Proceedings of

Spring School in Combinatorics

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(edited by Jiří Fiala)

Contents

Chief paper

Epilogue

Just a few memories...

Spring School is a meeting organized every year by the Combinatorial Seminar of Charles University. Despite of the fact that this is a traditional this is the first written report which we decided to publish. Thus perhaps it is appropriate to give a short outline of the history and of the scope of these Spring Schools. But in order to do so we have to start a bit more broadly.

Combinatorial Seminar exists in its present from at Charles University since 1970 and I spent a lot of time and energy organiziring it singlehandedly for cca 15 years. I always considered it as a key to my scientific and educational activity at Charles University. That may seem strange from outside but things somehow developed this way. Several factors contributed to it. If I should isolate two of them I would state the quality of our students and the fact that mathematical community at Charles University is closely tight. One cannot give full list of excellent students which started their scientific carrier in the Combinatorial Seminar. Still I cannot resist to name in a moreless chronological order just a few now internationally known names. This all started with V. Müller, J. Pelant and V. Rödl and continued with S. Poljak, D. Turzík, then with J. Oldřich, D. Volný, then with J. Kratochvíl then with R. Thomas, I. Kříž and J. Matoušek and M. Loebl still later with P. Valtr and M. Klazar and then followed an era of P. Hliněný and P. Čížek which brings us to the present time of several what I strongly believe future stars. Let us hope that the list will continue. Presently the seminar has approximately 15 members. They are all our undergraduates. The senior members of KAM consider the seminar of pivotal importance and the seminar is guided by all teachers which are present i.e. a subset of M. Klazar, J. Kratochvíl, M. Loebl, J. Matoušek, J. Nešetřil, P. Valtr. I state all these facts just to give an idea what combinatorial seminar is and what meaning it has for us. In such a setting it was only natural that an idea occurred to have an informal meeting where we could work indisturbed for a week, where we could summarize and outline future prospects of the activity of our Combinatorial Seminar. So, Spring School started and although there were some isolated events such as three springschools in giant mountains in 1976-8 in its present form the first Spring School was organized in 1983 in a small Slovak village Východná (where parents of one of the members of the seminar had a small cottage). This was a very small meeting and in many aspects very primitive. But the several of the principles were already settled and did not change since:

- the students speak either about their own results or thoroughly cover papers assigned them earlier;
- the participation is limited to those who are prepared to speak (although sometimes there is not enough time for everybody);
- there are no other lectures (with isolated exception of occasional guests);
- teachers do not lecture and intervence minimally;
- students organize their time and style of lectures with a minimal supervision of a teacher;
- the usual program involves morning and night sessions with afternoon being reserved to trips and games;
- night sessions are best;
- spring school takes place usually in the mountains or hills;

Ofcourse such a style of the organization can be efficient or can be effectively functioning only if those involved have interests and abilities to work. So far we have been fortunate that Combinatorial Seminar attracted students of such qualities. Let us hope that we shall continue to be so fortunate in the future too. Without this our activity would have a little meaning.

Jaroslav Nešetřil

Insight from outside

Many former students of the combinatorial seminar have become tutors who regularly take part in the life of the seminar and in particular in the famous Spring Schools. As the oldest of these, I would like to complement the overview written by Jarik Nešetřil by a few glimpses and reminiscences "from inside out", describing the transition from being a student to becoming one of the supervisors.

The student combinatorial seminar in its present form started in 1981[1] I was a student at that time and I remember very well its spontaneous beginning. Jarik Nešetřil was teaching a class of Combinatorics and Graph Theory but the students were misbehaving and did not pay much attention. They (well, shame to say we) were discussing the (what in those days seemed) difficult question of choosing the facultative seminar (the curriculum was very firmly determined, we only could and had to choose one facultative lecture and one facultative seminar in each year of the university study). Instead of getting mad (as some of the other tutors would be) and yelling at us or walking out of the classroom, Jarik simply asked what was the matter and then he suggested to found a combinatorial seminar. This was an elegant and easy solution to our problem and most of us chose this seminar.

The seminar proved to be exciting, fruitful and most successful in the years that came after then, and I hope that our current students see it as such as well. At least I do. The first years of the seminar after 1980 were strongly influenced by Jarik's choice of problems to work on. The very first one were string graphs, intersection graphs of curves in the plane, and the work on this topic has led to successful submissions to student contests, conference presentations and scientific publications. Perhaps this is the right place where I could express my personal thanks to Jarik for the most interesting seminar of my university studies, for the choice of problems, for his stimulating supervision of our student work and for his encouraging guidance. All students of the seminar ben-

[1] To be more precise, the current connected component of the interval graph whose vertices correspond to students and the time intervals in which they were visiting the seminar, has started in the mentioned year.

efit from Jarik's foresight with which he distributed papers to be read and problems to be worked on, whether these were geometric problems or tree-width and graph minors, partially ordered sets or unprovable combinatorial statements. The intensive program of the Spring Schools always enhanced the influence on students and stressed the working atmosphere of the entire seminar. Though the scientific program is the most important and the first goal, the Spring Schools have developed a unique easy going atmosphere. I would like to add a few personal comments on these additional features of the Spring Schools.

Nowadays the Spring School is an integral part of the student seminar and we cannot even imagine a year without it. It was not so in the beginning and I remember how excited we were when Jarik first suggested this adventure in the second or third year of our seminar. Our first Spring School was (so far) the one organized at the greatest distance from Prague, in High Tatra Mountains in Slovakia. We stayed in an old village house kindly sublet to us by parents of Dušan Paleček, also a member of the seminar. This was a really adventurous event, starting with a hectic wake-up in a train which some of us left in pyjamas, missing the connecting train, sleeping in sleeping bags on the floor which was about the same size as the sum of our bodies and downhill skiing on cross-country skis at Solisko. The atmosphere of this school has influenced the next schools and since that time we aim at some mountain areas regularly. And also for a few years we kept alive the tradition of dramatic travels to the spring schools. Perhaps the most exciting was right the next one, when we were jumping one by one from a moving train upto the moment when Bruce Rothschild did not dare to jump and Svata Poljak did not dare to push him out. The two of them then continued to the next stop, without knowing the exact location of the Spring School. With a lot of travelling mathematicians' skills they came in time for dinner, but Lord knows how they found us.

This last described story happened in Smržovka, where we got our first blackboard. Our original idea was to borrow one in the local school, but the janitor donated an old blackboard to us. We then carried this board to Prague, back to Smržovka the year after, to Prague again and then to Filipova Huť, where it rests forever in the attic of what has now become a herberk-hostel. A tradition of table tennis tournaments started in Smržovka, when we used this board in the breaks for sports. In the old days we played with bare hands or books instead of paddles, and we were using meat cans as the net. Gone are these pioneer days, the hotel Kavalier in Borová Lada has a real table and the students are bringing professional paddles. And this year the landlord of hotel Kavalier has made a real blackboard for us, so that we do not have to bother the local schools. However, the tradition of nontrivial arrivals lives in our current students and every year several of them arrive on bikes after 80 km of exhausting exercise in Šumava hills.

The schedule of the talks traditionally allows free afternoons for outdoor activities. Since we always try to organize the Spring School in nice areas, afternoon hikes are very popular. I certainly don't remember the destinations of all the hikes, but just to mention some of them - rocky formations "Stone Boletus" in Kraslice, Stožecká kaple and Vltava at Soumarský Most in České Žleby, a hike to Borová Lada from Filipova Huť and many others. In the last years a short hike to the little lake at Chalupská slať is a must. The first school in High Tatras gave the impression that skiing might be the right activity, but since then the weather did not really prove in favor of this sport - either we brought the skis with us and then played soccer on green grass, or we gave up the skis and then found a meter of old snow in the mountains. In the last years the popularity has definitely shifted to bikes. Not only the students, also some of the adults are bringing bikes for afternoon trips to Knížecí Pláně or the springs of Vltava. Last year our Canadian visitor Mike Fellows was amazed by the beauty of Šumava mountains when seen from a bike. This year the biking was even more adventurous. Forty kilometers in 10 cm of fresh snow and occasional blizzards and snowstorms may not seem the right cup of tea for bikers, in particular when bikes have to be walked even in the flat parts of the tour, but going downhill at the speed of 20-25 kmh^{-1} is a real excitement. Fortunately, the 10 centimeters of snow are not only slippery but also a soft cushion.

For many years we kept alive the tradition of taking care of all cooking during the Spring Schools. In the very beginning it was the cheapest and probably the only way how to make the participation affordable to all students. Preparations always started several weeks before the event itself – the students collected coupons for menza, the tutors wrote an official letter to the menza manager that would allow the students to get things like rice, spaghetti, ketchup, luncheonmeat cans, dalmatian tchuftas etc. Then we distributed the weight into particular backpacks on the last seminar and transported all the stuff to the Spring School. And then the real adventure started. Fake chinese dishes from cans, dalmatian tchuftas with rice, pudding and porridge, and tons of spaghetti. With some invention and help of chemistry we sometimes entertained ourselves with blue pudding or rainbow colored soups. A person whose name is in my memories forever tied with cooking is Iva Dvořáková. As the only female at some of the Spring Schools, our male chauvinist education was sending her into the kitchen year after year. And nobody complained, Iva was an excellent cook. But year after year the same story repeated, she only cooked the first half of the week and then she quit with a lot of noise. The reason, however, was not that we (males) did not take our part in cooking or dishwashing, but that we always let the same man to do the dishwashing the whole half-week. Iva got really upset, gave us a speech with a pretty high voice and quit cooking. Maybe we did it on purpose, since at this time our experiments had started. Pancakes with mustard, sausage with apples, rice porridge with oranges and many other strange variations.

Perhaps a turnpoint started at Mariánská in 1992, when Štěpán Kasal brought 50 kilos of oatflakes and we had to eat porridge twice a day. Some of us were not able to stand this attack on our taste cells and ended up having dinners in nearby pubs. We then tried to save the cooking tradition in České Žleby (e.g., by yogurt Indian style potato dish a la Pankaj Agarwal prepared by Jirka Matoušek, a Chinese dish cooked from luncheonmeat by Jarik Nešetřil or fresh otherwise never-ever picked mushrooms prepared by myself), but we finally gave up and since 1994 we are catered by Mrs. Jirková from hotel Kavalier in Borová Lada. Gone are our experiments and we have sold our own creativity for the pleasure of being served coffins with whipped cream.

Other things are changing as well. New students bring in new ideas and new angles of view. Mathematics and music belong together and it is somewhat surprising that only this year we had a concert on the program for the very first time.

There are more things from the history of Spring Schools that could be added. In particular, I did not mention the foreign students we had at almost every School (but this was on purpose – I am not able to remember all of them, and I don't want to offend anybody by omitting them unintentionally). The presence of foreign students and visitors was always very stimulating and welcome. But the Spring School is devoted to and belongs to students, and they will have the rest of the space in the brochure. Let me just conclude with my personal feeling that this year's Spring School was one of the best schools I took part in and that it really deserves to have the proceedings. It is just right and fits the flavor of the school that the proceedings are prepared by the students themselves.

Jan Kratochvíl

PS. The list of Spring Schools from 1983

1. 1983 Východná
2. 1984 Smržovka u Tanvaldu
3. 1985 Smržovka u Tanvaldu
4. 1986 Filipova Huť
5. 1987 Filipova Huť
6. 1988 Filipova Huť
7. 1989 Filipova Huť
8. 1990 Nejdek
9. 1991 Kraslice
10. 1992 Mariánská
11. 1993 České Žleby
12. 1994 Borová Lada
13. 1995 Borová Lada
14. 1996 Borová Lada
15. 1997 Borová Lada

Personal experiences

Although the Spring School has a long tradition, everyone is different to others. I think that it would be good idea to remind us of an atmosphere and events of this year's one, from a student's point of view.

First, there was very surprising winter weather. This was first school I remember where we erected a real snowman[1]. It should not seem as a real problem, but winter weather in the mountains with 15 cms of snow and strong wind aren't good conditions for biking and hiking. Many of us regretted to leave our ski at home.

Sometimes the weather came little peaceful, so we could go for a trip to see nice Šumava nature (maybe from a lookout point at Bučina or at hill above Borová Lada). Walking in the countryside is a good occasion for talking with friends — I remember long discussions about Penrose's tiling with Robert Babilon or about the principle of Global Position System, which Martin Dlouhý brought with and we tried test its precision. You can have unusual experience on trips such as crossing river Vltava over a fallen-down tree or unexpected meeting a colleague kilometres far away from Borová Lada or losing orientation on the mountain footpath when you have to rely only on occasional tourist pathmarking[2].

Actually, walking is not the main part of Spring Schools, but lectures. I think they were well-prepared, some of them had unusual environment for spring school : cinema-like with a slide projector or circus-like with juggling oranges (Petr Hliněný) or eggs[3] (Doc. Kratochvíl). I also remember the two-language modified lecture by Petr Hliněný or the long introduction into a topic in style of "what it won't be about" by Jirka Vševěd Hanika. Some lectures went into natural talking or solving open problems. It took us about 2 days to find the winning strategy in a special case of a graph-triangle-colouring game, which existence someone asked for after Jirka Hanika's lecture on combinatorial games.

Then, thursday morning was dedicated to foreign guests and their

[1] Maybe you can see it on the snapshot, but I think that it's hidden just behind the edge.

[2] Also occasionally confusing.

[3] Hard boiled, as we recognized when Doc. Kratochvíl started paring and eating them.

talks. Firstly, it was little uncommon for some of us to listen to a lecture in English, but this feeling blew away after a few sentences. I think that all three guests soon got used to us (and Karen after several days proclaimed a time-bomb war to Petr). It was also interesting to speak with them about scientific news or foreign circumstances and habits — like study programs at Bonn's University or Dutch winter joys, too different from ours.

Finally, the most interesting parts of days at Spring School are late evenings after the last lecture. Someone can go out for a night walk, but usually there are lots of indoor activities like taking part in a card or desk game (our favourite ones were bridge and taroky) or having a match in midnight table-tennis, singing with the guitar, or simply having "beer-talking" about several also non-combinatorial topics. At this year's spring school there were two nice evening surprises. First one was the concert at which Lucka Bittnerová played the flute some Bach's sonatas and which continued with old spirituals sung by a voice quartet. It was very pleasant and all performers were rewarded by a big round of applause and special portion of sugar "coffins". Another unique event was publicating the crossword made by Robert Babilon especially for this year's Spring School. We solved it together with lot's of fun and you can try to do it too — this crossword is also published at the end of the proceedings.

In my opinion, the friendly and stimulating atmosphere on Combinatorial Spring School, is due to many factors. Maybe it is because of perfect background in boardinghouse Kavalier or nice surroundings of Borová Lada. I'm sure, that the biggest part in this success is shouldered on people, who take part at Spring School: students and chiefly at our tutors.

Jiří Fiala

Program

Monday April 14

— morning —

— evening —

Martin Dlouhý
Short Certificates for Tournaments

Ondřej Pangrác
Graphs Drawn with Few Crossings per Edge

Tuesday April 15

— morning —

Zuzana Honzíková
Pseudo-Interval Graphs

Stanislav Hencl
Fruit Salad

Jana Turzíková
Cycle and Path Covers of Graphs

— evening —

Lucie Bittnerová
A Combinatorial Algorithm for the Determinant

Jiří Hanika
Foundations of Positional Games

Wednesday April 16

— morning —

Jan Vondrák
Dense Graphs and Edge Reconstruction

Matěj Maxa
The Geometry of Coin-weighing Problem

Petr Hliněný
Computational Complexity of the Krausz Dimension of Graphs

— evening —

Helena Nyklová
A New Idea for Hamiltonian Problem

Robert Šámal, Jan Vondrák
The Limit Checker Number of a Graph

Thursday April 17

— morning —

J. Solymosi
Simple Counting

Ch. Vogt
Zero-sum Ramsey Numbers

— evening —

Robert Šámal
The Growth Rate of Vertex--transitive Planar Graphs

Robert Babilon
The 3–colorability of Rhomb Tilings of the Plane

Concerto piccolo

Friday April 18

— morning —

Jiří Fiala
Graph Covering

Cyril Sochor
Games, Randomness and Algorithms

Jan Fryščák
Applying Tabu Search to Determine New Ramsey Graphs

— evening —

Petra Špičková
Juggling Drops and Descents

Robert Babilon

The 3–colorability of Rhomb Tilings of the Plane

Theorem. *Let us have such a system (finite or infinite) of nonoverlapping rhombs in the plane in which the intersection of any two rhombs is either an empty set or a whole common edge. Then one can color all rhombs with one of three colors in such a way that every two adjacent rhombs (i. e. with common edge) have different color.*

Sketch of proof. At first we will prove this theorem for finite systems, namely by the mathematical induction for a number of rhombs. The proof is trivial for at most three rhombs. For more rhombs we can always find a rhomb which has at most two adjacent rhombs (we use here a fact that the opposite edges of a rhomb are parallel) and we can color this rhomb with such a color that no adjacent rhomb has.

The proof for infinite systems then follows from the compactness.

Remark. The theorem is also valid for parallelograms, because we do not use a fact that adjacent edges have the same length.

Lucie Bittnerová

A Combinatorial Algorithm for the Determinant

Paper by: M. Mahajan, V. Vinay
From: Proceedings of ACM-SIAM Symposium
on Discrete Algorithms 1997

This article deals with the first efficient combinatorial algorithm for the computation of the determinant. Hitherto, all (known) algorithms for determinant have been based on linear algebra. In contrast, this algorithm and its proof of correctness are totally combinatorial in nature. The algorithm requires no division and works on arbitrary commutative rings. It also lends itself to efficient sequential and parallel implementations.

Martin Dlouhý

Short Certificates for Tournaments

Paper by: N. Alon, M. Ruszinkó
From: The Electronic Journal Of Combinatorics 4 (1997)

An *isomorphism certificate* of labeled tournament T is a labeled subdigraph of T which together with an unlabeled copy of T allows the errorless reconstruction of T. It is shown that any tournament on n vertices contains an isomorphism certificate with at most $n \log_2 n$ edges. This answers a question of Fishburn, Kim and Tetali.

A *score certificate* of T is a labeled subdigraph of T which together with the score sequence of T allows its errorless reconstruction. It is shown that there is an absolute constant $\varepsilon > 0$ so that any tournament on n vertices contains a score certificate with at most $(1/2 - \varepsilon)n^2$ edges.

Jiří Fiala

Graph covering

Covering and partial covering

A *covering* projection f from a graph G to graph H is a vertex mapping such that each vertex u and all his neighbours map bijectively onto the image of u and all its neighbours. This projection is also known as "local isomorphism".

A *partial covering* g maps neighbours of any vertex u bijectively onto a subset of the neighbours of $g(u)$.

Theorem. *For any graphs G, H, if there exists covering $G \to H$ then any partial covering $G \to H$ is also "full" covering.*

Sketch of proof. Dividing vertices of G into classes by their degree and inductively by degree of its neighbours we got system of classes called *degree distribution* of vertices of graph G. We do the same operation for the graph H. Since there exists at least one covering $G \to H$, both systems have the same structure. Using ordering classes we prove that any vertex $u \in V(G)$ and its image in partial covering must belong to corresponding classes, so their degree has the same value.

There is a natural problem about existence of covering between two graphs G and H. After fixing graph H we have problem H-COVER which takes the graph G as input and answers the question, if there exists covering from the graph G to H.

In [1] class of graphs H is described for which problem H-COVER is NP-complete. We call these graphs *stable*.

Edge connectivity of stable graphs

For the graph G and its vertex u denotes the symbol G_u graph in which vertex u is split into pendant vertices of degree 1. Also if there exists covering $f : G \to H$ we denote partial covering $f_u : G_u \to H$ which maps pendant vertices onto the image of u and the rest vertices of G_u maps onto the same vertices from H as in projection f.

Definition. The regular graph H is *stable*, if for any graph G, vertex u and any partial covering $g : G_u \to H$ there exists covering $f : G \to H$ such that $f_u = g$.

Theorem. *The k-regular $\lfloor k/2 \rfloor + 1$-edge connected graph is stable.*

Sketch of proof. Counting number of vertices from G_u which maps onto one vertex from the graph H, we find that images of edges which consists pendant vertices divides the graph H into two partitions. One partition consists only of one vertex — the image of pendant vertices and also the image of split vertex u.

Remark. For any even k, there exists k-regular $\lfloor k/2 \rfloor$-edge connected graph which is not stable. It is any graph H with articulation u which splitting divides the graph H into two isomorphic components.

Partial covering from these components H_u onto one 2-vertex connected component of H contradicts definition of a stable graph.

In [2] a construction is shown for $4l$-regular $2l$-edge connected graph H and graph G such that graph G_u is connected and destifies to H is not stable.

References

[1] J. Kratochvíl, A. Proskurowski, J.A. Telle: *Covering regular graphs* (preprint)

[2] J. Fiala: *Graph covers* (in Czech), Master Thesis, Charles University, 1997

[3] N. Biggs: *Algebraic Graph Theory*, Cambridge University Press, 1974

Jan Fryščák

Applying Tabu Search to Determine New Ramsey Graphs

Paper by: Konrad Piwakowski
From: The Electronic Journal of Combinatorics 3 (1996)

In this note an adaptation of heuristic tabu search for finding Ramsey graphs is presented. As a result, seven new lower bounds for classical Ramsey numbers are established.

This method is based as a local search among graphs builded as conjunction of any subset of cyclic graphs on n vertices. A cyclic graph G_i for $i = 1, ..., \lfloor n/2 \rfloor$ has n vertices and for any edge (u, v) should hold $\min\{|u - v|, n - |u - v|\} = i$.

Jiří Hanika

Foundations of Positional Games

Paper by: József Beck
(preprint)

The article covers the foundations of a theory of non-trivial positional games, such as Tic-Tac-Toe, Hales-Jewett or Van der Waerden families of games. These games don't decompose and the author shows various ways how to approach their complicated structure, such as the strategy stealing argument, which allows disproving the existence of a winning strategy for the second player, or the resource counting argument, where a player is able to keep some metric too low for his opponent to win, thus reaching the draw (or even the win, if draw is impossible by definition). A special application of the resource counting method is the pairing strategy, where one of the players forces the draw by preventing every opponent's winning set (of moves) soon enough — each winning set contains a pair of moves, the first of which being taken by the opponent causes the defender to take the other one.

Another toolkit of methods are the so-called probabilistic approaches. The author conjectures that the typical result of a random game is often the same as the result between two "perfect" players — at least for the positional games (games where a certain "winning set" of moves must be taken by one of the players, while the other one is trying either to prevent this or to construct his own winning set). This allows counting the probability of creating a winning set during the course of the game (some approximation may be used); if the probability turns out to be very near to zero or one, we'll conclude that the first player most likely has a winning strategy, or the "perfect" game ends in a draw, respectively.

A basic classification of positional games is given as well and the applicability of the abovementioned methods to these classes is discussed.

Stanislav Hencl

Fruit Salad

Paper by: András Gyarfás
From: The Electronic Journal of Combinatorics 4, (1997)

This article dedicated to the memory of Paul Erdös talks about some problems given by him.

The first question is this Ramsey-type problem. Let $r(m,n)$ be the smallest p for which every graph of order p contains either a cycle of length at most m or n independent vertices. Using the properties of nearly bipartite graphs author of this article proved how much $r(n,n)$ is.

Another question: Assume that the edges of the complete graph with n vertices are colored with red and blue. Can we find a monochromatic subgraph of $\lceil 3n/4 \rceil$ vertices which has diameter 2? Author proved a little weaker statement that we can find one color and a subset A of at least $\lceil 3n/4 \rceil$ vertices such that for any two distinct vertices x, y of A there is a path of length at most two in this color with endpoints x, y.

The third problem is also Ramsey-type. Assume that the edges of the complete graph with n vertices are colored with red and blue and let $f(k)$ be the smallest n for which you can find two edge disjoint monochromatic complete subgraphs on k vertices. Author of the article proved that $f(4) = 19 = R(4) + 1$.

The last question of this article is about colorability of graphs. If each even cycle of a graph spans a bipartite subgraph then the graph is 3-colorable.

Petr Hliněný and Jan Kratochvíl

Computational Complexity of the Krausz Dimension of Graphs

Introduction

A *Krausz partition* of a graph $\boldsymbol{G}$ is a partition of the edge set $E(\boldsymbol{G})$ into complete subgraphs (that are also called the *clusters* of the partition). The number of clusters containing a vertex v is called the *order* of v (in the Krausz partition). The order of the partition is the maximum order over all vertices of $\boldsymbol{G}$. The *Krausz dimension* of $\boldsymbol{G}$ is defined as the minimum partition order over all Krausz partitions of $\boldsymbol{G}$, and denoted by $dim(\boldsymbol{G})$. Note that if $\boldsymbol{G}$ is not connected, its dimension is the maximum dimension over all of its components; so we consider only connected graphs in our paper.

Every graph can be partitioned just by taking each edge alone as a cluster — thus for every graph, $dim(\boldsymbol{G}) \leq \Delta(\boldsymbol{G})$. This bound is optimal for triangle-free graphs. On the other hand, complete graphs have dimension 1. Another important class of graphs, for which the Krausz dimension is known is described by Krausz's characterization of line graphs [6], which in fact inspired the notion of the Krausz dimension:

Theorem 1 *(Krausz) The Krausz dimension of a graph is at most* 2 *if and only if this graph is a line graph.*

Our paper studies the computational complexity of the Krausz dimension for various classes of graphs. The general problem of determining the dimension of a given graph is denoted by *KrauszDim*, the question whether the dimension is at most k is denoted by *KrauszDim*(k), and the same question restricted to graphs with maximum degrees at most d is denoted by $KrauszDim(k,d)$. These decision forms of the Krausz dimension problem clearly belong to NP.

Computational complexity

We summarize the main results of our paper in this section. We first consider general graphs and the relation between the maximum degree of a graph and its Krausz dimension.

Theorem 2 **(a)** *The problem* $KrauszDim(3,4)$ *is solvable in polynomial time.*
(b) *The problem* $KrauszDim(3,5)$ *is NP–complete, even when restricted to planar graph.*

Next we pay closer attention to various classes of chordal graphs. It may seem somewhat surprising that Krausz dimension remains difficult even for chordal graphs, though here we are only able to prove the hardness result for larger dimension:

Theorem 3 *The problem* $KrauszDim(6)$ *is NP–complete for chordal graphs.*

However, we have several results showing that special classes of chordal are easier:

Theorem 4 **(a)** *The problem* $KrauszDim$ *is polynomial for chordal graphs with bounded maximal clique size, and in particular for chordal graphs of bounded maximum degree.*
(b) *For any fixed* D*, the problem* $KrauszDim(D)$ *is polynomial for interval graphs.*

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[5] T. Kloks, Treewidth, computations and approximations, Lecture Notes in Computer Science, Springer-Verlag 1994.

[6] J. Krausz, *Démonstration nouvelle d'un théorème de Whitney sur les résaux* (Hungarian with French summary), Mat. Fiz. Lapok 50 (1943), 75–85.

Zuzana Honzíková

Pseudo-Interval Graphs

Paper by: E. O. Brauner, R. A. Brualdi, E. S. N. Sneyd
From: Journal of Graph Theory 20, No. 3 (1995) 309-318

In the lecture, I have presented these parts of the article:

Pseudo-interval graphs (PI-graphs) generalize interval graphs; their vertices correspond to intervals of a linearly ordered set, but a modified definition of intersection is used to determine edges. The complements of PI-graphs are comparability graphs; PI-graphs are weakly trinagulated (they do not contain induced circles of length k, or their complements, for any $k > 4$). Hence PI-graphs are perfect.

I have mentioned (without proofs) these results:

A characterization of trees and their complements which are PI-graphs. A table of all non-PI-graphs on eight or fewer vertices which are weakly triangulated and the complements of which are comparability graphs.

Matěj Maxa

The Geometry of Coin-weighing Problem

Paper by: N.Alon, D.N. Kozlov, V.H. Vu
(preprint)

Given a set of m coins out of a collection of coins of k unknown distinct weights, we wish to decide if all the m given coins have the same weight or not using the minimum possible number of weighings in a regular balance beam. Let $m(n,k)$ denote the maximum possible number of coins for which the above problem can be solved in n weighings. We show that

$$m(n,2) = n^{(1/2+o(1))n}$$

The proof has an interesting geometric flavour, and combines Linear Algebra techniques with geometric, probabilistic and combinatorial arguments.

Helena Nyklová

A New Idea for Hamiltonian Problem

Paper by: Z. M. Song, K. M. Zhang
From: Ars Combinatoria 43 (1996) 97-105

Let G be a 2-connected graph of order n with the connectivity k and the independence number α. In this paper, we show that if for each independent set S with $|S| = k+1$, there are $u, v \in S$ such that satisfying one of following conditions:

1. $d(u) + d(v) \geq n$; or $|N(u) \cap N(v)| \geq \alpha$; or $|N(u) \cup N(v)| \geq n - k$;

2. for any $x \in \{u, v\}, y \in V(G)$ and $d(x, y) = 2$, it implies that $max\{d(x), d(y)\} \geq n/2$,

then G is hamiltonian. This result reveals the internal relations among several well-known sufficient conditions: (1) it shows that it does not need to consider all pair of nonadjacent or distance two vertices in G; (2) it makes known that for the different pairs of vertices in G, it permits to satisfy the different condition.

Ondřej Pangrác

Graphs Drawn with Few Crossings per Edge

Paper by: J. Pach, G. Toth
(preprint)

We show that if graph of v vertices is drawn in the plane so that every edge crosses at most k others then the number of edges $e \leq 4.108\sqrt{k}v$. For $0 \leq k \leq 4$ we have a better bound: $e \leq (k + 3)(v - 2)$ and it is tight for $k = 0, 1, 2$. We also apply these estimates to provide a general lower bound for the crossing number of a graph in terms of its number of vertices and edges and another possibilities of application of these results.

Cyril Sochor

Games, Randomness and Algorithms

Paper by: József Beck
(preprint)

This text is about generalized Tic-Tac-Toe games. There are two players and one board (for example chess board) and winning sets (of fields). The players take turns; they alternatively take one free field. He wins who first reaches a winning set. This category of games includes for example: Ramsey game (the board is a large complete graph on N vertices and winning sets are complete subgraphs on $M < N$ vertices), Van der Waarden game (the board are all integers less then N and winning sets are all arithmetic progressions of M terms) and Hales-Jewett game (the board is d-dimensional cube of size N and winning sets are all lines of size N).

In this text, a strategy-stealing argument is obtained which says that the second player cannot win. There are two strategies for the second player (pairing strategy and better strategy used an exponential weight function). With these strategies the second player under some conditions (the wining sets are large) can force a draw. When can the first player win? It is less difficult to study the Maker-Breaker version (the first player (Maker), wins, when he succeeds in constructing any wining set, while the second player (Breaker) wins, when he manages to prevent that). The Maker-Breaker version is easier to study, because the Maker doesn't need to check against Breaker's making his own winning set, and he can therefore concentrate on his own winning sets only. With weight function strategy can Maker force a win (under some conditions).

József Solymosi

Mathematical Institute of the Hungarian Academy of Sciences

A simple counting method

In this short note we shall prove a theorem joint with J. Pach [PS], concerning the ε–straight lines in a point set of the d–dimensional Euclidean space. In our proof we shall use a counting method which is very useful in many situations where one is working with special geometrical structures.

In their classical paper written in 1935, Erdös and Szekeres, [ES] (see also [EA]) proved that for every $r \geq 3$, there exists an integer $f(r)$ such that any set of at least $f(r)$ points in the plane has r elements in convex position. (A set of points is said to be in a *convex* position if all the points are on the convex hull.) This result has inspired many results in combinatorial geometry and in Ramsey theory (see e.g. [BDV], [GRS], [H], [PA], [V]).

It follows that if n is much larger than $f(r)$, then every n-element point set P contains many r-tuples in convex position. For instance, one can prove that for a suitable constant $c_r > 0$, one can select a sequence of $c_r n$ distinct elements from P, whose any r consecutive members are in convex position [S]. In the case $r = 4$, Nielsen [N] and, in general, Bárány and Valtr [BV] proved the following stronger result.

Theorem A. *For any fixed $r \geq 4$, there is a constant $c_r > 2^{-2^{6r}}$ satisfying the following condition. Every n-element point set P in general position in the plane has r pairwise disjoint subsets P_i $(1 \leq i \leq r)$ such that $|P_i| \geq \lfloor c_r n \rfloor$ and choosing one point from each P_i, arbitrarily, they are in convex position.*

The method used to prove Theorem 1 below would also give a simple and easy proof of the Theorem A, also providing better constants. (We will not prove Theorem A here; the reader can try to modify the proof of Theorem 1 to get it. In [PS] we shall use this very method to get a more general result.)

Definition. A polygonal path $p_1p_2\dots p_r$ in the plane or in space, is called *ε–straight* if $\angle p_{i-1}p_ip_{i+1} > \pi - \varepsilon$, $1 < i < r$ (cf. [ES], [P]). The *length* of a polygonal path is the number of its *vertices.*

Theorem 1. *For every $d \geq 2, r \geq 3$ and $\varepsilon > 0$, there exists a positive constant $c = c^d_{r,\varepsilon}$ with the following property.*

Every n-element point set P in general position in Euclidean d-space has r pairwise disjoint subsets P_i $(1 \leq i \leq r)$ with at least $\lfloor cn \rfloor$ elements such that no matter how we pick a point from each P_i, they always form an ε-straight polygonal path.

To establish Theorem 1, we need the following straightforward generalization of a result from [ES].

Lemma 1.1. *There exists a constant $c > 0$ such that any set of at least $k^{(c/\varepsilon)^{d-1}}$ points in Euclidean d-space has k elements that form an ε-straight polygonal path of length k.*

Proof of Theorem 1. Let ε, d, and r be fixed, and set $k = 2r - 1$. By Lemma 2.1, there exists an integer $K = K(\varepsilon, d, r)$ such that any set of K points in d-space contains k elements that form the vertex set of an $\varepsilon/3$-straight polygonal path Π. Notice that if we skip every other vertex of P, then we obtain a polygonal path Π' with r vertices, which is ε-straight. The sequence formed by the $r-1$ vertices we skipped is called the *support* of Π.

Consider now any set P of n points in the plane. Clearly, P contains at least

$$\binom{n}{K} \Big/ \binom{n-k}{K-k} = \binom{n}{k} \Big/ \binom{K}{k}$$

different $\varepsilon/3$-straight polygonal paths of length k, and at least

$$\frac{\binom{n}{k}}{\binom{K}{k}} \frac{(n-r+1)!}{n!} > \frac{n^r}{K^{2r-1}}$$

of them must share the same support S.

Let P_i denote the set of all elements of P that occur as the $(2i-1)$-st vertex in some $\varepsilon/3$-straight polygonal path of length k, whose support is S $(1 \leq i \leq r)$. These sets meet the requirements of the theorem, and for every i, we have

$$|P_i| > \frac{n^r/K^{2r-1}}{\prod_{j \neq i} |P_j|} > n/K^{2r-1}. \qquad \blacksquare$$

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Robert Šámal

The Growth Rate of Vertex-transitive Planar Graphs

Paper by: László Babai
From: Proceedings of ACM-SIAM Symposium
on Discrete Algorithms, ACM 1997, 564-573

A graph is *vertex-transitive* if all of its vertices are equivalent under automorphisms. Confirming a conjecture of Jon Kleinberg and Éva Tardos, we prove the following trichotomy theorem concerning locally finite vertex-transitive planar graphs: the *rate of growth of a graph with these properties is either linear or quadratic or exponential.* The same result holds more generally for locally finite, *almost* vertex-transitive planar graphs (the automorphism group has a finite number of orbits). The proof uses the elements of hyperbolic plane geometry.

The first step of the proof uses the notion of ends of graph. It can be easily proved that the graphs under consideration can have either one or two or infinitely many ends. The case of two and infinitely many ends is quickly solved, the corresponding growth rates are linear and exponential, respectively.

The case of one end is the difficult one. The main idea is to embed the graph into an appropriate plane (Euclidean or some hyperbolic one) in such a way, that every face is a regular polygon with unit sidelength. The rate of growth of the graph is the same as the growth rate of the area of discs as a function of the radius, i.e., quadratic in the Euclidean plane and exponential in the hyperbolic planes.

Petra Špičková

Juggling Drops and Descents

Paper by: J. Buhler, D. Eisenbud, R. Graham, C. Wright
From: American Mathematical Monthly 101, No. 6 (1994)

There is described a juggling pattern in the first section of the article. It is supposed that b balls are thrown in a constant rhythm. The times t of throws are identified with integers and only one ball is thrown at any given time. A juggling pattern is a permutation $f : Z \to Z$ such that $f(t) \geq t$ for all $t \in Z$ (it corresponds to the time of next throw of the ball which is thrown at time t). The height function of a juggling pattern is $df(t) = f(t) - t$ (it corresponds to the height to which balls must be thrown).

The first result says that if the throw height is bounded, then the number of balls (defined as the number of infinite sets determined by f) is finite and can be calculated as the average value of the throw heights over large intervals.

In the second section some results about periodic juggling patterns are proved. A periodic-n juggling pattern is a juggling pattern with the height function satisfying $df(t + n) = df(t)$ for all $t \in Z$. The main theorem says that the number of periodic-n juggling patterns with fewer than b balls (where cyclic shifts are not counted as distinct) is b^n. The given proof is based on some properties of permutations on n elements and it is simple and elegant. By the theorem there are $(b+1)^n - b^n$ patterns of period n with exactly b balls if cyclic shifts are counted as distinct. The number $M(n, b)$ of patterns of period n with exactly b balls, where cyclic shifts are not counted as distinct, can be obtained by Möbius inversion from

$$(b+1)^n - b^n = \sum_{d|n} dM(d, b)$$

Jana Turzíková

Cycle and Path Covers of Graphs

Paper by: Karen Seyffarth
Doctoral thesis, Waterloo, Ontario, Canada 1989

Theorem. *(H. Li) Every simple graph G has a perfect path double cover.* (PPDC is a collection of paths such that each edge lies in exactly two of the paths, and each vertex appears precisely twice as an endpoint of the paths.)

We have proved the theorem by induction on $|E(G)|$. Let $e = (wv)$ be an edge of G. Set $G' = G \setminus e$. G' has a PPDC. Let $u_1, ..., u_k$ denote the neighbours of v in G'. We have constructed a special digraph D on a vertex set $\{u_1, ..., u_k, v, w\}$ and applied the following lemma:

Lemma. *Let D be a digraph with vertices $V(D) = \{u_1, ..., u_k, v, w\}$, where $deg_+(w) - deg_-(w) > \sum_{i=1,...,k} \max\{deg_-(u_i) - deg_+(u_i); 0\}$. Then D contains a directed path from w to v.*

In a similar way we have constructed a digraph D' on a vertex set $\{\bar{u}_1, .., \bar{u}_l, w, v\}$, (where $\bar{u}_1, ..., \bar{u}_l$ are the all neighbours of w in G') and applied the lemma again.

Finally we have modified the PPDC of G' to a PPDC of G.

Jan Vondrák

Dense Graphs and Edge Reconstruction

Paper by: L. Pyber
From: Combinatorica 16, No. 4 (1996) 521-525

A graph G on n vertices with m edges is edge reconstructible if it can be reconstructed from the following m graphs: for each edge e of the original graph, we get a graph which contains all edges of G except e, and its vertices are re-numbered arbitrarily.

By a well-known result of Nash-Williams, if G is not edge reconstructible, then for any subset of edges A, which differs from $E(G)$ in an even number of edges, there is a permutation f of the vertices of G such that $E(G) \cap E(Gf) = A$.

This lemma immediately implies Muller's previous result, that any graph with at least $n \log n$ edges is edge reconstructible. However, in this article a bipartite graph is constructed for any sufficiently large n, which has n vertices, more than $n/2 * \sqrt{\log n}$ edges, and satisfies the conditions of the Nash-Williams lemma. This doesn't imply that this graph is not edge reconstructible, but it suggests that the Nash-Williams lemma cannot provide a much better bound on the number of edges of graphs which are not edge reconstructible than $n \log n$. To prove, for example, than graphs with at least cn edges (for some large c) are edge reconstructible, essentially new methods would have to be developed.

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Zero-sum Ramsey numbers for graphs (and hypergraphs)

Zero-sum Ramsey theory is a variant of the classical Ramsey theory. In general zero-sum Ramsey theory problems can be formulated like this:

Take a combinatorial structure and map its elements into the additive group modulo k ($\mathbf{Z}_k$). The question is:

Does this structure contain a prescribed substructure with the property that the sum of the "colours" of its elements is 0 (in $\mathbf{Z}_k$)? (Let us call this property *the zero-sum property.*)

Normally the problem will appear the other way around:

Given a substructure, how "big" must the supreme structure be to make sure that for arbitrary colouring of its elements it contains a copy *of the prescribed substructure which has the zero-sum property?*

Let me give an example:

One of the first contributions to combinatorial zero-sum theory was the *Erdös-Ginzburg-Ziv theorem (1961)*:

Theorem 1. *Let $m \geq k \geq 2$ be two integers and suppose $k|m$. Then any sequence of $m+k-1$ integers contains a subsequence of cardinality m with the property that the sum over these elements is divisible by k.*

If the word integer in this theorem is replaced by "element of $\mathbf{Z}_k$" the statement of the theorem appears in the above mentioned form. (Please note that the assumption k divides m is important because otherwise a sequence of $(m+k-1)$-many ones would be a counterexample.)

The problem underlying the EGZ-Theorem can be formulated more generally:

Problem 1. *Let $m \geq k \geq 2$ be two integers and suppose $k|m$. Furthermore let H be an r-uniform hypergraph having n vertices and m edges. How many vertices must a complete r-uniform hypergraph (whose edges are coloured with elements of $\mathbf{Z}_k$ have to ensure that it contains a hypergraph isomorphic to H which has the zero-sum property?*

For a fixed hypergraph H and additive group $\mathbf{Z}_k$ the smallest number of vertices which meets this requirement is called the zero-sum Ramsey number $R(H, \mathbf{Z}_k)$. In other words:

Definition 1. The *zero-sum Ramsey number* $R(H, \mathbf{Z}_k)$ is the smallest integer t such that in any $\mathbf{Z}_k$-colouring of the edges of the complete r-uniform hypergraph on t vertices $K_t^{(r)}$ there exists a zero-sum copy of H.

(Please note that the degenerated case $r = 1$ corresponds to the statement of the EGZ-Theorem.)

The existence of the zero-sum Ramsey numbers follows from the existence of the classical Ramsey numbers. More generally speaking, the zero-sum Ramsey numbers can be estimated by $R(H, \mathbf{Z}_k) \leq R(H, k)$.

Contrary to the classical Ramsey numbers zero-sum Ramsey numbers do not share the *monotonicity property*: $G \subset H \Rightarrow R(G, k) \leq R(H, k)$.

For $r = 2$ (simple graphs) $R(C_4, \mathbf{Z}_2) = 4 < 5 = R(2K_2, \mathbf{Z}_2)$ but $2K_2 \subset C_4$ is a simple counterexample.

Until now, only a few particular facts about r-uniform hypergraphs (with arbitrary r) are known.

For the special case $r = 2$ quite a number of particular results have been found and (what is more interesting) there are also some more general results.

In 1992 Yair Caro from the university of Haifa (Israel) published *the zero-sum Ramsey numbers for stars* (Discrete Mathematics 104 (1992), p.1-6):

Theorem 2. *Let $m \geq k \geq 2, k|m$. Then*

$$R(K_{1,m}, \mathbf{Z}_k) = \begin{cases} m+k-1 & \text{if } m \equiv k \equiv 0 \ (mod\ 2) \\ m+k & \text{otherwise.} \end{cases}$$

A second more general result is also due to Caro (J. Comb. Theory Ser. A 68 (1994), p.205-211):

A complete characterization of the zero-sum Ramsey numbers (mod 2):

Theorem 3. *Let G be a graph on n vertices and an even number of edges. Then*

$$R(G, \mathbf{Z}_2) = \begin{cases} n+2 & \text{if } G = K_n, n \equiv 0,1 \ (mod\ 4) \\ n+1 & \text{if } G = K_p \cup K_q, \binom{p}{2} + \binom{q}{2} \equiv 0 \ (mod\ 2) \\ & \text{or if all degrees are odd} \\ n & \text{otherwise.} \end{cases}$$

Finally a little result of my own:

Theorem 4. *Let $G = (V, E)$, $n = v(G)$, $3|e(G)$ be a graph with at least two edges of degree 1 without a common vertex. Then*

$$R(G, \mathbf{Z}_3) = \begin{cases} n+2 & \text{if all degrees} \equiv 1 \ (mod\ 3) \\ n+1 & \text{if there are degrees} \equiv 1,2 \ (mod\ 3) \\ & \text{but not} \equiv 0 \ (mod\ 3) \\ & \text{or if exactly one degree} \equiv 0 \ (mod\ 3) \\ & \text{and the remaining degrees} \\ & \text{are pairwise congruent} \\ n, n+1 & \text{otherwise.} \end{cases}$$

Robert Šámal, Jan Vondrák

The Limit Checker Number of a Graph

Introduction

The Checker Number of a graph was introduced by Joseph Zaks in [1]. For a graph G, the checker number $\mathrm{CN}(G)$ is defined by the result of the following game:

1. We start with one pebble at each vertex of the graph
2. The allowed move is: we take a pebble at vertex u and jump over its neighbor v; the neighboring pebble v is removed and our pebble arrives at another vertex w which is adjacent to v. There must be at least one pebble at both u and v; negative pebble counts are not allowed, but there can be more than one pebble at one vertex.
3. The goal of the game is to accumulate as many pebbles as possible at a given vertex; the checker number is then such a number of pebbles that can be accumulated at any vertex of the graph. We denote this number $\mathrm{CN}(G)$. Explicitly denote by $\mathrm{CN}(G,x)$ the largest number of pebbles that can be gathered at vertex x. Then $\mathrm{CN}(G) = \min_{x\in G} \mathrm{CN}(G,x)$.

To get an upper bound for $\mathrm{CN}(G)$, we define a "weight" of the graph:

$$\begin{aligned} \mathrm{W}(G,x) &= \sum_{v\in G} \sigma^{d(x,v)} \\ \mathrm{W}(G) &= \min_{x\in G} \mathrm{W}(G,x), \end{aligned}$$

where $d(x,v)$ is the distance between x and v in G and σ is the positive solution of the quadratic equation $\sigma^2+\sigma-1=0$ ($\sigma \doteq 0.618$).

We can also extend this definition to reflect the current state of the game by counting $\sigma^{d(x,v)}$ not over vertices, but over all pebbles with their respective distances from the goal vertex x. Then the weight of the final state is at least the number of pebbles in the desired position (distance $= 0$), while the weight of the initial state is exactly $\mathrm{W}(G,x)$. But due to the careful choice of σ, the weight can only decrease or stay constant during the game. So we get an inequality $\mathrm{CN}(G,x) \leq \mathrm{W}(G,x)$ and therefore

$$\mathrm{CN}(G) \leq \mathrm{W}(G).$$

However, the weight of the graph doesn't provide any lower bound for $\mathrm{CN}(G)$; for example if G is a star with n vertices, then $\mathrm{W}(G) = 1 + \sigma + (n-2)\sigma^2$, but $\mathrm{CN}(G) = 2$. So for arbitrarily large $\mathrm{W}(G)$, $\mathrm{CN}(G)$ can still be constant.

We study the checker number of graphs generated as cartesian products of G; we shall see that the structure of these graphs makes it possible to exploit the graph weight much more efficiently.

The following facts have been proven by Joseph Zaks.

Claim.

1. $\mathrm{CN}(G \square H) \geq \mathrm{CN}(G)\mathrm{CN}(H)$
2. $\mathrm{W}(G \square H) = \mathrm{W}(G)\mathrm{W}(H)$

Proof. 1. Given a vertex (x,y) in $\mathrm{CN}(G \square H)$, we can simply apply the process leading to $\mathrm{CN}(G)$ pebbles at x in each copy of G, and then play the game in the x-copy of H, $\mathrm{CN}(G)$-times; each time getting $\mathrm{CN}(H)$ pebbles at (x,y).

$$2. \quad \begin{aligned} \mathrm{W}(G \square H, (x,y)) &= \sum_{v \in G, w \in H} \sigma^{d((x,y),(v,w))} = \sum_{v,w} \sigma^{d(x,v)+d(y,w)} \\ &= \sum_{v} \sigma^{d}(x,v) \sum_{w} \sigma^{d}(y,w) = \mathrm{W}(G,x)\mathrm{W}(H,y) \end{aligned}$$

Now consider the n-th power of G, $G^n = G\Box G\Box\ldots\Box G$; we know that

$$\mathrm{CN}(G)^n \leq \mathrm{CN}(G^n) \leq \mathrm{W}(G^n) = \mathrm{W}(G)^n,$$

i.e. the checker number grows at least as the n-th power of $\mathrm{CN}(G)$, but at most as the n-th power of $\mathrm{W}(G)$; so it seems reasonable to study the behaviour of the n-th root of $\mathrm{CN}(G^n)$, which we shall denote by $\mathrm{C}(n)$. Also the following definition resembles the Shannon capacity of a graph (see [2]).

Definition. Our object of interest will be the limit of $\mathrm{C}(n)$ (if it exists), which we call the *limit checker number*:

$$\begin{aligned} \mathrm{LCN}(G) &= \lim_{n\to\infty} \mathrm{C}(n) \\ \mathrm{C}(n) &= \sqrt[n]{\mathrm{CN}(G^n)} \end{aligned}$$

By rewriting the equation above we get:

$$\mathrm{CN}(G) \leq \mathrm{C}(n) \leq \mathrm{W}(G)$$

The following is the main result of this paper.

Theorem. *For any graph G,* $\quad LCN(G) = W(G)$.

First we'll prove two Lemmas.

Lemma 1. *$LCN(G)$ always exists and $LCN(G) = \sup C(n)$.*

Proof. For any $m \geq kn$:

$$\mathrm{CN}(G^m) \geq \mathrm{CN}(G^{kn}) \geq \mathrm{CN}(G^n)^k.$$

If $kn \leq m < (k+1)n$,

$$\mathrm{C}(m) = \sqrt[m]{\mathrm{CN}(G^m)} \geq \sqrt[(k+1)n]{\mathrm{CN}(G^n)^k} = \mathrm{CN}(G^n)^{\frac{k}{n(k+1)}} = \mathrm{C}(n)^{k/(k+1)}.$$

As $\lim_{k\to\infty} \mathrm{C}(n)^{k/(k+1)} = \mathrm{C}(n)$, for any $\varepsilon > 0$, there is a sufficiently large k, so that:

$$\begin{aligned} \text{for all } m \geq kn: \qquad \mathrm{C}(m) &\geq \mathrm{C}(n) - \varepsilon, \\ \text{therefore} \qquad \liminf \mathrm{C}(m) &\geq \mathrm{C}(n) \end{aligned}$$

This holds for any n, thus we get:

$$\liminf \mathrm{C}(m) \geq \sup \mathrm{C}(n) \geq \limsup \mathrm{C}(n)$$

which implies $\lim \mathrm{C}(m) = \sup \mathrm{C}(n)$.

Corollary. $CN(G) \leq LCN(G) \leq W(G)$

Lemma 2. *Unless $CN(G) = 1$, there is a constant $\alpha > 0$, so that for any $n \geq 1$:*

$$CN(G^{n+\lfloor \alpha n \rfloor}) \geq \frac{\sigma}{2} CN(G^n)\, W(G^{\lfloor \alpha n \rfloor})$$

Proof. As $\mathrm{CN}(G) \geq 2$, we can choose $\alpha > 0$ so small that

$$\mathrm{CN}(G)\sigma^{\alpha D} > \tau,$$

where D is the diameter of G (the largest distance of two vertices), and $\tau = 1/\sigma \doteq 1.618 < 2$ is the golden ratio.

Now we perform the following action: considering $G^{n+\lfloor \alpha n \rfloor}$ as the cartesian product of $H = G^n$, and $K = G^{\lfloor \alpha n \rfloor}$, we can move $\mathrm{CN}(H)$ pebbles to a selected vertex v in each copy of H. However, instead of this, by omitting some of the final moves arriving at v, we can distribute the $\mathrm{CN}(H)$ pebbles arbitrarily between v and its neighbors. We can do this in such a manner that we get at least $\lfloor \sigma \mathrm{CN}(H) \rfloor$ pebbles at v, and $\lfloor \sigma^2 \mathrm{CN}(H) \rfloor$ pebbles in its neighborhood (because $\sigma + \sigma^2 = 1$). Then we move with these two groups of pebbles along the shortest path towards the goal vertex x in K, jumping over each other alternately. By induction, we'll get at least $\lfloor \sigma^{d(x,v)+1} \mathrm{CN}(H) \rfloor$ pebbles at x:

let's suppose we have $\lfloor \sigma^{n+1} \mathrm{CN}(H) \rfloor$ and $\lfloor \sigma^{n} \mathrm{CN}(H) \rfloor$ pebbles at two consecutive vertices along the path. By jumping with the first group over the second, we get $\lfloor \sigma^{n+1} \mathrm{CN}(H) \rfloor$ pebbles at the next vertex, and

$$\lfloor \sigma^{n} \mathrm{CN}(H) \rfloor - \lfloor \sigma^{n+1} \mathrm{CN}(H) \rfloor \geq \lfloor (\sigma^{n} - \sigma^{n+1}) \mathrm{CN}(H) \rfloor = \lfloor \sigma^{n+2} \mathrm{CN}(H) \rfloor$$

at the middle vertex; so with each step along the path, the exponent of σ increases by 1, and finally we'll get

$$\lfloor \sigma^{d(x,v)+1} \mathrm{CN}(H) \rfloor \geq \frac{\sigma}{2} \sigma^{d(x,v)} \mathrm{CN}(H)$$

pebbles at x. The rounding operation cannot decrease the number of pebbles more than by a factor of 2, because

$$\sigma^{d(x,v)+1}\mathrm{CN}(H) \geq \sigma^{\alpha nD+1}\mathrm{CN}(G)^n = \sigma(\sigma^{\alpha D}\mathrm{CN}(G))^n \geq \sigma\tau^n \geq 1.$$

In total, we get

$$\mathrm{CN}(H\Box K) \geq \frac{\sigma}{2}\mathrm{CN}(H)\sum_v \sigma^{d(x,v)} = \frac{\sigma}{2}\mathrm{CN}(H)\mathrm{W}(K).$$

Theorem. *For any graph G, $\quad LCN(G) = W(G)$.*

Proof. We can assume $\mathrm{CN}(G) > 1$, because if G is a single vertex, the equation is trivial, and if G contains at least one edge, we'll prove the theorem for G^2 ($\mathrm{CN}(G^2) > 1$), which implies

$$\mathrm{LCN}(G) = \sqrt{\mathrm{LCN}(G^2)} = \sqrt{\mathrm{W}(G^2)} = \mathrm{W}(G).$$

Now let's suppose $\mathrm{LCN}(G) < \mathrm{W}(G)$, i.e.

$$\mathrm{LCN}(G) = \omega\mathrm{W}(G), \quad 0 < \omega < 1$$

Then we'll find n such that:

$$\mathrm{C}(n) > \mathrm{LCN}(G)\sqrt[n]{\frac{2}{\sigma}\omega^{\lfloor \alpha n\rfloor}},$$

where $\mathrm{C}(n) = \sqrt[n]{\mathrm{CN}(G^n)}$, and α is the constant required in Lemma 2. This is possible, because:

$$\lim \sqrt[n]{\frac{2}{\sigma}\omega^{\lfloor \alpha n\rfloor}} = \omega^\alpha < 1,$$

$$\text{so} \qquad \mathrm{LCN}(G)\sqrt[n]{\frac{2}{\sigma}\omega^{\lfloor \alpha n\rfloor}} < \mathrm{LCN}(G) - \varepsilon,$$

$$\text{whereas} \qquad \mathrm{C}(n) > \mathrm{LCN}(G) - \varepsilon \qquad \text{for } n \text{ sufficiently large.}$$

So we have:

$$\begin{aligned}\mathrm{CN}(G^n) &= \mathrm{C}(n)^n > \mathrm{LCN}(G)^n \frac{2}{\sigma}\omega^{\lfloor \alpha n \rfloor} \\ &= (\omega \mathrm{W}(G))^n \frac{2}{\sigma}\omega^{\lfloor \alpha n \rfloor} \\ &= \frac{2}{\sigma}\mathrm{W}(G^n)\omega^{n+\lfloor \alpha n \rfloor}\end{aligned}$$

Using Lemma 2 for this n yields:

$$\begin{aligned}\mathrm{CN}(G^{n+\lfloor \alpha n \rfloor}) &\geq \frac{\sigma}{2}\mathrm{CN}(G^n)\mathrm{W}(G^{\lfloor \alpha n \rfloor}) \\ &> \mathrm{W}(G^n)\omega^{n+\lfloor \alpha n \rfloor}\mathrm{W}(G^{\lfloor \alpha n \rfloor}) \\ &= (\mathrm{W}(G)\omega)^{n+\lfloor \alpha n \rfloor} = (\mathrm{LCN}(G))^{n+\lfloor \alpha n \rfloor}\end{aligned}$$

i.e. $\mathrm{C}(n + \lfloor \alpha n \rfloor) > \mathrm{LCN}(G)$, which, however, contradicts the fact that $\mathrm{LCN}(G) = \sup \mathrm{C}(n)$.

References

[1] Joseph Zaks: The Checker Number of a Graph, 1996 (manuscript)

[2] Lászlo Lovász: On the Shannon Capacity of a Graph, IEEE Trans. Inform. Theory 25, 1–7 (1979)

Písmenná polodokreslovka se skrytou tajenkou (8 písmen)

	1	2	3	4	5	6	7	8	9	10
A										
B										
C										
D										
E										
F										
G										
H										
I										
J										

VODOROVNĚ: **A.** Spojka (neboť); prostorná místnost; ekonomka (zkr.); **B.** Čisticí prášek (zn.); východosibiřská řeka; český herec (Jiří); **C. První tajenka**; **D.** Rozsáhlé prozaické dílo; slov. pádová otázka; pohybovat se vozidlem; **E.** sin/cos; překvapení; značka vysavačů; **F.** Zařídit (rébusově); krtci (zast.); SPZ Ostravy; **G.** Slovensky “jaké”; velké pole; velká schůze; **H. Druhá tajenka**; **I.** Zámezí (ve fotbale i jinde); 9; 1101; **J.** 3; islámské rodové tradice; anglicky “olej”;

SVISLE: **1.** Dostat se jízdou na něco; židovská sobota; **2.** Část cyklistického závodu; jeden dým; **3.** Šlechtic; asfalt; ukazovací zájmeno; **4.** Německé mužské jméno (Jan); pukrle; **5.** Američan; Tamil; **6.** Snažit se o zaplnění křížovky písmeny; pitomá ženská; **7.** Patřící Armandovi (domácky); sjezd sokolů; **8.** 45; tyče u vozu; oním směrem; **9.** Zuřit; matematici; **10.** Tiskové chyby s chybou; smalt (nebo druh pošty);

Nezapomeňte nalézt skrytou tajenku !!!