The Amalgamation of Diagrams and the Dimension

Jiří Otta
Dept. of Applied Mathematics,
MFF, Charles University,
Malostranské nám. 25, 118 00 Praha 1, Czech Republic

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Abstract

The importance of posets in theoretical computer science increases in last years. The useful data structure to store posets are Hasse diagrams. In this paper we show an operation on the diagrams: amalgamation. We investigate some types of amalgamation of the diagrams and show bounds on the dimension of them.

1 Introduction

A *partially ordered set* (*poset*) $\mathbf{P}$ consists of a pair (X,P), where X is a *ground set* (always finite in this paper) and P is a reflexive, antisymmetric and transitive relation on X. The relation P is called *partial order* on X. To emphasize the order concept we write $x \leq y$ in P instead of $(x,y) \in P$. We will also write $y \geq x$ in P and, when $x \neq y$, $x < y$ in P. When the poset remains fixed throughout a discussion, we will sometimes abbreviate $x \leq y$ in P by just writing $x \leq y$. Distinct points $x, y \in X$ are said to be *incomparable*, denoted $x \parallel y$, if neither $x \leq y$ nor $y \leq x$ is in P. A pair of points (x,y) for which $x \parallel y$ is called *incomparable pair*. When for P is no incomparable pair, then P is called *a linear order* on X and $\mathbf{P}$ is called a *linearly ordered set* or a *chain*. If $x > z \geq y$ in P implies $z = y$, then we say x *covers* y and write $y \prec x$ in P.

When $\mathbf{P} = (X, P)$ is a poset and Y is nonempty subset of X, the *restriction* of P to Y, denoted $P(Y)$, is a partial order on Y and we call $(Y, P(Y))$ a *subposet* of $\mathbf{P}$. A point $x \in X$ is called *maximal* point (*minimal* point respectively) if there is no point $y \in X$ with $x < y$ in P ($x > y$ in P respectively). If P has the only one maximal point, this point is called a *greatest* point. Similarly the only one minimal point is called a *least* point.

The two partial orders (X, P) and (Y, Q) are *isomorphic* if there exists a bijection $f : X \to Y$ such that for $x, y \in X$ $x \leq y$ in P if and only if $f(x) \leq f(y)$. The *dual* of a partial order P on a set X is defined by $P^d = \{(x, y) | (y, x) \in P\}$.

With a poset we associate *cover graph* $G = (X, E)$. The edges of the cover graph consist of those pairs xy for which $x \prec y$ or $y \prec x$ in P. The poset is completely determined by suitable diagram of cover graph in the Euclidean plane. We require that the y-coordinate of the point corresponding to y be larger than the y-coordinate of the point corresponding to x whenever $x \prec y$ in P and the edges are monotone in y-coordinate. Such diagrams are called *Hasse diagrams* (or just *diagrams*). We will often identify the points of the diagram with the points of the corresponding poset.

One can characterize diagrams as those oriented acyclic graphs which has no *guasicycle* i. e. a set of edges of the form

$$(x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n), (x_1, x_n)$$

for some $n > 2$. If such a quasicycle exists, we call the edge (x_1, x_n) a *transitive edge*. The overview about drawing the diagrams is [2].

If P and Q are partial orders on X and $P \subseteq Q$ then Q is called an *extension* of P. If Q is also a linear order, then Q is called a *linear extension* of P.

When $\mathbf{P} = (X, P)$ is a poset, the *dimension* of $\mathbf{P}$ denoted $\dim(\mathbf{P})$ is the least positive integer t for which there exists a family $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of P so that

$$P = \bigcap \mathcal{R} = \bigcap_{i=1}^{t} L_i$$

The concept of dimension was introduced by Dushnik and Miller [1] in a paper which continues to have significant impact on combinatorics and set theory. The overview about the dimension theory is the famous book of Trotter [3]. A family $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of linear orders on X is called a *realizer* of a partial order P on X if $P = \cap\mathcal{R}$. It is easy to see that if $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ is family of linear extensions of P then it is a realizer of P if and only if for every incomparable pair (x, y) there exists an extension L_i in which is $y < x$. We say that L_i *invert* (x, y).

When $\mathbf{P} = (X, P)$ is a poset, we will frequently want to specify a linear extension L of P satisfying certain properties. At times we will construct L explicitly. At times we will not be entirely precise in defining linear extension. For example suppose $X = A \cup B$ is a partition, L_1 is a linear extension of $P(A)$. Then we write

$$L = L_1(A), B$$

to denote any linear extension of P satisfying

1. $x_1 < x_2$ in L whenever $x_1 \in A$, $x_2 \in B$
2. $L(A) = L_1$

We also say that the set A is before the set B and B is after A in a linear order L.

Throughout this paper we define some types of amalgamations of the diagrams. Notice that the amalgamation of the diagrams is different from the amalgamation of the posets in general, although in special cases (and these are all cases which we investigate in this paper) it is the same. The difference is that in amalgamation of the posets the same structure is in the common parts. On the other hand in amalgamation of the diagrams we only identify the vertices of the diagrams thus it is possible that in first diagram the pair of the vertices is comparable, while its copy in the second poset is incomparable.

2 The amalgamation of two diagrams over a vertex

The simplest case of amalgamation of diagrams is the amalgamation over one vertex. We say that the poset $\mathbf{P} = (X, P)$ (its diagram $H = (X, E)$ respectively) was obtained by *amalgamation of the Hasse diagrams* $H_1 = (X_1, E_1)$ *and* $H_2 = (X_2, E_2)$ *over the vertex* if the Hasse diagram of $\mathbf{P}$ was obtained from the Hasse diagrams H_1 and H_2 by identifying the pair of vertices $x \in X_1$, $y \in X_2$ in graph theoretical sense.

It is easy to see that the preceding definition is correct, that means the resulting oriented graph H is acyclic and does not contain a quasicycle.

The following theorem gives the bound on the dimension of the poset which was obtained by the amalgamation of two diagrams over the vertex.

Theorem 1 *Let* $\mathbf{P_1} = (X_1, P_1)$ *be the poset with Hasse diagram* $H_1 = (X_1, E_1)$ *and led* $\mathbf{P_2} = (X_2, P_2)$ *be the poset with Hasse diagram* $H_2 = (X_2, E_2)$*. Let the poset* $\mathbf{P} = (X, P)$ *with the diagram* $H = (X, E)$ *was obtained by the amalgamation of the diagrams* H_1 *and* H_2 *over the vertex. Then the following inequality holds*

$$\dim(\mathbf{P}) \leq \max(\dim(\mathbf{P_1}), \dim(\mathbf{P_2})) + 1$$

Proof: Let $x_1 \in X_1$ and $x_2 \in X_2$ are the vertices which was identified in the amalgamation of the diagrams H_1 and H_2. The vertex which was created by their identification we call x and in the following we use the symbol x for both the vertices x_1 and x_2. We then define the following sets

$$A_1 = \{y \in X_1 | y < x\}$$

$$A_2 = \{y \in X_2 | y < x\}$$

$$B_1 = \{y \in X_1 | y > x\}$$

$$B_2 = \{y \in X_2 | y > x\}$$

$$C_1 = \{y \in X_1 | y \parallel x\}$$

References

[1] B. Dushnik, E. W. Miller: *Partially Ordered Sets*, Amer. J. Math. 63, 600–610, 1941

[2] Ivan Rival: *Reading, Drawing and Order*, Algebras and Orders, 359–404, 1993

[3] W. T. Trotter: *Combinatorics and Partially Ordered Sets: Dimension Theory*, The Johns Hopkins University Press, Baltimore, 1992

[4] W. T. Trotter, J. I. Moore: *The Dimension of Planar Posets*, Journal of Combinatorial Theory, Vol. 22, No. 1, 54–67, 1977

6 Open problems

Since the area of amalgamations of diagrams has not been much investigated so far, there are a lot of open problems. We present only some of them.

Problem 1 *Let* $\mathbf{P} = (X, P)$ *be a poset which was obtained by an amalgamation of diagrams* H_1 *of the poset* $\mathbf{P_1}$ *and* H_2 *of the poset* $\mathbf{P_2}$ *over one vertex. What is the characterization of the posets* $\mathbf{P_1}$ *and* $\mathbf{P_2}$ *for which* $\dim(\mathbf{P}) = \max(\dim(\mathbf{P_1}), \dim(\mathbf{P_2}))$ *holds?*

Problem 2 *Does there exist for any integer* k *and posets* $\mathbf{P_1}$ *and* $\mathbf{P_2}$ *such that* $\dim(\mathbf{P_1}) = k$, $\dim(\mathbf{P_2}) \leq k$ *and after the amalgamation of diagrams of* $\mathbf{P_1}$ *and* $\mathbf{P_2}$ *over one vertex a poset* $\mathbf{P}$ *is obtained such that* $\dim(\mathbf{P}) = k + 1$*?*

The second problem can be easier than the first one, but it also seems to be difficult. In our opinion if the answer is positive then we need a quite new class of posets with unbounded dimension to prove it. In this paper we only showed that the answer is positive for $k = 2$ and $k = 3$.

The whole group of problems arises from the amalgamation of diagrams over a set of general vertices. We present only the principal one.

Problem 3 *Let* $\mathbf{P} = (X, P)$ *be a poset which was created by the amalgamation of diagrams* H_1 *of the poset* $\mathbf{P_1}$ *and* H_2 *of the poset* $\mathbf{P_2}$ *over a general set of vertices of the size* k*. Does it exists a function* $f(\dim(\mathbf{P_1}), \dim(\mathbf{P_2}), k)$ *such that*

$$\dim(\mathbf{P}) \leq f(\dim(\mathbf{P_1}), \dim(\mathbf{P_2}), k)$$

for any $\mathbf{P_1}$ *and* $\mathbf{P_2}$*?*

Similar problems as for the amalgamation of the two diagrams over one vertex can be said for the amalgamation of more diagrams over one vertex.

$$C_2 = \{y \in X_2 | y \parallel x\}$$

Let $n = \max(\dim(\mathbf{P_1}), \dim(\mathbf{P_2}))$. We know that there exists a realizer

$$\mathcal{R}_1 = \{L_1^1, \ldots, L_n^1\}$$

of the poset $\mathbf{P_1}$ and a realizer

$$\mathcal{R}_2 = \{L_1^2, \ldots, L_n^2\}$$

of the poset $\mathbf{P_2}$. At the first of these realizers in each L_i^1 the set A_1 is before x and B_1 is after x. Similarly in the second one A_2 is before x and B_2 is after x. So we can for $i = 1, 2, \ldots, n$ correctly define linear orders

$$L_i' = L_i^2(A_2), L_i^1(A_1), x, L_i^2(B_2), L_i^1(B_1)$$

These linear orders we extend to the orders $L_1, \ldots, L_n$ in such a way that:

1. For every $c \in C_1, y \in X_1$ if y covers c in L_i^1 then y covers c in L_i, for $i = 1, \ldots, n$.
2. For every $c \in C_2, y \in X_2$ if c covers y in L_i^2 then c covers y in L_i for $i = 1, \ldots, n$.

$L_1, \ldots, L_n$ are then extensions of P. Let

$$L_{n+1} = A_1, C_1, A_2, x, B_1, C_2, B_2$$

For each $i = 1, \ldots, n$ is $L_i(X_1) = L_i^1$ and $L_i(X_2) = L_i^2$. Hence all the incomparable pairs of the type (y, z) where $y, z \in X_1$ or $y, z \in X_2$ are inverted. Furthermore all the incomparable pairs of the type (y, z) where $y \in A_1$ and $z \in A_2$ or $y \in B_1$ and $z \in B_2$ are inverted. Furthermore there are inverted all the following incomparable pairs:

1. $y \in X_1$ and $z \in C_2$, because there exist i for which is $z < x$ in L_i, thus less then any point of X_1.
2. $y \in C_1$ and $z \in X_2$, because there exist i for which is $y > x$ in L_i, thus greater then any point of X_2.

It remains to invert the following types of incomparable pairs (y, z):

1. $y \in A_2$ a $z \in A_1$
2. $y \in B_2$ a $z \in B_1$
3. $y \in X_2$ a $z \in C_1$
4. $y \in C_2$ a $z \in X_1$

These incomparable pairs are inverted in the linear extension L_{n+1}.

Extensions $L_1, \ldots, L_{n+1}$ are thus the realizer of the poset $\mathbf{P}$. ☐

The following example shows that the inequality is tight.

Example: The posets $\mathbf{P_1}$ and $\mathbf{P_2}$ on the figure 1 have dimensions 2. After the amalgamation which is shown on the figure, we obtain a poset $\mathbf{P}$,which has dimension 3. ☐

3 Amalgamation of more diagrams over common vertex

We say that the poset $\mathbf{P} = (X, P)$ (its diagram $H = (X, E)$, respectively) was obtained by *amalgamation of the Hasse diagrams* $H_1 = (X_1, E_1)$, $H_2 = (X_2, E_2)$,...,$H_k = (X_k, E_k)$ *over common vertex* if the Hasse diagram of $\mathbf{P}$ was obtained from the Hasse diagrams H_1, H_2 ,...,H_k by identifying the k-tuple of vertices $x_1 \in X_1$, $x_2 \in X_2$,...,$x_k \in X_k$ in graph theoretical sense.

Theorem 2 *For an integer $k \geq 2$ let $\mathbf{P_1} = (X_1, P_1)$ be a poset with a Hasse diagram $H_1 = (X_1, E_1)$, $\mathbf{P_2} = (X_2, P_2)$ be a poset with a Hasse diagram $H_2 = (X_2, E_2)$,..., $\mathbf{P_k} = (X_k, P_k)$ be a poset with a Hasse diagram $H_k = (X_k, E_k)$. Further let a poset $\mathbf{P} = (X, P)$ with a diagram $H = (X, E)$ was obtained by amalgamation of the diagrams H_1, H_2,...,H_k over common vertex. Then*

$$\dim(\mathbf{P}) \leq \max(\dim(\mathbf{P_1}), \dim(\mathbf{P_2}), \ldots, \dim(\mathbf{P_k})) + 2$$

rest of the diagrams everything holds, which holds above in the proof of the special case. Hence all the incomparable pairs are inverted.

This completes the proof. ☐

In the preceding we presented an example in which the dimension increase by two in the amalgamation of diagrams over a common vertex. We now show another example in which each point is a common vertex of at most two diagrams.

Example: A poset which diagram is on the figure 6 has dimension three. It can be built by an amalgamation of five diagrams over one vertex. ☐

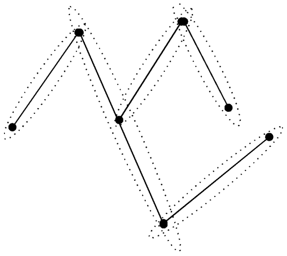

Figure 6: 3-dimensional poset obtained by amalgamation of chains

We showed that the inequality in the theorem 4 is tight and thus we solved the problem of dimension of the posets obtained by the amalgamation of diagrams over one vertex.

Suppose that the vertex H_i of the tree G has the immediate successors

$$H_{j_1}, \dots, H_{j_l}$$

such that

$$X_i \cap X_{j_1} \cap \dots \cap X_{j_l} = \{x_j\}$$

Then we work with the diagrams $H_{j_1}, \dots, H_{j_l}$ as with one diagram in the amalgamation, which has just one common vertex x_j with its immediate predecessor. This diagram (we call it $H_i = (X_i, E_i)$) has by the theorem 2 a realizer

$$L_i^1, L_i^2, \dots, L_j^n, L_j^{n+1}, L_j^{n+2}$$

which extensions are the same as in the proof of the theorem 2 (especially the last two ones).

In the construction of the linear extensions $L_1, \dots, L_n$ we use, instead of a realizer of H_i, the linear extensions $L_i^1, \dots, L_i^n$. In the construction of the linear extension L_{n+1} we use instead the order $L_i'(A_i), L_i'(C_i)$, the order

$$L_i'(A_{i_l}), L_i'(C_{i_l}), L_i'(A_{i_{l-1}}), L_i'(C_{i_{l-1}}), \dots, L_i'(A_{i_1}), L_i'(C_{i_1})$$

and, instead of $L_i'(B_i)$, we use the order

$$L_i'(B_{i_1}), L_i'(B_{i_2}), \dots, L_i'(B_{i_l})$$

Similarly in the construction of L_{n+2} we use, instead of $L_i'(A_i)$ the order

$$L_i'(A_{i_l}), L_i'(A_{i_{l-1}}), \dots, L_i'(A_{i_1})$$

and, instead of $L_i'(C_i), L_i'(B_i)$, we use the order

$$L_i'(C_{i_1}), L_i'(B_{i_1}), L_i'(C_{i_2}), L_i'(N_{i_2}), \dots, L_i'(C_{i_l}), L_i'(B_{i_l})$$

Thus for $1 \leq j \leq n+2$

$$L_j(\bigcup_{m=1}^{l} X_{i_m}) = L_i^j$$

holds and thus all the incomparable pairs of the type (y, z) where $y, z \in X_i$ are inverted. At the same time for the diagram H_i and the

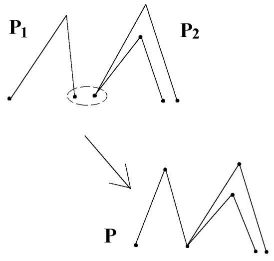

Figure 1: The dimension can increase after the amalgamation diagrams over one vertex

Proof: Let $x_1 \in X_1$, $x_2 \in X_2$,…,$x_k \in X_k$ be the vertices which was identified in the amalgamation of the diagrams H_1, H_2,…,H_k. The vertex which was created by the amalgamation we call x and in the following by x we also mean the vertices $x_1, x_2, \dots, x_k$. For $i = 1, 2, \dots, k$ we define the following sets

$$A_i = \{y \in X_i | y < x\}$$

$$B_i = \{y \in X_i | y > x\}$$

$$C_i = \{y \in X_i | y \parallel x\}$$

Let

$$n = \max(\dim(\mathbf{P_1}), \dim(\mathbf{P_2}), \dots, \dim(\mathbf{P_k}))$$

There exist a realizers $L_1^i, \dots, L_i^n$ of P_i for each $i = 1, 2, \dots, k$.

For $i = 1, \dots, k$ and $j = 1, \dots, n$ let $D_i^j \subset X_i$ be the set such that for $y \in D_i^j$ $y < x$ in L_i^j and $U_i^j \subset X$ such that $y \in U_i^j$ $y > x$ in L_i^j. We construct a realizer of the poset $\mathbf{P}$ of size $n+2$.

For $i = 1, 2, \dots, n$ let

$$L_i = L_1^i(D_1^i), L_2^i(D_2^i), \dots, L_k^i(D_k^i), x, L_k^i(U_k^i), L_{k-1}^i(U_{k-1}^i), \dots, L_1^i(U_1^i)$$

and

$$L_{n+1} = A_k, C_k, A_{k-1}, C_{k-1}, \dots, A_1, C_1, x, B_1, B_2, \dots, B_k$$

$$L_{n+2} = A_k, A_{k-1}, \dots, A_1, x, C_1, B_1, C_2, B_2, \dots, C_k, B_k$$

In $L_1, L_2, \dots, L_n$ are inverted all the incomparable pairs of the type (z, y) where z, y are both from the same X_j for some $1 \leq j \leq k$, because $L_i(X_j) = L_j^i$. It remains to show that $L_1, L_2, \dots, L_{n+2}$ invert all the incomparable pairs of the type (z, y) where $z \in X_l$, $y \in X_m$ for $1 \leq l, m \leq k$, $l \neq m$. We investigate all the possible cases:

1. $z \in A_l, y \in A_m$. If $l < m$ then $y < z$ in L_{n+1}. In the opposite case $y < z$ in L_1.

2. $z \in A_l, y \in B_m$. Then $z < y$ in P and (z, y) is not an incomparable pair.

construction of L_{n+2} (because $x_{i_1} < x_{j_1}$), that $B_{i_1} > B_{j_1}$ in L_{n+2}. Further $B_r > B_{i_1}$ and $C_r > B_{i_1}$ in L_{n+2}. Because in the construction of L_{n+2} we put the points of X_{i_2} after the points of X_{i_1} and X_{j_1} and in the construction of L_{n+2} we put the points of B_{j_2} just after X_{j_1}, $X_{j_2} < B_{i_1}$ in L_{n+2} holds. Similarly for $X_{j_3}, \dots, X_{j_q} = X_s$. We proved that $B_r > X_s$ a $C_r > X_s$ in L_{n+2}.

It remains to show that $A_s < X_r$ and $C_s < X_r$ in L_{n+1}. It can be done by the symmetrical way to the preceding steps. We can use the fact that $A_{i_1} < A_{j_1}$ in L_{n+1}.

We proved that if $x_r < x_s$ then all the incomparable pairs of the type (y, z), where $y \in X_r$ and $z \in X_s$ are inverted.

2. $x_r \parallel x_s$. In this case all the points of the set X_r are incomparable with all the points of X_s. By the same way as in the item 1 we can show that $B_r > X_s$ and $C_r > X_s$ in L_{n+2} and that $A_s < X_r$ and $C_s < X_r$ in L_{n+1}. It remains to show that $B_s < A_r$.

   Without lost of generality we assume that $x_s \not> x_{j_1}$. Let x_{i_l} be the first vertex from the sequence $x_{j_0}, \dots, x_q = x_s$, which is greater than its immediate predecessor. Then $x_{j_l} \in C_{j_{l-1}} \cup A_{j_{l-1}}$ and thus in L_{n+1} $x_{j_l} < X_{j_{l-2}}$ and thus $x_{j_l} < X_t$.

   Since $x_{i_1} < x_{j_1}$ in P, it is $A_{j_1} < A_{i_1}$ in L_{n+1}. Further in L_{n+1} is $A_{i_2} > A_{j_1}$ in L_{n+1} and similarly for $A_{i_3}, \dots, A_{i_p}$ we obtain $A_r > A_{j_1}$ in L_{n+1} and thus $x_{j_l} < A_r$ in L_{n+1}. For B_{j_l} it holds, that its points are in L_{n+1} just after x_{j_l} and thus $B_{j_l} < A_r$ in L_{n+1}. That means $X_{j_l} < A_r$. Similarly for $B_{j_{l+1}}, \dots, B_{j_q} = X_s$ we obtain $B_s < A_r$ in L_{n+1}.

   We proved that all the incomparable pairs of the type (y, z), where $y \in X_r$ and $z \in X_s$ are inverted.

Under the assumption that every vertex is a common vertex of at most two original diagrams we proved that the linear orders $L_1, L_2, \dots, L_{n+}$ invert all the incomparable pairs from P.

Now we describe the general case. The construction of linear orders is almost the same as in the preceding special case.

common vertex x_{i_j}. In the sequence

$$x^r = x_{i_2}, x_{i_3}, \ldots, x_{i_q} = x_s$$

let x_{i_t} is the first vertex, which is not less then its predecessor. Then $x_{i_t} \in B_{i_{t-1}} \cup C_{i_{t-1}}$ and thus in $L_{n+2} isx_{i_t}$ behind X_r. Then no vertex of

$$X_{i_t} \cup X_{i_{t+1}} \cup \cdots \cup X_s$$

is less than any vertex of X_r and in L_{n+2} are these vertices behind X_r. Thus $y > z$ in L_{n+2} for every $z \in X_r$ and thus all the incomparable pairs with the vertex y and any vertex of X_r are inverted.

3. For $y \in B_s$ is the proof symmetrical to the preceding case.

Now let H_r and H_s are not on the same oriented path in the tree G. Let H_t be the nearest common predecessor of H_r and H_s. Let

$$H_t = H_{i_0}, H_{i_1}, H_{i_2}, \ldots, H_{i_p} = H_r$$

be a path from the vertex H_t to the vertex H_r in G and

$$H_t = H_{j_0}, H_{j_1}, H_{j_2}, \ldots, H_{j_q} = H_s$$

a path from the vertex H_t to the vertex H_s in G. Recall that the diagrams H_{i_m} and $H_{i_{m-1}}$ have the common vertex x_{i_m} for $m \geq 1$.

Suppose that $x_{i_1} \parallel x_{j_1}$ in P_t. Then all the points of X_r are greater then the immediate predecessor of x_{i_1} in each linear order from $L_t^1, \ldots, L_t^n$ (the realizer of P_t) and less then the immediate successor. The same holds for the points of X_s and x_{j_1}. The order of x_{i_1} and x_{j_1} are different in at least two orders from $L_t^1, \ldots, L_t^n$ and thus also the order of the sets X_r and X_s are different.

Hence we can assume that $x_{i_1} < x_{j_1}$. Thus $X_r < X_s$ in all the orders $L_1, \ldots, L_n$. All the incomparable pairs of the type (y, z) where $y \in X_s$ and $z \in X_r$ are inverted. We distinguish two possibilities.

1. $x_r < x_s$ in P. Then any point of A_r is less then any point of B_s in P. We prove that for the sets B_r and C_r the facts $B_r > X_s$ and $C_r > X_s$ in L_{n+2} holds. It follows from the

3. $z \in A_l, y \in C_m$. If $l < m$ then $y < z$ in L_{n+1}. In the opposite case, because $y \parallel x$ in P_m, there exists j such that $y < x$ in L_m^j. Hence $y \in D_m^j$. At the same time $z \in D_l^j$. Thus $y < z$ in L_j.
4. $z \in B_l, y \in A_m$. Then $z > y$ in P and (z, y) is not an incomparable pair.
5. $z \in B_l, y \in B_m$. If $l > m$ then $y < z$ in L_{n+1}. In the opposite case $y < z$ in L_1.
6. $z \in B_l, y \in C_m$. Then $y < z$ in L_{n+1}.
7. $z \in C_l, y \in A_m$. Then $y < z$ in L_{n+2}.
8. $z \in C_l, y \in B_m$. If $l > m$ then $y < z$ in L_{n+2}. In the opposite case, because $z \parallel x$ v P_m, there exists j such that $y > x$ in L_m^j. Hence $y \in U_m^j$. At the same time $z \in U_l^j$. Thus $y < z$ in L_j.
9. $z \in C_l, y \in C_m$. If $l > m$ then $y < z$ in L_{n+2}. In the opposite case $y < z$ in L_{n+1}.

All the incomparable pairs are inverted and it completes the proof.

□

4 Square posets

To show that the inequality in the preceding theorem is tight, we now define a new class of posets. For an integer $n \geq 2$ we define the *square poset* $T_n = (X, P)$ such that

$$X = \{x, a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n\}$$

and for each $i = 1, 2, \ldots, n$:

1. $a_i < x$
2. $a_i < c_i$
3. $a_i < b_i$
4. $c_i < b_i$

5. $x < b_i$

in P. No other comparabilities are in P. The diagram of square poset T_4 is on the figure 2.

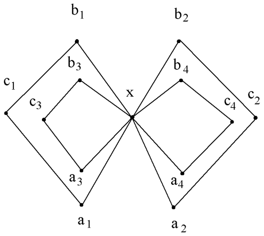

Figure 2: The square poset T_4

We now show what is the dimension of the square poset T_n.

Theorem 3 *Let $T_n = (X, P)$ be the square poset. Then*

1. *For $n = 1, 2$ it is $\dim(T_n) = 2$.*
2. *For $n = 3, 4, 5, 6$ it is $\dim(T_n) = 3$.*
3. *For $n \geq 7$ it is $\dim(T_n) = 4$.*

Proof: For each case we have to show that there exists a realizer of the needed size and does not exist a realizer of the less size.

1. For $n = 1, 2$ the poset T_n is not a chain thus it has the dimension at least 2. On the other hand when we add the greatest and the least point to T_n, it stay planar, thus it has the dimension two.

- In L_{n+1} the points of A_s and C_s are "before" the diagram H_r and each point of X_s is "as low as possible".
- In L_{n+2} the points of B_s and C_s are "after" the diagram H_r and each point of X_s is "as high as possible".

The extensions $L_1, L_2, \ldots, L_n$ invert all the incomparable pairs of the type (y, z) where $y, z \in X_j$ for $j = 1, \ldots, k$ because

$$L_i(X_j) = L_j^i$$

To see that

$$\mathcal{R} = \{L_1, L_2, \ldots, L_n, L_{n+1}, L_{n+2}\}$$

is a realizer of P, we have to check if all the incomparable pairs of the type (y, z) where $y \in X_r$, $z \in X_s$ for $1 \leq r, s \leq k$ and $r \neq s$ are inverted.

At first we investigate the case, when H_r is a predecessor of H_s in G.

We investigate three cases:

1. For $y \in C_s$ it is $y < z$ in L_{n+1} and $y > z$ in L_{n+2} for every $z \in X_r$ and so there are inverted all the incomparable pairs with y and any element of X_r.
2. For $y \in A_s$ it is $y < z$ in L_{n+1} for every $z \in X_r$. Let $x^r \in X_r$ is the common vertex of H_r and its common successor on the path from H_r to H_s in G. There are two cases:
   (a) $x_s \leq x^r$ in P. Then for every $z \in X_r$, which is incomparable with y in P it is either $z \parallel x^r$ or $z < x^r$ in P_r. For such z there exists $1 \leq i \leq n$ such that $x^r > z$ in L_r^i and so $y > z$ in L_i. Hence all the incomparable pairs of the type (y, z) and (z, y) where $y \in A_s$ and $z \in X_r$ are inverted.
   (b) $x_s \not\leq x^r$ in P. Then $y \parallel z$ in P for every $z \in X_r$. Let

$$H_r = H_{i_1}, H_{i_2}, \ldots, H_{i_q} = H_s$$

is the sequence of the vertices of G on the path from P_r to P_s. Let us recall that the diagrams P_{i_j} and $P_{i_{j-1}}$ have

step m Let q be such that in Y_{m-1} are points of the set X_q (vertices of H_q) but not vertices of diagrams which are immediate successors of H_q in the tree G. Let d be the outdegree of the vertex H_q in the tree G and

$$H_{i_1}, H_{i_2}, \dots, H_{i_d}$$

be the immediate successors of H_1 such that $x_{i_r} \not> x_{i_s}$ in P_1 for $r < s$. Then

$$Y_m = Y_{m-1} \cup X_{i_1} \cup X_{i_2} \cup \cdots \cup X_{i_d}$$

and let

$$Y^d_{m-1} = \{y \in Y_{m-1} | y < z \in M_{n+1,m-1} \text{ for every } z \in X_q\}$$

In $M_{n+1,m}$ is the order

$$M_{n+1,m-1}(Y^d_{m-1}), L'_{i_d}(A_{i_d}), L'_{i_d}(C_{i_d}), L'_{i_{d-1}}(A_{i_{d-1}}),$$
$$L'_{i_{d-1}}(C_{i_{d-1}}), \dots, L'_{i_1}(A_{i_1}), L'_{i_1}(C_{i_1}), M_{n+1,m-1}(Y_{m-1} \setminus Y^d_{m-1})$$

and for $1 \leq u \leq d$ the points of B_{i_u} we put (in order of L'_{i_u}) immediately after x_{i_u}.

Similarly let

$$Y^u_{m-1} = \{y \in Y_{m-1} | y > z \in M_{n+2,m-1} \text{ for every } z \in X_q\}$$

In $M_{n+2,m}$ is the order

$$M_{n+2,m-1}(Y_{m-1} \setminus Y^u_{m-1}), L'_{i_d}(C_{i_d}), L'_{i_d}(B_{i_d}), L'_{i_{d-1}}(C_{i_{d-1}}),$$
$$L'_{i_{d-1}}(B_{i_{d-1}}), \dots, L'_{i_1}(C_{i_1}), L'_{i_1}(B_{i_1}), M_{n+2,m-1}(Y_{m-1})$$

and for $1 \leq u \leq d$ the points of A_{i_u} we put (in order of L'_{i_u}) immediately before x_{i_u}.

Finally we set $L_{n+1} = M_{n+1,l}$ and $L_{n+2} = M_{n+2,l}$.

Observe the following: If H_r is a predecessor of H_s in the tree G then:

- In $L_1, \dots, L_n$ all the points of X_s are "inside" the diagram H_r.

2. For $n = 3, 4, 5, 6$ the poset T_n contains both the posets from the figure 3 which are both 3-dimensional (see [4]).

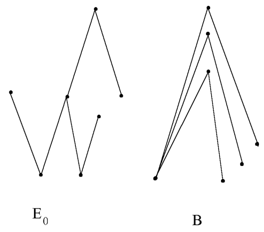

Figure 3: The 3-dimensional posets

On the other hand the realizer of T_6 is

$$L_1 = a_2, c_2, a_3, a_5, c_3, a_6, c_5, a_1, a_4, x, b_3, b_2, c_6, b_5, c_4, b_6, b_4, c_1, b_1$$

$$L_2 = a_4, c_4, a_5, a_1, c_5, a_2, c_1, a_3, a_6, x, b_5, b_4, c_2, b_1, c_6, b_2, b_6, c_3, b_3$$

$$L_3 = a_6, c_6, a_1, a_3, c_1, a_4, c_3, a_5, a_2, x, b_1, b_6, c_4, b_3, c_2, b_4, b_2, c_5, b_5$$

and for $n = 3, 4, 5$ the poset T_n is the subposet of T_6 and thus it has dimension 3, too.

3. For $i < j$ is T_i the subposet of T_j. To show that for $n \geq 7$ a realizer of the size 3 does not exist, it is enough to show this for T_7. Suppose for the contrary that $\mathcal{R} = \{L_1, L_2, L_3\}$ is a realizer of T_7.

   For $i = 1, 2, \dots, 7$ are c_i and x incomparable in P and one of the following possibilities holds

(a) c_i is in L_k and in L_l greater than x and in L_m less than x for $k, l, m \in \{1, 2, 3\}$, k, l, m are pairwise different.
(b) c_i is in L_k and in L_l less than x and in L_m greater than x for $k, l, m \in \{1, 2, 3\}$, k, l, m are pairwise different.

Suppose that the first case holds. The point c_i is incomparable with the point a_j for all $j \neq i$. Because each such a_j is less than x in T_7, it is less than c_i in L_k and in L_l. Because there are inverted all the incomparable pairs of the type (a_j, c_i) for $j \neq i$, it must hold $c_i < a_j$ in L_m. Thus a_i is the least point in L_m, because $a_i < c_i$ in T_n.

This situation can not hold for three different values of i. For at most three values of i the first situation holds. Similarly one can show that for at most three values of i the second situation holds. But we have seven values of i. The contradiction completes the proof.

On the other hand, for T_n there exists a realizer of the size 4:

$$\begin{aligned}
L_1 &= a_1, c_1, a_2, c_2, \ldots, a_n, c_n, x, b_1, b_2, \ldots, b_n \\
L_2 &= a_n, c_n, a_{n-1}, c_{n-1}, \ldots, a_1, c_1, x, b_1, b_2, \ldots, b_n \\
L_3 &= a_1, a_2, \ldots, a_n, x, c_1, b_1, c_2, b_2, \ldots, c_n, b_n \\
L_4 &= a_1, a_2, \ldots, a_n, x, c_n, b_n, c_{n-1}, b_{n-1}, \ldots, c_1, b_1
\end{aligned}$$

Thus $\dim(T_n) = 4$ for $n \geq 7$.

We investigated all the cases and the proof is complete. □

Notice that the diagram of T_n is planar for all n (see figure 4). For $n \geq 7$ T_n is 4-dimensional planar poset.

The following example shows that the inequality in the theorem 2 is tight.

Example: Let $T_2^1, T_2^2, T_2^3, T_2^4$ are isomorphic copies of T_2. By the amalgamation of these posets over the common vertex, in which we identify the copies of the vertex x, we obtain an isomorphic copy of the poset T_8 which has dimension 4. Thus the dimension increases by two.

Let us remark, that by the amalgamation of two copies of T_2 or T_4 over the copies of x, the dimension increase by one. □

for $j = 2, \ldots, k$

step j Let x_{j-1} be the common vertex of the sets X^{j-1} and X_j. Further let $D_j^i \subset X_j$ be the set of points which are in L_j^i less than x_{j-1} and similarly $U_j^i \subset X_j$ set of points which are greater, $\mathcal{D}_j^i \subset X^{j-1}$ set of points which are in $M_{i,j-1}$ less than x_{j-1} and $\mathcal{U}_j^i \subset X^{j-1}$ set of points which are greater. Then

$$M_{i,j} = M_{i,j-1}(\mathcal{D}_j^i), L_j^i(D_j^i), x_{j-1}, L_j^i(U_j^i), M_{i,j-1}(\mathcal{U}_j^i)$$

where $X^j = X^{j-1} \cup X_j$.

Finally we set $L_i = M_{i,k}$.

Before we define the rest two linear extensions L_{n+1} and L_{n+2} we define the following labeling. For each j such that $2 \leq j \leq k$ we mean by x_j the vertex for which is $x_j \in X^j \cap X_j$, that means the common vertex of H_j and its immediate predecessor in the tree G. Further

$$A_j = \{y \in X_j | y < x_j \mathrm{in} P_j\}$$

$$C_j = \{y \in X_j | y \parallel x_j \mathrm{in} P_j\}$$

$$B_j = \{y \in X_j | y > x_j \mathrm{in} P_j\}$$

Let $L_1, L_2, \ldots, L_k$ be any linear extensions of $P_1, P_2, \ldots, P_k$ respectively. Further let t is the number of leaves of G and $l = k - t + 1$. The extensions L_{n+1} and L_{n+2} we construct in such a way that we construct in l steps sequences

$$M_{n+1,1}, M_{n+1,2}, \ldots, M_{n+1,l}$$

and

$$M_{n+2,1}, M_{n+2,2}, \ldots, M_{n+2,l}$$

on the sets $Y_1, Y_2, \ldots, Y_l$. In the following we suppose $1 \leq m \leq l$

step 1 $M_{n+1,1}$ a $M_{n+2,1}$ are any linear extensions of P_1 on the set $Y_1 = X_1$

For $m = 2, \ldots, l$

in L_d.

If we identified in the amalgamation vertices $x \in X_i$ and $y \in X_j$ for $1 \le i, j \le k$, then we say that the diagrams $H_i = (X_i, E_i)$ and $H_j = (X_j, E_j)$ has common vertex.

Let
$$n = \max(\dim(\mathbf{P_1}), \dim(\mathbf{P_2}), \dots, \dim(\mathbf{P_k}))$$
We know that there exists a realizer $L_1^1, \dots, L_1^n$ of the poset $\mathbf{P_1}$, a realizer $L_2^1, \dots, L_2^n$ of the poset $\mathbf{P_2}$, etc., a realizer $L_k^1, \dots, L_k^n$ of the poset $\mathbf{P_k}$.

At first we prove the special case of the theorem in which for any i, j, l pairwise different, $1 \le i, j, l \le k$ holds: If x is a common vertex of the diagrams H_i and H_j then x is not a vertex of H_l. Hence at most two diagrams are amalgamate over one common vertex. Lately we show that the proof works with small adaptations also for the general case.

We show that there exists a realizer
$$\mathcal{R} = \{L_1, L_2, \dots, L_n, L_{n+1}, L_{n+2}\}$$
of $\mathbf{P}$.

For $i = 1, 2, \dots, n$ we construct an extension L_i in such a way that we construct linear orders
$$M_{i,1}, M_{i,2}, \dots, M_{i,k}$$
where for $1 \le j \le k$ is $M_{i,j}$ linear order on the set
$$X^j = X_1 \cup X_2 \cup \dots \cup X_j$$
which is an extension of $P(X^j)$ and $M_{i,k} = L_i$. Notice that the order L_d gives that the sets X^j and X_{j+1} has a common point for $1 \le j \le k-1$.

In the following is $1 \le i \le n$ an index for the extensions and $1 \le j \le k$ an index for the diagrams. We then construct linear orders
$$M_{i,1}, M_{i,2}, \dots, M_{i,k}$$
in the following way:

step 1 $M_{i,1} = L_1^i$, where $X^1 = X_1$

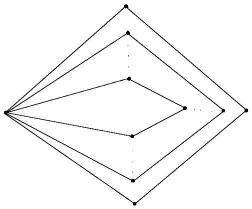

Figure 4: Planar drawing of T_n

5 The amalgamation over one vertex

We say that the poset $\mathbf{P} = (X, P)$ (its diagram $H = (X, E)$ respectively) was obtained by *amalgamation of the Hasse diagrams* $H_1 = (X_1, E_1)$, $H_2 = (X_2, E_2)$,…,$H_k = (X_k, E_k)$ *over a vertex* if there exists their ordering
$$H_{i_1}, H_{i_2}, \dots, H_{i_k}$$
such that the diagram H was obtained in $k-1$ steps by the following way:

At the first step a diagram H^1 was obtained by amalgamation of diagrams H_{i_1} and H_{i_2} over one vertex.

At the second step a diagram H^2 was obtained by amalgamation of diagrams H^1 and H_{i_3} over one vertex.

Etc.

At the $k-1$ step a diagram H was obtained by amalgamation of diagrams H^{k-2} and H_{i_k} over one vertex.

In the definition we repeatedly amalgamate two diagrams and thus the definition is correct, H is a Hasse diagram. W. T. Trotter in [3] defines an amalgamation of a family of rooted posets over a poset. In fact it is a special case of our amalgamation over a vertex. We now present a bound on the dimension of a poset obtained by amalgamation diagrams over a vertex, which is a generalization of Trotter's one.

Theorem 4 *For an integer $k \geq 2$ let $\mathbf{P_1} = (X_1, P_1)$ be a poset with a Hasse diagram $H_1 = (X_1, E_1)$, $\mathbf{P_2} = (X_2, P_2)$ be a poset with a Hasse diagram $H_2 = (X_2, E_2)$ etc., $\mathbf{P_k} = (X_k, P_k)$ be a poset with a Hasse diagram $H_k = (X_k, E_k)$. Further let a poset $\mathbf{P} = (X, P)$ with a diagram $H = (X, E)$ was obtained by amalgamation of diagrams H_1, H_2,…,H_k over a vertex. Then*

$$\dim(\mathbf{P}) \leq \max(\dim(\mathbf{P_1}), \dim(\mathbf{P_2}), \ldots, \dim(\mathbf{P_k})) + 2$$

Proof: The resulting diagram H has a tree structure in the sense: Let $G = (V_G, E_G)$ be a graph such that

$$\begin{aligned} V_G &= \{H_1, H_2, \ldots, H_k\} \\ E_G &= \{\{H_i, H_j\} | H_i, H_j \in V_G, \text{a vertex of the diagram } H_i \\ &\quad \text{was identified with a vertex of the diagram } H_j\} \end{aligned}$$

then G is a tree. We use this tree structure. We choose any diagram among the diagrams H_1, H_2,…,H_k, say H_1 to be a root of G and the edges we orient in direction from the root to leaves (see figure 5). From this moment G is oriented rooted tree.

The orientation of G gives a partial order on the set of the diagrams H_1, H_2,…,H_k. For vertices $H_i, H_j \in V_G$ we say that the vertex H_i is *successor* of the vertex H_j and H_j is *predecessor* of H_i if an oriented path from H_j to H_i exists in G. Further we say that H_i is *immediate successor* of H_j and H_j is *immediate predecessor* of H_i if $(H_j, H_i) \in E_G$ is an oriented edge in G. Let L_d be any linear extension of the partial order on V_G. Without lost of generality we suppose that

$$H_1 \leq H_2 \leq \cdots \leq H_k$$

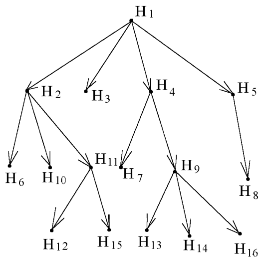

Figure 5: A tree G