The Dimension of Unicyclic Posets

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Abstract

A poset is called unicyclic, when the covering graph of that poset is a unicyclic graph. We show that the dimension of unicyclic poset is at most four. It is a partial answer to the Graham Brightwell's question. We also show two classes of unicyclic posets of dimension at most three. But we don't know any unicyclic poset of dimension four.

1 Introduction

A *partially ordered set* (*poset*) **P** consists of a pair (X, P), where X is a *ground set* (always finite in this paper) and P is a reflexive, antisymmetric and transitive relation on X. The relation P is called *partial order* on X. To emphasise the order concept we write $x \leq y$ in P instead of $(x, y) \in P$. We will also write $y \geq x$ in P and, when $x \neq y$, $x < y$ in P. When the poset remains fixed throughout a discussion, we will sometimes abbreviate $x \leq y$ in P by just writing $x \leq y$. Distinct points $x, y \in X$ are said to be *incomparable*, denoted $x \parallel y$, if neither $x \leq y$ nor $y \leq x$ is in P. A pair of points (x, y) for which $x \parallel y$ is called *incomparable pair*. When for P is no incomparable pair, then P is called *a linear order* on X and **P** is called a *linearly ordered set* or a *chain*. If $x > z \geq y$ in P implies $z = y$, then we say x *covers* y and write $y \prec x$ in P.

A point $x \in X$ is called *maximal* point (*minimal* point respectively) if there is no point $y \in X$ with $x < y$ in P ($x > y$ in P respectively). If P has the only one maximal point, this point is called a *greatest* point. Similarly the only one minimal point is called a *least* point.

The two partial orders (X, P) and (Y, Q) are *isomorphic* if there exists a bijection $f : X \to Y$ such that for $x, y \in X$ $x \leq y$ in P if and only if $f(x) \leq f(y)$. The *dual* of a partial order P on a set X is defined by $P^d = \{(x, y) | (y, x) \in P\}$. If (X, P) is a poset and $Y \subseteq X$ then $P(Y)$ is the restriction of P to Y.

With a poset we associate *cover graph* $G = (X, E)$. The edges of the cover graph consist of those pairs xy for which $x \prec y$ or $y \prec x$ in P.

The poset is completely determined by suitable diagram of cover graph in the Euclidean plane. We require that the y-coordinate of the point corresponding to y be larger than the y-coordinate of the point corresponding to x whenever $x \prec y$ in P and the edges are monotonic in the y-coordinate. Such diagrams are called *Hasse diagrams* (or just *diagrams*). We will often identify the points of the diagram with the points of the corresponding poset. The overview about drawing the diagrams is [3].

If P and Q are partial orders on X and $P \subseteq Q$ then Q is called an *extension* of P. If Q is also a linear order, then Q is called a *linear extension* of P.

When $\mathbf{P} = (X, P)$ is a poset, the *dimension* of $\mathbf{P}$ denoted $\dim(\mathbf{P})$ is the least positive integer t for which there exists a family $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of P so that

$$P = \bigcap \mathcal{R} = \bigcap_{i=1}^{t} L_i$$

The concept of dimension was introduced by Dushnik and Miller [2] in a paper which continues to have significant impact on combinatorics and set theory. The overview about the dimension theory is the famous book of Trotter [4]. A family $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of linear orders on X is called a *realizer* of a partial order P on X if $P = \cap \mathcal{R}$. It follows from the definition that if $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ is a realizer of P, then for every incomparable pair (x, y) there exists an extension

[2] B. Dushnik, E. W. Miller: *Partially Ordered Sets*, Amer. J. Math. 63, 600–610, 1941

[3] Ivan Rival: *Reading, Drawing and Order*, Algebras and Orders, 359–404, 1993

[4] W. T. Trotter: *Combinatorics and Partially Ordered Sets: Dimension Theory*, The Johns Hopkins University Press, Baltimore, 1992

[5] W. T. Trotter, J. I. Moore: *The Dimension of Planar Posets*, Journal of Combinatorial Theory, Vol. 22, No. 1, 54–67, 1977

L_i in which is $y < x$.

There are several theorems which shows how the dimension of the poset depends on Hasse diagram. We present the theorem of Trotter and Moore [5]:

Theorem 1 *Let* $\mathbf{P} = (X, P)$ *be a planar poset with the least point. Then* $\dim(\mathbf{P}) \leq 3$.

2 Tree Posets

We call a poset *tree poset* if its covering graph is a tree in graph–theoretic sense[1]. For example posets, whose Hasse diagrams are drawn in figure 1 are tree posets. They have dimension three (see [5]) and the notation is from [4].

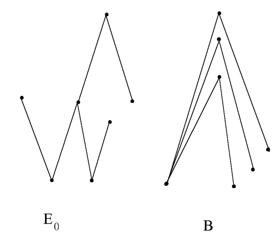

Figure 1: Tree posets

[1] In set theory, the word tree is used to describe a partially ordered set $\mathbf{P} = (X, P)$ for which $\{x \in X | x \leq y \text{ in } P\}$ is a well–ordered subposet of $\mathbf{P}$ for every $y \in X$. In another setting it is only required that $\{x \in X | x \leq y \text{ in } P\}$ is a chain.

Let S is a set in the Euclidean plane. Then $\pi_1(S)$ is the projection of the S to the x-coordinate and $\pi_2(S)$ is the projection to the y-coordinate.

For the tree posets we prove the following theorem.

Theorem 2 *Let* $\mathbf{P} = (X, P)$ *be a tree poset with diagram* $H = (X, E)$ *and* $x \in X$. *Let* $\mathbf{P}'$ *was obtained from* $\mathbf{P}$ *by adding the least point 0. Then there exists a planar drawing of diagram* $H' = (X \cup \{0\}, E')$ *of the poset* $\mathbf{P}'$ *such that*

$$\pi_1(x) < \pi_1(y)$$

for all $y \in X, y \neq x$.

Proof: The proof of this theorem without the condition on the vertex x is in [5]. But in the following we will need this stronger result. We proceed by induction on the number of vertices. If $\mathbf{P}$ is a poset on one point, such drawing trivially exists. We then assume the validity for all tree posets on the at most k points and $\mathbf{P}$ be a poset on $k+1$ vertices.

Let $y_1, \ldots, y_i$ be the vertices from which there is an edge to x in diagram H and $z_1, \ldots, z_j$ to which there is an edge from x in diagram H (we call them neighbours of x). When the vertex x is deleted from H, then H splits into $i+j$ connected components, each of them is tree poset and they corresponds to the neighbours of x (exactly one neighbour of x is in each of them). By induction hypothesis, there exists a planar drawing of their diagrams, such that the neighbour of x has the least x-coordinate in each of them and we can add the 0 vertex and the diagrams will stay planar. That diagrams we call

$$H_1, H_2, \ldots, H_i, H_{i+1}, H_{i+2}, \ldots, H_{i+j}$$

and assume that

$$y_1 \in H_1, y_2 \in H_2, \ldots, y_i \in H_i, z_1 \in H_{i+1}, \ldots, z_j \in H_{i+j}$$

We draw this diagrams in the plane in such a way that

$$\max(\pi_1(H_l)) < \min(\pi_1(H_{l+1}))$$

b_1^j and in L_3 than b_n^j and a_1. Hence all the incomparable pairs of the type (x, y), where $x = a_1^2$ are inverted. Also for $j = 0, 1, 2$ the point a_1^2 is in L_1 less than all the points of the set $X \setminus \{b_1^j, a_1^j, a_2^j\}$, in L_2 less than b_1^j and a_1, and in L_3 less than a_2^j. Hence all the incomparable pairs in which is the point a_1^2 are inverted.

5. Similarly we can do the proof for the point a_n^2.

6. The point b_1^2 is in L_1 the least point. Furthermore it is in L_2 greater than all the points of the set $X \setminus \{b_1, b_1^1\}$ and in L_3 it is $b_1^2 > b_1$. Hence all the incomparable pairs in which is the point b_1^2 are inverted.

7. Similarly we can do the proof for the point b_{n-1}^2

We investigated all the possibilities for the incomparable pairs of the poset $\mathbf{P} = (X, P)$ and every incomparable pair is inverted. Thus the linear orders L_1, L_2, L_3 are the realizer of the poset $\mathbf{P} = (X, P)$ and its dimension is at most three. □

5 Concluding Remarks

The question of the least upper bound of the dimension of unicyclic posets remains still open. The second class of 3-dimensional unicyclic posets gives in our opinion a chance, that our upper bound can be made better. Unfortunately it seems to be difficult to enlarge that class by the way, we prove its upper bound.

One possibility is to use induction on the number of vertices. That leads to the question, what happens to the dimension of poset, if we delete the vertex from its diagram. In general, the dimension can dramatically increase or decrease. But it does not say anything about the case of unicyclic posets.

References

[1] *Prague Midsummer Combinatorial Workshop*, KAM Series No. 93–254, page 7, Prague, 1994

orders L_1, L_2, L_3 the following holds:

For $i = 3, 4, \ldots, n-1$ point a_i^2 is in L_1 immediate successor of b_{i-1}^1 and in L_2 immediate successor of b_i^1. For $i = 2, 3, \ldots, n-2$ point b_{i-1}^2 is in L_1 immediate predecessor of a_i^1 and in L_2 immediate predecessor of a_{i-1}^1.

It can be easily seen that linear orders L_1, L_2, L_3 are extensions of P. It remains to show, that linear orders L_1, L_2, L_3 are realizer of P. To this, it remains to show that the incomparable pairs of the type (x, y), where at least one point is from the set $A^2 \cup B^2$ are inverted. We investigate all the cases for such incomparable pairs. We will sometimes write a_i^0 (b_i^0 respectively) instead of a_i (b_i respectively).

1. The point b_n^2 is either in L_1 or in L_2 greater then any point of the set $X \setminus \{b_n^j | j = 1, 2\}$. Hence all the incomparable pairs of the type (b_n^2, y) are inverted. The point b_n^2 is the least point in L_3. Hence all the incomparable pairs which contains b_n^2 are inverted.

2. For $i = 2, \ldots, n-1$ and $j = 0, 1, 2$ a point a_i^2 is in L_1 greater than a_k^j and b_k^j for $k < i$ and greater than a_i, and in L_2 greater than a_{k+1}^j and b_k^j for $k \geq i$. Hence all the incomparable pairs of the type (x, y), where $x \in A^2 \setminus \{a_1^2, a_n^2\}$ are inverted. Also the point a_i^2 is in L_1 less than a_{k+1}^j and b_k^j for $k \geq i$ and in L_2 less than a_k^j and b_k^j for $k < i$ and in L_3 less than a_i. Hence all the incomparable pairs in which is a point from the set $A^2 \setminus \{a_1^2, a_n^2\}$ are inverted.

3. Similarly for $i = 2, \ldots, n-2$ and $j = 0, 1, 2$ a point b_i^2 is in L_1 greater than a_k^j and b_{k-1}^j for $k \leq i$ and in L_2 greater than a_{k+1}^j and b_{k+1}^j for $k \geq i$ and in L_3 greater than b_i. Hence all the incomparable pairs of the type (x, y), where $x \in B^2 \setminus \{b_1^2, b_{n-1}^2, b_n^2\}$ are inverted. Also the point b_i^2 is in L_1 less than a_{k+1}^j and b_{k+1}^j for $k \geq i$ and less than b_i and in L_2 less than a_k^j and b_{k-1}^j for $k \leq i$. Hence all the incomparable pairs in which is a point from the set $B^2 \setminus \{b_1^2, b_{n-1}^2, b_n^2\}$ are inverted.

4. The point a_1^2 is in L_2 greater than all the points from the set X instead of a_1, b_n^j, b_1^j for $j = 0, 1, 2$ and in L_1 is greater than

and

$$\max(\pi_2(H_l)) < \min(\pi_2(H_{l+1}))$$

for $l = 1, \ldots, i+j$. In such a drawing we can add the common least vertex and the diagrams will stay planar. Then we draw the vertex x in such a way that

$$\max(\pi_2(H_i)) < \pi_2(x) < \min(\pi_2(H_{i+1}))$$

and

$$\pi_1(x) < \min(\pi_1(H_1))$$

Then we draw edges connecting vertex x and its neighbours as straight line segments and the diagram will stay planar. □

The following theorem is due to Trotter and Moore (see [5]).

Theorem 3 *Let* $\mathbf{P} = (X, P)$ *be a tree poset. Then* $\dim(\mathbf{P}) \leq 3$.

Proof: Proof of this theorem is an easy consequence of the preceding theorem and the theorem 1. □

Let us remark, that the two tree posets on the figure 1 characterise three dimensional tree posets in the sense, that every three dimensional tree poset contains a division of E_0 or B or their dual. (See [5]).

3 Upper Bound on the Dimension of Unicyclic Posets

In 1992 during the first Prague Midsummer Workshop [1] Graham Brightwell asked, if the dimension of unicyclic poset is bounded. He conjectured, that it is bounded by three. W. T. Trotter proved that dimension of unicyclic posets is bounded by five. The following theorem improve his result.

Theorem 4 *Let* $\mathbf{P} = (X, P)$ *be an unicyclic poset. Then* $\dim(\mathbf{P}) \leq 4$.

Proof: We show four linear extension of the poset **P**, which forms its realizer. Let $G = (X, E)$ be the covering graph of the poset **P**. We will assume that G contains exactly one cycle. In the case that G does not contain a cycle, **P** is a tree poset and its dimension is at most tree. We will also assume that G is connected.

Let C be the cycle in G. Pick any vertex $x \in \mathrm{V}(C)$ which is maximum in the cycle C, i. e. for every vertex $y \in \mathrm{V}(C)$ holds either $y < x$ or $y \parallel x$. Let e_1, e_2 be the two edges from $\mathrm{E}(C)$, for which vertex x is their endpoint. See figure 2.

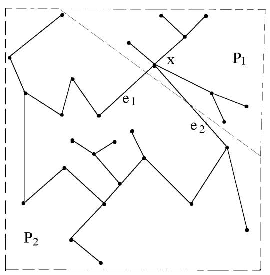

Figure 2: Division of unicyclic graph to two posets

When we delete the edges e_1, e_2 from the covering graph, we obtain a covering graph of another poset, which consists of two connected graphs of posets $\mathbf{P_1} = (X_1, P_1)$ and $\mathbf{P_2} = (X_2, P_2)$. Formally: For $a \in X$ let $T(a)$ is a shortest path in G with endpoints a and $t(a) \in V(C)$.

$$X_1 = \{a \in X | t(a) = x\}$$

$$L_1^1 = a_1, a_2, b_1, a_3, b_2, a_4, b_3, \ldots, b_{n-2}, a_n, b_n, b_{n-1}$$

$$L_2^1 = a_n, a_{n-1}, b_{n-1}, a_{n-2}, b_{n-2}, a_{n-3}, b_{n-3}, \ldots, b_2, a_1, b_n, b_1$$

$$L_3^1 = a_1, a_n, b_n, a_2, a_3, \ldots, a_{n-1}, b_1, b_2, \ldots, b_{n-1}$$

of L_1, L_2, L_3 are a realizer of the suborder C, which form the cycle in the covering graph of **P**. Let A be the set of minimal points in the cycle C and B be the maximal points in the cycle C.

The points of A are in L_1^1 and in L_2^1 in the reverse order, so they are incomparable. The same holds for the points of B except b_n. But b_n is in L_1^1 or in L_2^1 greater then any other point of B and in L_3^1 less then any other point of B. For $i = 1, 2, \ldots, n-1$ the point b_i is greater than $a_1, \ldots, a_{i+1}$ and less than $a_{i+2}, \ldots, a_n$ in L_1^1, greater than $a_n, \ldots, a_i$ and less than $a_{i-1}, \ldots, a_1$ in L_2^1. Point b_n is greater than $a_1, \ldots, a_n$ in L_1^1 and less than $a_2, \ldots, a_{n-1}$. Thus all the incomparable pair of the type (x, y) where $x \in A$ and $y \in B$ or $x \in B$ and $y \in A$ are inverted.

Linear orders L_1^2, L_2^2, L_3^2 are also suborders of L_1, L_2, L_3. They are build from L_1^1, L_2^1, L_3^1 in following way. We add points of the sets $A^1 = \{a_1^1, \ldots, a_n^1\}$ and $B^1 = \{b_1^1, \ldots, b_n^1\}$ in such a way that for every $i = 1, 2, \ldots, n$ the point a_i^1 is immediate predecessor of the point a_i and the point b_i^1 is immediate successor of the point b_i. Because every point a_i^1 (b_i^1 respectively) is incomparable just with these points from $A \cup B$, with which the point a_i (b_i respectively) is incomparable, there are inverted all the incomparable pairs, in which are one point from $A \cup B$ and one from $A^1 \cup B^1$. The poset $(A^1 \cup B^1, P(A^1 \cup B^1))$ is isomorphic with $(A \cup B, P(A \cup B))$ and for $j = 1, 2, 3$ is $(A^1 \cup B^1, L_j^2(A^1 \cup B^1))$ isomorphic with $(A \cup B, L_j^1)$. Thus all the incomparable pair from $A^1 \cup B^1$ are inverted. Linear orders L_1^2, L_2^2, L_3^2 are thus realizer of the poset

$$(A \cup B \cup A^1 \cup B^1, P(A \cup B \cup A^1 \cup B^1))$$

Linear orders L_1, L_2, L_3 are built from L_1^2, L_2^2, L_3^2. We add points of the sets $A^2 = \{a_1^2, \ldots, a_n^2\}$ and $B^2 = \{b_1^2, \ldots, b_n^2\}$. For the linear

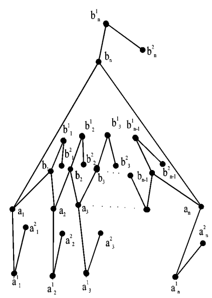

Figure 6: Class of 3-dimensional unicyclic posets

$$X_2 = X \setminus X_1$$

$$P_1 = P(X_1)$$

$$P_2 = P(X_2)$$

Both posets $\mathbf{P_1}$ and $\mathbf{P_2}$ are tree posets, so they have dimension at most three. Let L_1^1, L_2^1, L_3^1 are linear orders, which are realizer of poset $\mathbf{P_1}$. Similarly let L_1^2, L_2^2, L_3^2 are linear orders, which are realizer of poset $\mathbf{P_2}$.

The realizer of $\mathbf{P}$ is L_1, L_2, L_3, L_4, where $L_1 = L_1^2, L_1^1$, that means: points from X_2 ordered by L_1^2, followed by points from X_1 ordered by L_1^1. Similarly $L_2 = L_2^2, L_2^1$ and $L_3 = L_3^2, L_3^1$.

These three linear extensions contains a realizer of the poset $\mathbf{P_1}$ and thus they invert all the incomparable pairs of the type (a, b) where $a, b \in X_1$. Similarly they invert all the incomparable pairs of the type (a, b) where $a, b \in X_2$. Furthermore they invert all the incomparable pairs of the type (a, b) where $a \in X_1$ and $b \in X_2$. It remains to invert incomparable pairs of the type (a, b) where $a \in X_2$ and $b \in X_1$. These incomparable pairs inverts the extension L_4.

Before we construct this extension L_4, we define the following sets:

$$A_1 = \{a \in X_1 | a > x \text{ in } P\}$$

$$B_1 = \{a \in X_1 | a \not> x \text{ in } P\}$$

$$A_2 = \{a \in X_2 | a < x \text{ in } P\}$$

$$B_2 = \{a \in X_2 | a \not< x \text{ in } P\}$$

Let L_4^1 be any linear order on the set X_1, which is an extension of P_1 and in which every point of the set B_1 is less then x and every point of the set A_1 is greater then x. Further let L_4^2 is any liner order on the set X_2 which is extension of P_2 and in which every point of the set A_2 is less then x and every point of the set B_2 is greater then x.

Now we can construct the linear extension L_4:

$$L_4 = L_4^1(B_1), L_4^2(A_2), x, L_4^1(A_1), L_4^2(B_2)$$

That means: points of B_1 ordered by L_4^1 followed by points of A_2 ordered by L_4^2 followed by points of A_1 ordered by L_4^1 followed by points of B_2 ordered by L_4^2. It is easy to see that it is really an extension of P.

Now we prove that the linear extension L_4 inverts all the incomparable pairs, which are not inverted by extensions L_1, L_2, L_3. Let (a, b) be any incomparable pair where $a \in X_2$ and $b \in X_1$. The case $a \in A_2$ and $b \in A_1$ is impossible because then $a < b$ and it is not incomparable pair. It remains to investigate the following three cases:

1. $b \in B_1$, then b is less than any point of the set X_2 in L_4.
2. $a \in B_2$, then a is greater than any point of the set X_1 in L_4.
3. $b = x$, then x is less then any point of the set A_2 in L_4, so less than any point of the set X_2, which is incomparable with x

These three cases are all the possibilities for the incomparable pair of the type (a, b) where $a \in X_2$ and $b \in X_1$. In all these cases such incomparable pair is inverted.

Thus linear orders L_1, L_2, L_3, L_4 invert all the incomparable pairs from the poset $\mathbf{P} = (X, P)$ and hence they are its realizer. The proof is now complete. $\square$

Unfortunately, the preceding theorem does not give the full answer to the Brightwell's question. We don't know any unicyclic poset of dimension four, so the question if the least upper bound is three or four is still open.

4 Classes of 3-dimensional Unicyclic Posets

We now show two classes of unicyclic posets which has dimension at most three.

Let $\mathbf{P} = (X, P)$ be an unicyclic poset. Let C be a cycle in the covering graph $H = (X, E)$ of the poset $\mathbf{P}$. We say that poset $\mathbf{P}$ contains *maxhook* if there exist three points $a_1, a_2, a_3 \in X$ such that: a_1 is maximal point in the suborder $(V(C), P(V(C)))$, $a_2 > a_1$, $a_3 < a_2$ and $a_3 \parallel a_1$. Similarly poset $\mathbf{P}$ contains *minhook* if there exist three points $b_1, b_2, b_3 \in X$ such that: b_1 is minimal point in subposet $(X_C, P(X_C))$, $b_2 < b_1$, $b_3 > b_2$ and $b_3 \parallel b_1$. See figure 3.

4. For $i = 1, 2, \ldots, n-1$, c is a vertex of the path

$$a_{i+1} = d_1, d_2, \ldots, d_{k_i} = b_i$$

and $c \neq b_i$ or c is a vertex of the path

$$a_1 = d_1, d_2, \ldots, d_k = s$$

To this drawing of the diagram H we can add the least vertex and the diagram will stay planar. So $\dim(\mathbf{P}) \leq 3$.

In similar way we can show that if at most one of the points $a_1, a_2, \ldots, a_n$ is in minhook, than we can add the greatest point to $\mathbf{P}$ and its diagram will stay planar. Again $\dim(\mathbf{P}) \leq 3$ $\square$

The condition that we can add the greatest or the least point to unicyclic poset $\mathbf{P}$ and its diagram will stay planar is sufficient but not necessary for the dimension to be at most three. We now show one class of 3-dimensional unicyclic posets to which neither the least neither the greatest point can be added.

Example: For $n \geq 2$ let $\mathbf{P} = (X, P)$ be a poset whose diagram is on the figure 6.

We show that this poset has dimension at most three. We will prove that following three posets form the realizer of the poset $\mathbf{P}$.

$$\begin{aligned} L_1 &= b_1^2, a_1^1, a_1, a_2^1, a_2, b_1, b_1^1, a_1^2, a_2^2, \ldots, b_{i-1}^2, \\ &\quad a_i^1, a_i, b_{i-1}, b_{i-1}^1, a_i^2, b_i^2, a_{i+1}^1, a_{i+1}, \ldots, a_n^1, \\ &\quad a_n^2, a_n, b_n, b_n^2, b_n^1, b_{n-1}^2, b_{n-1}, b_{n-1}^1 \end{aligned}$$

$$\begin{aligned} L_2 &= b_{n-1}^2, a_n^1, a_n, a_{n-1}^1, a_{n-1}, b_{n-1}, b_{n-1}^1, a_n^2, a_2^{n-1}, \ldots, b_{i-1}^2, \\ &\quad a_{i-1}^1, a_{i-1}, b_{i-1}, b_{i-1}^1, a_{i-1}^2, b_{i-2}^2, a_{i-2}^1, a_{i-2}, \ldots, a_1^1, \\ &\quad a_1^2, a_1, b_n, b_n^2, b_n^1, b_1^2, b_1, b_1^1 \end{aligned}$$

$$\begin{aligned} L_3 &= b_n^2, a_1^1, a_1, a_n^1, a_n, b_n, b_n,^1, a_1^2, a_n^2, a_2^1, a_2^2, a_2, a_3^1, a_3^2, a_3, \ldots, a_{n-1}^1, \\ &\quad a_{n-1}^2, a_{n-1}, b_1, b_1^2, b_1^1, b_2, b_2^2, b_2^1, \ldots, b_{n-1}, b_{n-1}^2, b_{n-1}^1 \end{aligned}$$

At first we show that suborders

$$\max(\pi_2(H'_c)) - \min(\pi_2(H'_c)) \leq \frac{1}{2}$$

and furthermore

$$\min(\pi_1(H'_c)) > \pi_1(c) + \frac{1}{2}$$

$$\max(\pi_1(H'_c)) < \pi_1(c) + 1$$

$$\min(\pi_2(H'_c)) > \pi_2(c) - \frac{1}{4}$$

$$\max(\pi_2(H'_c)) < \pi_2(c) + \frac{1}{4}$$

see figure 5.

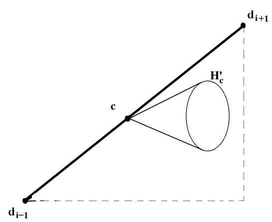

Figure 5: Drawing of the diagram H_c

In such drawing, H'_c is in outerface of C and

$$\max(\pi_1(H'_c)) < \pi_1(d_{i+1})$$

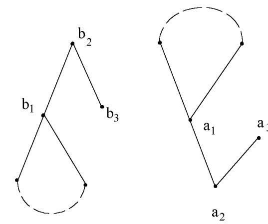

Figure 3: Maxhook and minhook

The following theorem shows one of the classes of unicyclic 3-dimensional posets.

Theorem 5 *Let* $\mathbf{P} = (X, P)$ *be an unicyclic poset and* C *a cycle in covering graph* $G = (X, E)$ *of* $\mathbf{P}$. *Let* $a_1, a_2, \ldots, a_n$ *be minimal and* $b_1, b_2, \ldots, b_n$ *be maximal points of subposet* $(V(C), P(V(C)))$. *If at most one point of* $b_1, b_2, \ldots, b_n$ *is in maxhook (one point from* $a_1, a_2, \ldots, a_n$ *is in minhook, respectively) then* $\dim(\mathbf{P}) \leq 3$.

Proof: Suppose that at most one point from $b_1, b_2, \ldots, b_n$ is in maxhook (for example b_n) then we show that if we add the least point to the poset $\mathbf{P}$, the poset will stay planar. Let $s, t \in \mathrm{V}(C)$ be vertices for whichá $e_1 = \{s, b_n\} \in \mathrm{E}(C)$ and $e_2 = \{t, b_n\} \in \mathrm{V}(C)$, $s > a_1$ and $t > a_n$ in P. We can draw the cycle C as on the figure 4. Edges are strigh line segments with the angle $\pi/4$ or $3\pi/4$ with x-axis. The length of edges with vertex in the set $\{b_1, \ldots, b_n\}$ is at least $2\sqrt{2}$ and the length of the rest of the edges is at least $\sqrt{2}$. Furthermore

$$\pi_2(s) > \max\{\pi_2(b_i) | i = 1, 2, \ldots, n-1\}$$

$$\pi_2(t) > \max\{\pi_2(b_i) | i = 1, 2, \ldots, n-1\}$$

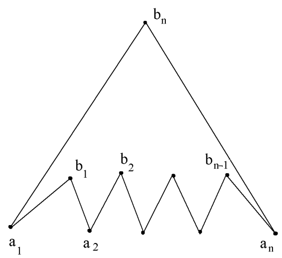

Figure 4: Drawing of the cycle C

For $c \in V(C)$ let

$$\begin{aligned} X_c &= \{c\} \cup \{v \in X | \text{there exists a path } T \text{ in } G\ , \\ &\quad v \in \mathrm{V}(T), \mathrm{V}(T) \cap \mathrm{V}(T) = \{c\}\} \end{aligned}$$

and H_c is a Hasse diagram of the suborder of $\mathbf{P}$ on the set X_c. Furthermore let H_c' is a subdiagram of H on the vertex set $X_c \setminus \{c\}$. Theorem 2 prooves that there exist a drawing of H_c such that c has the least (greatest) x-coordinate and we can add the least point and the diagram will stay planar. We discuse the following cases:

1. $c = b_n$. Then we draw H' in such a way that the vertex c has the least x-coordinate. The vertex c we identify with its copy in the cycle and the rest of the vertices of H' we draw in such a way that

$$\min(\pi_1(H_c')) > \pi_1(t) + 1$$

$$\min(\pi_2(H_c')) > \pi_1(t) + \frac{1}{2}$$

2. $c = b_i$, $i = 1, 2, \ldots, n-1$, then for $x \in X'$, $x > c$ in P' does not exist $y \in X'$ such that $y < x$ and $y \parallel c$ in P', because c is not in maxhook. Let H_c^1 is induced subdiagram of H_c on the vertex set

$$\{u \in X_c | u \geq c \text{ v } P\}$$

The diagram H_c^1 we draw in such a way that

$$\min(\pi_1(H_c^1)) > \pi_1(c) - \frac{1}{2}$$

$$\max(\pi_1(H_c^1)) > \pi_1(c) + \frac{1}{2}$$

$$\max(\pi_2(H_c^1)) > \pi_2(c) + 1$$

Let H_c^2 is an induced subdiagram of H_c on vertex set

$$\{u \in X_c | u \geq c \text{ v } P\}$$

We choose a drawing of H_c^2 in which c has the least x-coordinate and

$$\min(\pi_1(H_c^1)) > \pi_1(c) - \frac{1}{2}$$

$$\max(\pi_1(H_c^1)) > \pi_1(c) + \frac{1}{2}$$

$$\max(\pi_2(H_c^1)) > \pi_2(c) + 1$$

3. For $i = 1, 2, \ldots, n-1$, c is a vertex of the path

$$a_i = d_1, d_2, \ldots, d_{k_i} = b_i$$

and $c \neq b_i$ or c is a vertex of the path

$$a_n = d_1, d_2, \ldots, d_k = t$$

Let's say $c = d_i$. We draw H_c in such a way that c has the least x-coordinate and it holds

$$\max(\pi_1(H_c')) - \min(\pi_1(H_c')) \leq \frac{1}{2}$$