Structure of the graph homomorphisms I.

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Abstract

We consider three aspects of homomorphisms of graphs and hypergraphs which are related to the structure of color classes: 1. density, 2. fractal property and 3. generating color classes. In particular we prove the density theorem for hypergraphs and we show that for connected oriented graphs all jumps are balanced (and give an example that the connectivity is needed here. We also show that a Hajos-type theorem holds for any color class of undirected graphs thus further contributing to the well known "non-effective" character of Hajos theorem.

1 Introduction and statement of results

Graph theory receives its mathematical motivation connection from the two main areas of mathematics: algebra and geometry (topology -and the graph notion stood at the birth of algebraic topology). Conse quently various operations and comparisions for graphs stress either its algebraical part (e.g. various products) or geometrical part (e.g. contraction, subdivision). It is only natural that the key place in the modern graph theory is played by (fortunate) mixtures of both approaches as exhibited best by the various modifications of the notion of graph minor. However from the algebraical point pf view perhaps the most natural graph notion is the notion of a homomorphism:

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Given two graphs $G = (V, E)$ and $G' = (V', E')$ a homomorphism f of G into G' is any mapping $f : V \longrightarrow V'$ which satisfies the following condition: (1) $[x, y] \in E$ implies $[f(x), f(y)] \in E'$ The condition (1) should

be understood as follows: on both sides of the implication one considers the same type of edges (undirected, directed). The analogous definitions give notions of the homomorphism for hypergraphs (set systems) and relational systems.

Homomorphism is an algebraical notion which in graph theory found its way to problems related to products, reconstruction and chromatic polynomial , to name just a few. Our approach here is motivated by the chromatic number connection expressed by the following observation which holds for every undirected graph G: $G \longrightarrow K_k$ iff $\chi(G) \leq k$.

Motivated by this we call a homomorphism $G \longrightarrow H$ a *H-coloring* of G and given H we call the class of all graphs G which are H-colorable the *color class determined by* H. The color class determined by H is denoted by $\longrightarrow H$ or by $\mathcal{C}_H$. Thus $\longrightarrow H$ is the class of all H-colorable graphs G : $\longrightarrow H = G; G \longrightarrow H$.

The class of all color classes determines the partially ordered class $\mathcal{C}$ ordered by the inclusion. The structure of $\mathcal{C}$ is the subject of this paper.

Note first that the inclusion of the color classes $\longrightarrow H \subset \longrightarrow H'$ is equivalent to $H \longrightarrow H'$. Thus the graphs H and H' determine the same color class iff H and H' are homomorphism equivalent (by this we mean that both $H \longrightarrow H'$ and $H' \longrightarrow H$). This is (for finite graphs) best to express by means of the notion of the *core* (core of H is the minimal subgraph of H which is a homomorphic image of H;[HN1]) by saying that H and H' determine the same color class iff H and H' have the same core.

The color classes corresponding to chromatic number are the color classes determined by complete graphs. They form a chain isomorphic to N. This simplistic ilusion is quickly destroyed by the moment of thought and it appears that to the contrary of the first evidence the class $\mathcal{C}$ is a very rich class. The following are extremal results in this direction, see [PT] and [W]:

Theorem 1.1 *The class $\mathcal{C}$ is universal for all partially ordered classes. Explicitely, every partially ordered class is isomorphic to an indused subclass of $\mathcal{C}$. Moreover the class of all color classes determined by finite graphs is universal for all countable partially ordered sets.*

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Fig.3

As an example let us consider $H = C_5$. In this case $\mathcal{H} = K_3$ and a triangle-join has a form depicted on Fig.3. Except of Hajos theorem and this particular case we do not know any other example when the color class $\not\longrightarrow H$ is generated by a single graph.

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Theorem 1.2 *The class of all color classes of finite undirected graphs is (order) dense with the the unique exception of the pair (K_1, K_2). Explicitely for every pair of graphs G_1, G_2 which is not homomorphicaly equivalent to the pair K_1, K_2, there exists a graph G such that G is not homomorphically equivalent to neither G_1 nor G_2 but $G_1 \longrightarrow G$ and $G \longrightarrow G_2$.*

The original proof of Welzl [W] is a difficult ad hoc argument. M.Perles and J.N.found independently (at about 1990) a much simpler and perhaps more natural proof. As neither of us published it (but several times lectured about it) the proof eventually got to a forthcomming survey by G. Hahn and C. Tardiff. Thus it is perhaps fitting to include the original proof here as that proof allows us to prove several stronger statements. They may be formulated as follows

Theorem 1.3 *The class of all color classes of undirected graphs is (order) dense with the unique exception of the pair K_1, K_2.*

Theorem 1.4 *The class of all color classes of hypergraphs is dense.*

Thus for hyperhraphs there are no exceptional pairs which we may call (order) jumps. Explicitely, given a partially ordered class $\mathcal{K}$ we call a pair A, B of its elements a *jump* if $AleB$ but there is no C with $A < C < B$.

In another direction we generalize Theorem 2 as follows:

Theorem 1.5 *Every jump in the class of all connected directed graphs is a pair of balanced graphs.*

(Recall that a directed graph is called *balanced* if every its cycle contains equall number of forwarding and backwarding arcs. Alternatively, a graph G is balanced if there s a homomorphism of G to a monotonne path.)

The problem to characterize all the jumps in the class of directed graphs seems to be very difficult. It is known that jumps exist. Except of the trivial pair single vertex and single arc and another easy pair (single arc, monotonne path of length 2) there are infinitely many jumps of the form P_{k+1}, P_k where the graph P_k has $4 + 2(k-1)$ vertices and it is depicted on Fig.1. Let us call all these jumps *standard jumps.*

Fig.1

This has been proved by author and X. Zhu in [NZ] where moreover the following (presently difficult) result has been proved

Theorem 1.6 *The standard jumps are the only jumps in the class of all finite oriented paths.*

Even for the class of all finite directed trees the characterization of all jumps is presently unknown. However, viewing the richness of the homomorphisms between the trees (see e.g.[HNZ1] and [HNZ2], this is perhaps hardly surprising.

But perhaps there is more and more evidence that the standard jumps are the only jumps even for the class of all connected oriented graphs.

It is interesting to note that for not connected graphs the above Theorem 5 does not hold:

Proposition 1.7 *Consider the unbalanced cycle C_{2k+3} formed by the alternating path of length $2k+1$ together with the terminal vertex joined with the first and the last vertex of this path. Then the disconnected graphs G_1 with components C_{2k+3} and P_{k+1} and G_2 with components C_{2k+3} and P_k form a jump in the class of all oriented graphs. There are infinitely many such examples even for a fixed pair of paths.*

Let us state yet another result related to both Density Theorem 2 and Universality Theorem 1. We call it *Fractal Property* of the class $\mathcal{C}$. Before

3 Generating color classes

Here we prove Theorem 8.

Proof. Let H be a fixed undirected graph with independence number k. Denote by $\mathcal{H}$ the set of all inclusion minimal graphs H' with the following two properties: the independence number of H' is equall $k+1$and there is no homomorphism of H' into H. There is an absolute bound on the size of H' (which we obtain ,say by Ramsey theorem) and thus the set $\mathcal{H}$ is a finite set. We shaw that this set generates by means of the four stated operations exactly the color class $\longrightarrow H$. First, we prove that the above 4 operations preserve membership in $\not\longrightarrow H$. However this is obvious for 3 of them (as in the Hajos theorem case). Thus it suffices to consider H-join only. So suppose that G_{ij}, G'_{ij} and edges e_ij and e'_ij and sign pattern $sign(i,j)$ be given (we preserve all the notation in the above definition of the H-join). Let the resulting H-join be G and suppose for the contradiction that there is a homomorphism $f : G \longrightarrow H$.We consider the possible images of the amalgamation vertices $0, 1, 2, ..., k$. Observe that for distinct i, j we have $f(i)$ and $f(j)$ distinct as if $f(i) = f(j)$ then the restriction of f to the vertex set of G'_{ij} is a homomorphism from $G'_{ij} \longrightarrow H$. By the same token the set $f(i); i = O, 1, ..., k$ is an independent set in H which is a contradiction with $k > \alpha(H)$. Now we prove by a variation of the standard argument that the class $\not\longrightarrow H$ coincides with the class of of all constructible graphs. We proceed by the double induction on the number of vertices and the number of non-edges of the given graph G. The boundary cases are easy to handle. So let us assume that G is a graph , $G \not\longrightarrow H$ such that the graph $G+e$ can be constructed for every non-edge $e \notin G$. If $\alpha(G) \leq \alpha$ then G contains a minimal subgraph G' with $G' \not\longrightarrow H$. As it is again $\alpha(G') \leq \alpha(H)$ and thus $G' \in \mathcal{H}$ and thus also G is constructable. So let $\alpha(G) > \alpha(H) = k$. Assume without loss of generality that $0, 1, 2, 3, ..., k$ is an independent set in G. Observe that we may assume that for every i there exists a vertex i' such that i, i' is an edge of G. Define G_ij as the graph G with the edge i, j added and the graph G'_{ij} as the graph G with the edge i, j'.It is a routine to check that G arises from these graphs as H-join followed by some identification of non-edges.

G is not balanced and $G \longrightarrow G'$ then also G' fails to be balanced (this is best to see by the characterization of balanced graphs as those graphs which can be colored by monotonne paths; see above). It follows that if one of the graphs G_i is unbalanced then also G_2 is unbalanced. Let G_2 be connected and k denotes the length of the shortest unbalanced cycle in G_2. Now we can similarily to the above proof of Theorem 2: Let H be a directed graph with chromatic number (of its symmetrization) $> n^{n'}$ while every unbalanced cycle in H has length $> k$ (here we can use again shift like graphs). Then $G = G_1 \cup (HxG_2)$ is the desired graph. The arrows $G_1 \longrightarrow G$ and $G \longrightarrow G_2$ we get similarly as above. Also the non-existence of a homomorphism $G \longrightarrow G_1$ can be proved analogously to the above. Finally, the non-existence of the mapping $G_2 \longrightarrow G$ follows from the connectivity of G_2 together with the above observation about non-balanced graph-homomorphism.

□

Proof. Theorem 4:

In the proof we use the following hypergraph product: Given hypergraphs $H = (X, \mathcal{M})$ and $H' = (X', \mathcal{M}')$ we define their product $H \times H' = (X \times X', M \times M'; M \in \mathcal{M}, M' \in \mathcal{M}')$. It is easy to check that the projections are homomorphisms. Now given two hypergraphs H_1 and H_2 with homomorphism $H_1 \longrightarrow H_2$ and no homomorphism $H_2 \longrightarrow H_1$ we define the desired hypergraph H as follows: First we find a hypergraph $H_0 = (Y, \mathcal{N})$ with the following properties: the chromatic number of H_0 is $> n^{n'}$ where n and n' is the number of points of hypergraphs H_1 and H_2 ; ii. every hyperedge $N \in \mathcal{N}$ has size $> n'$ (any sufficiently large $n' + 1$ uniform hypergraph will do). The desired hypergraph H will be constructed as the disjoint union of H_1 and $H_0 \times H_2$. Obviously $H_1 \longrightarrow H \longrightarrow H_2$. There is no homomorphism $H_2 \longrightarrow H$ as the set system H_2 contains a hyperedge of size $\leq n'$ while the set system $H_0 x H_2$ has no such edge. The fact that there is no homomorphisms $H \longrightarrow H_1$ proceeds in a complete analogy with the above proof of Theorem 2. (We only have to use the different product but all the homomorphism properties are preserved.)

□

stating it let us introduce the following: A mapping $F : \mathcal{C} \longrightarrow \mathcal{C}$ is said to be *embedding* if F is 1-1 and for any pair of color classes A and B holds: $A \leq B$ iff $F(A) \leq F(B)$. Given a pair A, B of color classes we denote by $[A, B]$ the class of all color classes C satisfying $A \leq C \leq B$. We call $[A, B]$ the *interval* in $\mathcal{C}$. Now we can state the fractal property of the color classes $\mathcal{C}$:

Theorem 1.8 *Let A, B be an interval of calC where at least one of A, B is not balanced (consequently B is not balanced). Then there is an embedding of $\mathcal{C}$ into the interval $[A, B]$.*

Thus each non balanced interval of $\mathcal{C}$ contains a copy of the whole class $\mathcal{C}$. We shall prove this theorem elsewhere. Let us note that the fractal property is not known to be true in some cases where the density theorem is valid. This is the case e.g. for the class of all finite paths. Let us close this paper by a small illustration of the usefullness of the approach by means of homomorphisms to graph theory problems. One of the classical results of the graph theory is Hajos Theorem [] . The result states that a graph G has chromatic number $\geq k$ iff G can be contructed from K_k by recursively applying the following four very simple operations: $+v$ (vertex addition), $+e$ (edge addition), $.e$ (identification of two vertices not joined by an edge; i.e. contraction of an non-edge) and so called *Hajos join* which is the following operation: given to (disjoint) graphs G and G' we first select two edges x, y in G and x', y' in G' and we build the new graph, called Hajos join of G and G', by identyfying vertices x and x', by deleting edges x, y and x', y' and by adding the new edge x', y'; see the schematic Fig.2:

Fig.2

While on the first glance very interesting and even surprising the Hajos theorem is a bit surprisingly not a very usefull tool and its usefullness in chromatic graph theory is still beeing disscussed. On the other hand it has been proved by Pitassi that the minimal length of the construction of a graph by means of Hajos theorem is related to some hard problems of proof theory[]. This may clarify some aspects of Hajos theorem. Another indication of the same phaenomena is perhaps the following:

Theorem 1.9 *Let H be an undirected graph. Then the class of all graphs which are not H-colorable (i.e. the class of all graphs G for which there is no homomorphism to H) can be generated from a finite set $\mathcal{H}$ of graphs by means of recursive application of the following four very simple operations: vertex and edge addition, contraction of an non-edge and $\mathcal{H}$ -join. Only H-join has to be defined: First denote by k the independence number $\alpha(H)$, the definition of H-join will depend on k only. Next select $k+1$ points, say $0, 1, 2, ..., k$ and for each unordered pair $i, j, ineqj$, let G_{ij} and G'_{ij} be graphs . In each of these graphs select edges e_{ij} and e'_{ij} and with respect of these edges perform Hajos join thus obtaining a graph G_{ij}, let a_{ij} be the identified vertex (of edges e_{ij} and e'_{ij}) and let b_{ij} be one of other vertices (say the other vertex of e_{ij}). Furthermore, let for each $i < j$ be specified $sign(i, j)$ (either + or -). Then H-join arises from Hajos-joins G_{ij} by identifying a_{ij} with i and b_{ij} with j providing $sign(i, j) = +$ and by identifying a_{ij} with j and b_{ij} with i providing $sign(i, j) = -$. Appart from these identifications the graphs G_{ij} are disjoint.*

Thus in essence H-join is a multiple Hajos-join: we form $\binom{k}{2}$ Hajos joins and amalgam them according to a tournament on given set of k+1 vertices (expressed by the sign pattern).

Thus the Hajos theorem is a special case of a general theorem which holds for every color class of undirected graphs. Viewing this generality , perhaps one should not expect too fine structure and deep applications of Hajos theorem either.

The paper is organized as follows: In section 2 we give a short proof of the density Theorem 2 together with Theorems 3.4.5. In Section 3 we prove Theorem 9.

2 Density theorems

We begin with an easy proof of Density Theorem 2.

Proof. (M.Perles and J.N.): Let G_1 and G_2 be given undirected graphs, let $f : G_1 \longrightarrow G_2$ be a homomorphism , let there be no homomorphism $G_2 \longrightarrow G_1$. As this pair is not equivalent to the jump K_1, K_2 every component of the graph G_2 has the chromatic number > 2. At least one of these components fails to be G_1 colorable and let it contain an odd cycle of length k. Now choose a graph H with the following properties: H contains no odd cycle of length lek and the chromatic number of H is $> n^{n'}$ where n and n' denotes the number of vertices of the graphs G_1 and G_2. This graph exists by the celebrated Erdős Theorem [E]. Now let $G = G_1 \cup (H \times G_2)$. Here x denotes the direct product of two graphs and union sign means the disjoint union. We shall prove that G is the desired graph. Obviously $G_1 \longrightarrow G$ and $G \longrightarrow G_2$ follows as the second projection of $H \times G_2$ is a homomorphism into G_2. On the other hand there is no homomorphism G_2 into G as homomorphisms preserve odd cycles and they cannot increase the length of the shortest of them. Thus it suffices to prove that there is no homomorphism $G \longrightarrow G_1$. Let us suppose for the contradiction that there is a homomorphism $f : H \times G_2 \longrightarrow G_1$. Thus for any vertex x of H we have an induced mapping $f_x : V(G_2) \longrightarrow V(G_1)$ defined by $f_x(y) = f(x, y)$. (This mapping need not be a homomorphism.) As there are at most $n^{n'}$ of such mappings there are vertices x and x' forming an edge of H such that the mappings f_x and f'_x are identical equall , say, to g. However in this case g is a homomorphism of G_2 into G_1, contrary to our assumption.

Proof. Theorem 3

We proceede exactly as in the above proof for finite graphs. The only difference is that we need a graph H without odd cycles of length lek with chromatic number larger than a given cardinal number. This is another folkloristic result of Erdős and Hajnal which is easy to prove by considering so called iterated shift graphs together with the transfinite Ramsey theorem, see [EH].

□

Proof. Theorem 5

Assume that G_1, G_2 are directed graphs with a homomorphism $G_1 \longrightarrow G_2 but$ no homomorphism $G_2 \longrightarrow G_1$. We use the following observation: If