Figure 1: Trees and paths vs. the undirected path $p(\ell)$

3 Algorithm for trees

In this section we prove that the ranking number of a directed tree can be determined by a polynomial-time algorithm.

Assuming that a natural number k and a tree T with n vertices, rooted at a vertex r, are given, our next goal is to decide by an efficient algorithm if $\chi_r(T) \leq k$.

We shall use the following notation. For a vertex u of T, denote by T_u the subtree rooted at u and induced by those vertices from which the path (in the underlying undirected graph of T) to the root of T passes through u. If u is not the root, then u^+ denotes the first vertex on the path from u to the root r. The vertices adjacent to u other than u^+ are called the children of u.

The algorithm described below scans recursively the vertices of T from the leaves to the root and computes a set system $S(u)$ for every $u \in V(T)$. Each $S(u)$ is a family of subsets of $\{1, 2, \ldots, k\}$, storing essential information concerning the feasible rankings of the subtree rooted at u. Also, the values of auxiliary functions $Up(u)$, $Down(u)$, and $Compose(\mathcal{A}, \mathcal{B})$ are collections of subsets of $\{1, 2, \ldots, k\}$. In the subroutine *Compose*, we assume $\max \emptyset = 0$.

```
Algorithm TREE($k$)
Function $Up(u)$:
Let $u_1, u_2, \ldots, u_t$ be the children of $u$ such that $u_i u \in E(T)$.
$Up := \{\emptyset\}$;
for $j := 1$ to $t$ do $Up := \{A \cup B : A \in Up, B \in S(u_j)\}$.
Function $Down(u)$:
Let $u_1, u_2, \ldots, u_t$ be the children of $u$ such that $u u_i \in E(T)$.
$Down := \{\emptyset\}$;
for $j := 1$ to $t$ do $Down := \{A \cup B : A \in Down, B \in S(u_j)\}$.
Function $Compose(\mathcal{A}, \mathcal{B})$:
$Compose := \emptyset$;
for $A \in \mathcal{A}$ do
  for $B \in \mathcal{B}$ do
    for $i := \max(A \cap B) + 1$ to $k$ do
      if $i \notin A \cup B$ then $Compose := Compose \cup \{(A \cap \{i+1, i+2, \ldots, k\}) \cup \{i\}\}$.
Function $S(u)$:
if $u \neq r$ and $uu^+ \in E(T)$
  then $S := Compose(Up(u), Down(u))$
  else $S := Compose(Down(u), Up(u))$.
Program body:
if $S(r) = \emptyset$
  then $\chi^r(T) > k$
  else $\chi^r(T) \leq k$.
```

For a vertex u and a path $P = u_1 \ldots u_j$, $u_j = u$, we say that a color i is *visible on P from u* if some vertex u_h on this path receives color i and no vertex u_ℓ, $\ell = h+1, \ldots, j$ is colored with a color higher than i.

Proposition 4 *If u is not the root of T and $uu^+ \in E(T)$, then $S \in S(u)$ if and only if T_u admits a ranking such that S is the set of colors visible (from u) on directed paths from the inside of T_u to u. Otherwise (i.e., if u is the root or if $u^+u \in E(T)$), $S \in S(u)$ if and only if T_u admits a ranking such that S is the set of colors visible (from u) on directed paths leading from u into T_u.*

Next we show that all central vertices are located in the same bipartition class of G. If this is not the case, let x, y be central vertices belonging to distinct classes and being at minimum distance apart. (Recall that G is connected.) Now, any shortest x–y path has odd length and is alternating, for otherwise G would contain two central vertices in distinct classes closer to each other than x and y.

(4) ⇒ (1) Let $V(G) = A \cup B$ be a bipartition of G such that all central vertices belong to A. Then the mapping that assigns 1 to the vertices in B and 2 to the vertices in A is a 2-ranking of G. □

Remarks. 1. Algorithmically it is very easy to decide whether a digraph G is 2-rankable. Indeed, the answer is negative whenever G is not bipartite, and otherwise it suffices to test separately in each connected component if some of the two possible 2-colorings is a 2-ranking. Cf. also condition (4).

2. Similar types of problems have been studied in the framework of precoloring extension in several papers. Good characterizations are known for the existence of k-colorings of trees with any number of prescribed monochromatic independent sets [2, 3], and also for one prescribed monochromatic independent set in *perfect* graphs [5]. (As we have mentioned before, the problem for bipartite graphs with at least three precolored vertices of distinct colors is algorithmically hard [4], and so is for two monochromatic vertex pairs in distinct colors, too.) For an extensive survey on this subject, see [7].

3. Some small subgraphs excluded by the degenerate ‘alternating’ path of length 1 are:

- the cyclic triangle $y_1 \to y_2 \to y_3 \to y_1$, where any two of the y_i are adjacent central vertices and also each edge joins a starting vertex with an ending vertex,
- the transitive triangle $y_1 \to y_2 \to y_3 \leftarrow y_1$, where y_1y_3 is an edge from a starting vertex to an ending vertex (and $y_2y_3y_1y_2$ is an odd alternating walk from the central vertex y_2 to itself),
- the path $y_1 \to y_2 \to y_3 \to y_4$ of length 3, where the edge y_2y_3 joins a starting vertex with an ending vertex, both of which are central as well.

Moreover, chordless odd cycles of lengths ≥ 5 (with any orientation) are also excluded by the longer alternating paths or by the entire cycle as an alternating walk, according to the conditions (2) and (3) for longer paths/walks. Note that the characterization of 2-rankable digraphs in terms of forbidden subgraphs involves an *infinite* family of minimal configurations, which is not the case for undirected rankings.

Theorem 9 *For every digraph $G = (V, E)$, the following conditions are equivalent.*

1. *G is 2-rankable.*
2. *G contains no alternating path of odd length from a starting vertex to an ending vertex.*
3. *G contains no alternating walk of odd length with both endpoints being central vertices.*
4. *G admits a proper 2-coloring in which the set of central vertices is monochromatic.*

Proof.

(1) $\Rightarrow$ (2) Suppose that G is 2-rankable. Since P_3 has the unique 2-ranking 121, every starting and ending vertex must get the same color 1 in G. Consequently, *every* path P (not only the alternating ones) joining two such vertices must have even length, for otherwise the endpoints of P should get distinct colors in every proper 2-coloring (not only in the 2-rankings) of G.

(2) $\Rightarrow$ (3) Let G be a graph satisfying the condition (2), and suppose on the contrary that some $W = x_1x_2\dots x_{2t} \subset G$ is an alternating walk of odd length, $2t-1$, where both x_1 and x_{2t} are (possibly identical) central vertices. By definition, there exist directed paths of length 2, $P' = u'v'z'$ and $P'' = u''v''z''$, with $v' = x_1$ and $v'' = x_{2t}$. Denoting $x_0 := z'$ and $x_{2t+1} := u''$, observe that $W^* := u''W^{-1}z' = x_{2t+1}x_{2t}x_{2t-1}\dots x_1x_0$ is an alternating walk of odd length $2t+1$ from the starting vertex x_{2t+1} to the ending vertex x_0. Now (2) implies that W^* cannot be a path, i.e., $x_i = x_j$ holds for some $0 \le i < j \le 2t+1$. Assuming that $j-i$ is as small as possible, we find i and j so that $C := x_ix_{i+1}\dots x_j$ is a cycle.

We distinguish between two simple cases, depending on the parity of $i-j$. If $i-j$ is even, then C is an *odd* cycle in which x_i is the middle vertex of a directed P_3, namely either $x_{i+1}x_ix_{j-1}$ or $x_{j-1}x_ix_{i+1}$. Thus, $C - x_i$ is an alternating path of odd length from the starting vertex of this P_3 to its ending vertex, a contradiction to (2). On the other hand, if $i-j$ is odd, then removing the segment $x_{j-1}x_{j-2}\dots x_{i+2}x_{i+1}$ from W^* we obtain a shorter alternating walk of odd length from x_{2t+1} to x_0, and repeating the same argument we eventually get a final contradiction.

(3) $\Rightarrow$ (4) Let G be a connected graph satisfying condition (3). We first show that G is bipartite. Suppose on the contrary that $C = x_1x_2\dots x_{2k+1}$ is a cycle of odd length in G. By the assumption on parity, at least two consecutive edges are oriented in the same direction, and thus at least one vertex of C is central. It follows that, taking subscript addition modulo $2k+1$, there exist two subscripts i and j (possibly $j = i+2k+1$) such that $j-i$ is odd, both x_i and x_j are central vertices, and no vertex x_k, $i<k<j$, is central. Then the walk $x_ix_{i+1}\dots x_j$ (or its inverse, $x_jx_{j-1}\dots x_i$) is alternating.

Proof. We will prove the statement by induction. If u is a leaf, then any function $f_i : u \to i$ $(i = 1, 2, \dots, k)$ is a proper ranking of T_u, and $\{i\}$ is the set of visible colors in such a case. Indeed, $Up(u) = Down(u) = \{\emptyset\}$ and $Compose(\emptyset, \emptyset) = \{\{1\}, \{2\}, \dots, \{k\}\}$.

For the inductive step, suppose u is an inner vertex of T and $uu^+ \in E(T)$. Let $u_1, u_2, \dots, u_s$ be the children of u such that $u_ju \in E(T)$ $(j = 1, 2, \dots, s)$, and let $v_1, v_2, \dots, v_t$ be the children of u such that $uv_j \in E(T)$ $(j = 1, 2, \dots, t)$.

Suppose first that $f : V(T_u) \to \{1, 2, \dots, k\}$ is a ranking of T_u and $f(u) = i$. Let A_j be the set of colors visible from u_j on directed paths from within T_{u_j} to u_j $(j = 1, 2, \dots, s)$, and let B_l be the set of colors visible from v_l on directed paths from v_l into T_{v_l} $(l = 1, 2, \dots, t)$. By the induction hypothesis, $A_j \in S(u_j)$ and $B_l \in S(v_l)$. Then $A = \bigcup_{j=1}^{s} A_j \in Up(u)$ and this is exactly the set of colors visible from the children of u on directed paths from within T_u to u (not counting u itself). Similarly, $B = \bigcup_{l=1}^{t} B_l \in Down(u)$. Since f is a ranking of T_u, $i > \max(A \cap B)$, $i \notin A \cup B$, and $(A \cap \{i+1, \dots, k\}) \cup \{i\}$ is the set of colors visible from u on the directed paths from within T_u to u. And indeed, the definition of the function *Compose* gives $(A \cap \{i+1, \dots, k\}) \cup \{i\} \in S(u)$.

On the other hand, if $S \in S(u)$, then $S = (A \cap \{i+1, \dots, k\}) \cup \{i\}$ for some $A \in Up(u)$ and $B \in Down(u)$ such that $i > \max(A \cap B)$ and $i \notin A \cup B$. It follows from the definition of *Up* and *Down* that $A = \bigcup_{j=1}^{s} A_j$ and $B = \bigcup_{l=1}^{t} B_l$ for some $A_j \in S(u_j)$, $j = 1, 2, \dots, s$, and $B_l \in S(v_l)$, $l = 1, 2, \dots, t$. By the induction hypothesis each T_{u_j} has a ranking f^j such that A_j is the set of colors visible from u_j on directed paths from within T_{u_j} to u_j $(j = 1, 2, \dots, s)$. Similarly, each T_{v_l} has a ranking g^l such that B_l is the set of colors visible from v_l on directed paths from v_l into T_{v_l} $(l = 1, 2, \dots, t)$. Since $i > \max(A \cap B)$ and $i \notin A \cup B$, the function $f : V(T_u) \to \{1, 2, \dots, k\}$ defined by

$$f(x) = \begin{cases} f^j(x) & \text{if } x \in V(T_{u_j}) \\ g^l(x) & \text{if } x \in V(T_{v_l}) \\ i & \text{if } x = u \end{cases}$$

is a ranking of T_u, and S is the set of colors visible from u on directed paths from within T_u to u.

The proof for $u^+u \in E(T)$ or $u = r$ is analogous. $\square$

Corollary 5 *The algorithm* TREE(k) *gives the correct answer to the question whether $\chi_r(T) \le k$.*

Proposition 6 *The running time of the algorithm* TREE(k) *is at most $cnk^2\,2^{2k}$, for some absolute constant c independent of k.*

Proof. The function *Up* (which is a dynamic programming version for computing the set of all unions of type $\bigcup_{j=1}^{s} A_j$ for $A_j \in S(u_j)$) needs at most 2^{2k} set unions in each of the s steps. Hence, *Up* on a vertex with s ingoing children runs in $O(sk\, 2^{2k})$ time. The analogous property holds for *Down* as well. Throughout the entire tree T, there are as many children of processed vertices as the number of edges of T, and therefore *Up* and *Down* will consume in total at most $O(nk\, 2^{2k})$ steps.

The procedure *Compose* requires at most $O(k^2\, 2^{2k})$ steps, and being performed for every vertex, it requires running time at most $O(nk^2\, 2^{2k})$. $\Box$

In conclusion, we obtain

Theorem 7 *For any directed tree T on n vertices, the directed ranking number of T can be determined in time $O(n\,\ell^2 \log^3 \ell)$, where $\ell \geq 2$ is the length of a longest directed path in T.*

Proof. We know from Theorem 2 that $1 \leq \chi_r(T) - 1 \leq \log \ell$. Therefore, it suffices to run the algorithm TREE(k) for at most $\log \ell$ values of $k \leq \log \ell + 1$, and for each of them, TREE(k) takes at most $O(n\, \log^2 \ell\, 2^{2 \log \ell}) = O(n\, \ell^2\, \log^2 \ell)$ time. $\Box$

4 Ranking number of bipartite acyclic digraphs

Here we consider the algorithmic problem on DAGs (**d**irected **a**cyclic **g**raphs).

Theorem 8 *The problem* DIRECTED RANKING *is NP-complete on DAGs with planar bipartite underlying graphs, even for fixed ranking number $k = 3$.*

Proof. We show a reduction from the PRECOLORING EXTENSION problem of (undirected) bipartite graphs. It is known [4] that the following problem is NP-complete:

> *Given a planar bipartite graph with some of its vertices properly colored with three colors, does G admit a proper 3-coloring that extends the precoloring?*

One can observe that, without loss of generality, all the precolored vertices can be assumed to belong to the same vertex class of G. Indeed, for each precolored vertex v not in the proper vertex class, we create two new precolored vertices of degree 1, adjacent to v and assigned to the two colors different from the one prescribed for v ; then v can be made precolorless, as its precolored pendant neighbors force it to get the originally prescribed color.

Given such a bipartite graph $G = (A \cup B, E)$ with precolored vertex set $Z \subseteq A$ and precoloring $\phi : Z \to \{1, 2, 3\}$, we construct a directed graph D with vertex set

$$V(D) = A \cup B \cup \{z_i^j : z \in Z,\ 1 \leq i \leq 7,\ 1 \leq j \leq 2\}$$

and arc set

$$E(D) = \bigcup_{\substack{u \in A, v \in B \\ uv \in E}} \{uv\} \ \cup \bigcup_{\substack{z \in Z \\ 1 \leq i \leq 6 \\ 1 \leq j \leq 2}} \{z_i^j z_{i+1}^j\} \ \cup \bigcup_{z \in Z} \{z z^1_{i_1(z)}, z z^2_{i_2(z)}\}$$

where

$$i_1(z) = \begin{cases} 6 & \text{if } \phi(z) = 1 \\ 7 & \text{if } \phi(z) = 2 \vee 3 \end{cases} \qquad i_2(z) = \begin{cases} 4 & \text{if } \phi(z) = 1 \vee 2 \\ 6 & \text{if } \phi(z) = 3 \end{cases}$$

Obviously, D is acyclic, and it also remains planar and bipartite whenever so is G. We claim that D is 3-rankable if and only if G admits a precoloring extension with 3 colors.

Suppose first that D is 3-rankable, and let $f : V(D) \to \{1, 2, 3\}$ be a feasible ranking. Since the paths $P_{z,j} = z_1^j z_2^j \ldots z_7^j$ ($z \in Z$, $j = 1, 2$) are uniquely 3-rankable induced subgraphs of D, we must have $f(z_1^j) = f(z_3^j) = f(z_5^j) = f(z_7^j) = 1$, $f(z_2^j) = f(z_6^j) = 2$, and $f(z_4^j) = 3$. In this way, each $P_{z,j}$ excludes one well-defined color from its neighbor in A, and the total effect is that precisely the two colors distinct from $\phi(z)$ get excluded at each $z \in Z$. It follows that $f(z) = \phi(z)$ holds, and therefore f is a proper 3-coloring of G extending the precoloring ϕ.

On the other hand, any proper precoloring extension of ϕ together with the color sequence 1213121 on each $P_{z,j}$ gives a feasible 3-ranking. $\Box$

5 Directed 2-rankable graphs

Here we investigate directed rankings with $k = 2$ colors. For the structural characterization of 2-rankable digraphs the following concept will be convenient to introduce. By an *alternating walk of length ℓ* we mean a sequence $P = x_0 x_1 \ldots x_\ell$ of (not necessarily distinct) vertices such that its orientation is $x_0 \to x_1 \leftarrow x_2 \to x_3 \leftarrow \ldots$, i.e., $x_{2i} x_{2i+1} \in E$ for all $0 \leq i < \ell/2$ and $x_{2i} x_{2i-1} \in E$ for all $1 \leq i \leq \ell/2$. An alternating walk is an *alternating path* if its vertices are mutually distinct. Moreover, we say that a vertex v is *starting*, *central*, or *ending*, if there is a *directed path* $P_3 = x_1 x_2 x_3$ with $x_1 = v$, $x_2 = v$, or $x_3 = v$, respectively. In the present context, alternating paths and cycles of *odd* lengths will be crucial.