A Theory Of Pfaffian Orientations II: T-joins, Edge-Cuts, And A Duality of Enumeration.

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Abstract

This is a continuation of our paper 'A Theory of Pfaffian Orientations I'. We present a new combinatorial way to compute the generating functions of T-joins and edge-cuts of graphs. As a consequence we show that the computational problem to find the size of a maximum edge-cut is polynomially solvable for the graphs which may be drawn on an arbitrary fixed orientable surface. We also survey the related results concerning a duality of the Tutte polynomial and we give interesting expressions for the chromatic polynomial and the flow polynomial of a graph. In a continuation of this paper which is in preparation we present an application to the Ising problem of three-dimensional crystal structures.

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1 Introduction

We start by repeating basic notions and theorems of [3]. $G = (V, E)$ will always be a graph and x_e will be a variable associated with each edge e of G. We let $x = (x_e; e \in E)$ be the vector of the variables associated with the edges of G. For $M \subset E$ let $x(M)$ denote the product of the variables of the edges of M. An *orientation* of graph $G = (V, E)$ is a digraph obtained from G by giving an orientation to each edge of G.

Let $A \Delta B$ denote the symmetric difference of the sets A and B.

Definition 1.1 *The generating function of the perfect matchings of G is polynomial $\mathcal{P}(G, x)$ equal to the sum of $x(P)$ over all perfect matchings P of G.*

Given a graph $G = (V, E)$, an eulerian subgraph of G is any graph $G' = (V, E')$ with $E' \subset E$ and all degrees even.

Definition 1.2 *The generating function of eulerian subgraphs of G is polynomial $\mathcal{E}(G, x)$ equal to the sum of $x(U)$ over all eulerian subgraphs U of G.*

Given a graph $G = (V, E)$ and $T \subset V$, a T-join is any graph $G' = (V, E')$ such that $E' \subset E$ and the degree of a vertex v of G' is odd iff $v \in T$. Eulerian subgraphs are T-joins for $T = \emptyset$.

Definition 1.3 *Let $G = (V, E)$ be a graph and $T \subset V$. The generating function of T-joins of G is polynomial $\mathcal{T}_T(G, x)$ equal to the sum of $x(W)$ over all T-joins W of G.*

Next we consider multicuts.

Definition 1.4 *Let $k \geq 1$ and let $G = (V, E)$ be a graph. A pair $(\{V_1, ..., V_k\}, E')$ is called k-cut if $\{V_1, ..., V_k\}$ is a partition of V into k non-empty disjoint subsets and E' is the set of all edges with the end-vertices in different parts V_i, $i = 1, ..., k$.*

Definition 1.5 *The generating function of k-cuts is polynomial $\mathcal{C}_k(G, x)$ equal to the sum of $x(C)$ over all k-cuts $(\{V_1, ..., V_k\}, C)$ of G.*

Definition 1.6 *Let G be a graph and let D be an orientation of G. Let M be a perfect matching of G. For each perfect matching P of G let $sgn(D, M\Delta P) = (-1)^n$ where n equals to the number of clockwise even alternating cycles of $M \cup P$ and let $\mathcal{P}(D, M)$ equals the sum of $sgn(D, M\Delta P)x(P)$ over all perfect matchings P of G.*

Definition 1.7 *Let G be a graph and D be an orientation of G. Denote by $A(D)$ the skew-symmetric matrix with the rows and the columns indexed by $|V(G)| = 2n$, where $a_{vw} = x_{(v,w)}$ in case (v,w) is an arc of D, $a_{vw} = -x_{(v,w)}$ in case (w,v) is an arc of D, and $a_{vw} = 0$ otherwise.*

The Pfaffian of the skew-symmetric matrix $A(D)$ is defined as

$$Pf(A(D)) = \sum_P s^*(P) a_{i_1 j_1} \cdots a_{i_n j_n}$$

where $P = \{\{i_1 j_1\}, \cdots, \{i_n j_n\}\}$ is a partition of the set $\{1, \ldots, 2n\}$ into pairs and $s^*(P)$ equals to the sign of the permutation $(i_1 j_1 \ldots i_n j_n)$.

Each nonzero term of the expansion of the Pfaffian of $A(D)$ equals $x(P)$ or $-x(P)$ where P is a perfect matching of G. Let $s(D,P)$ equal the sign of the term $x(P)$ so that

$$Pf(A(D)) = \sum_P s(D,P) x(P)$$

The following theorem was proved by Kasteleyn ([5]).

Theorem 1.8 *Let G be a graph and D an orientation of G. Let P, M be two perfect matchings of G. Then*

$$s(D,P) = s(D,M) sgn(D, M \Delta P).$$

Hence,

$$Pf(A(D) = \sum_P s(D,P)x(P) = s(D,M) \sum_P sgn(D, M\Delta P) x(P) = s(D,M)\mathcal{P}(D,M)$$

.

The following theorem is well-known (see [7]).

Theorem 1.9 *Let $G = (V,E)$ be a graph and let D be its orientation. Then $Pf^2(A(D)) = det(A(D))$.*

Kasteleyn ([5]) introduced the following notion.

Definition 1.10 *Graph G is called Pfaffian if it has a Pfaffian orientation, i.e. an orientation such that each alternating cycle with respect to an arbitrary fixed perfect matching M of G has an odd number of edges oriented clockwise.*

Hence if a graph G has a Pfaffian orientation D then $\mathcal{P}(G,x)^2 = det(A(D))$.

Kasteleyn ([5]) also observed that the planar graphs have a Pfaffian orientation.

Theorem 1.11 *Each planar graph has a Pfaffian orientation.*

Proof. Let G be a planar graph, and let M be its perfect matching. Consider G drawn on the plane. Orient edges of G so that each face, except possibly the outer one, is clockwise odd. Each such face 'encircles' no vertex of G. Observe that the orientation has the property that a cycle C of G is clockwise odd iff C encircles an even number of vertices. Each alternating cycle with respect to M encircles an even number of vertices and hence it is clockwise odd.

□

Definition 1.12 *We define surface S_g , g positive integer, as follows. It consists of a base B_0 and $2g$ bridges B^i_j, $i = 1, ..., g$ and $j = 1, 2$. B_0 is a convex $4g$-gon with vertices $a_1, ..., a_n$, $n = 4g$, numbered clockwise. Bridge B^i_1 is a 4-gon with vertices $x^i_1, x^i_2, x^i_3, x^i_4$ numbered clockwise . It is glued with B_0 so that edge (x^i_1, x^i_2) of B^i_1 is identified with edge $(a_{4(i-1)+1}, a_{4(i-1)+2})$ of B_0 and edge (x^i_3, x^i_4) of B^i_1 is identified with edge $(a_{4(i-1)+3}, a_{4(i-1)+4})$ of B_0 .*

Bridge B^i_2 is a 4-gon with vertices $y^i_1, y^i_2, y^i_3, y^i_4$ numbered clockwise . It is glued with B_0 so that edge (y^i_1, y^i_2) of B^i_2 is identified with edge $(a_{4(i-1)+2}, a_{4(i-1)+3})$ of B_0 and edge (y^i_3, y^i_4) of B^i_2 is identified with edge $(a_{4(i-1)+4}, a_{4(i-1)+5 mod 4g})$ of B_0 .

Definition 1.13 *We say that a graph G is a g-graph if it may be drawn on S_g so that all the vertices belong to the base B_0, and each edge uses at most one bridge. The set of the edges drawn on the base will be denoted by $E_0 = E_0(G)$ and the set of edges drawn on bridge B^i_j will be denoted by $E^i_j = E^i_j(G)$.*

If moreover the following conditions are satisfied then we say that G is a proper g-graph.

1. *The outer face of the subgraph embedded on B_0 is a cycle, and it is embedded on the boundary of B_0,*
2. *Each vertex is incident with at most one edge out of E_0,*
3. *The subgraph embedded on B_0 has a perfect matching.*

If G is a proper g-graph then we denote by C_0 the cycle which forms the outer face of E_0, and we denote by M_0 a perfect matching of the subgraph of G embedded on B_0.

For each g-graph we fix its embedding on S_g.

Definition 1.14 *Let $G = (V, E)$ be a proper g-graph. The graphs $G_0 = (V_0, E_0)$ and $G^i_j = (V, E_0 \cup E^i_j)$ are planar. We define orientations of G_0 and G^i_j as follows: orientation D_0 of G_0 is Pfaffian and such that each face of G_0 is clockwise odd (as in the proof of Theorem 1.11). Next we draw G^i_j on the plane so that the drawing of G_0 is unchanged, and edge (x^i_1, x^i_4) ((y^i_1, y^i_4) respectively)of B^i_j belongs to the outer face of the drawing of G^i_j. Now complete D_0 to an orientation of G^i_j according to Theorem 1.11, i.e. so that each face is clockwise odd. This defines orientation $+D^i_j$ of the edges of E^i_j.*
$-D^i_j$ is defined obviously by reversing the orientation of $+D^i_j$.

Remark 1.15 *If G is a proper g-graph and Pfaffian orientation D_0 of G_0 is fixed, then for each ij $+D^i_j$ is uniquelly determined.*

Definition 1.16 *Let G be a proper g-graph, $g \geq 1$. An orientation D of G which equals to $+D^i_j$ or $-D^i_j$ on E^i_j and to D_0 on E_0 is called relevant. We define its type $r(D) \in \{+1, -1\}^{2g}$ as follows. For $i = 0, ..g-1$ and $j = 1, 2$, $r(D)_{2i+j}$ equals to $+1$ or -1 according to the sign of D^{i+1}_j in D.*

Definition 1.17 *Let $G = (V, E)$ be a proper g-graph and let A be a subset of its edges. We define its type $t(A) \in \{+1, -1\}^{2g}$ as follows. For $i = 0, ..g-1$ and $j = 1, 2$, we let $t(A)_{2i+j}$ equal to $(-1)^{s(A)_{2i+j}}$, where $s(A)_{2i+j}$ equals to the number of edges of A which belong to E^{i+1}_j.*

Definition 1.18 *Let $r = (r_1, ..., r_{2g})$ and $r_i \in \{1, -1\}$. We let $c(r) = (-1)^n 2^{-g}$, where $n = |\{i; r_{2i+1} = r_{2i+2} = -1\}|$.*

The following theorem and its corollary are the main results of [3].

Theorem 1.19 *Let G be a proper g-graph. Then $\mathcal{P}(G, x)$ equals to linear combination $\mathcal{L}'_g(G, x)$ of $Pf(A(D))$, where D is a relevant orientation of G. The coefficient of $\mathcal{L}'_g$ at $Pf(A(D))$ equals to $s(D, M_0)c(r(D))$. Moreover $s(D, M_0)$ is constant for each relevant orientation D of G.*

Corollary 1.20 *Let G be a graph of orientable genus g. Then $\mathcal{P}(G, x)$ may be expressed using Theorem 1.19.*

The following proposition is a consequence of Theorem 1.11.

Proposition 1.21 *Let D be a relevant orientation of a proper g-graph G and let P, M be two perfect matchings of G_0. Then $s(D, P) = s(D, M)$.*

This Proposition implies that the following definition is correct.

Definition 1.22 *Let D be a relevant orientation of a proper g-graph G. Then we let $c'(D) = s(D, M_0)c(r(D))$ be the coefficient of $Pf(A(D))$ in $\mathcal{L}'_g(G, x)$.*

2 T-joins.

In this section let $G = (V, E)$ be a graph and let $T \subset V$.

We generalize a construction of Kasteleyn ([6]).

Definition 2.1 *Let G be a g-graph. Construct g-graph G^T_1 from G so that each vertex v of G is replaced by a cycle $C(v)$ of length $d(v)$. If $deg_G(v)$ is even and $v \notin T$ or $deg_G(v)$ is odd and $v \in T$ then $d(v) = deg_G(v)$. Otherwise $d(v) = deg_G(v) + 1$.*

If $e_1, ..., e_k$ are the edges of G incident with v, listed in the clockwise order, and $(z_1, ..., z_{d(v)})$ is an order of the vertices of $C(v)$ then we let e_i incident with z_i in G^T_1. In this way each edge of G incident with v is incident with one vertex of $C(v)$ in G^T_1. G^T_1 is a g-graph and each $C(v)$ is a face of its planar part $E_0(G^T_1)$.

Next, let G^T be the graph obtained from G^T_1 by adding all the edges between the vertices of each $C(v)$, v vertex of G. Hence in G^T, each $C(v)$ is replaced by a complete graph with the same vertices. This complete graph will be denoted by $K(v)$. Finally we let $x^T_e = 1$ for each edge of $G^T - G$ and $x^T_e = x_e$ for each edge of $G \cap G^T$. We denote by x^T_1 the restriction of x^T to the edges of G^T_1.

Let F_1 denote a Pfaffian orientation of $E_0(G^T_1)$ such that each face of $E_0(G^T_1)$ is clockwise odd (as in Theorem 1.11). Particularly each $C(v)$ is clockwise odd in F_1.

Let e be an edge of $K(v) - C(v)$. Orient it so that both cycles it forms with $C(v) \subset F_1$ are clockwise odd. Note that this is always possible since $C(v)$ is clockwise odd in F_1. This gives an orientation of G_0 and of the edges of all $K(v)$, v vertex of G. Denote it by F_2.

Let D be an orientation of G^T. We denote by D^1 the restriction of D to G^T_1. We will call an orientation D of G^T T-relevant if $F_2 \subset D$ and the restriction D^1 of D is a relevant orientation of G^T_1.

Theorem 2.2 *Let G be a graph such that G^T_1 is a proper g-graph.*

Then $\mathcal{T}_T(G, x)$ equals to linear combination $\mathcal{L}^T_g$ of $Pf(A(D))$, where D is a T-relevant orientation of G^T and the coefficient of $Pf(A(D))$ equals to the coefficient $c'(D^1)$ of $Pf(A(D^1))$ in $\mathcal{L}'_g(G^T_1, x^T_1)$.

Proof. Let D be a T-relevant orientation of G^T. Each term of $Pf(A(D))$ equals to $x^T(P)$ or $-x^T(P)$ where P is a perfect matching of G^T. We remind that $s(D, P)$ denotes the sign of $x^T(P)$ in $Pf(A(D))$.

Let P be a perfect matching of G^T and let $P' = P \cap E(G)$. Then P' is a T-join of G, and $x(P') = x^T(P)$.

If W is a T-join of G then W is a matching of G^T which may be completed into a perfect matching. Let $\mathcal{R}(W)$ denote the set of the perfect matchings of G^T which contain W.

Each $\mathcal{R}(W)$ is non-empty and there are in general many ways in which T-join W may be completed into a perfect matching of G^T.

Let W be a T-join of G. Let $v \in V$. Let $V(W,v)$ be the set of vertices of $K(v)$ not incident with an edge of W. Then $|V(W,v)|$ is even. Let $K(W,v)$ be the complete subgraph of G^T induced on $V(W,v)$ and let $D(W,v)$ be the orientation of $K(W,v)$ induced by the T-relevant orientations of G^T.

Consider the arcs of $D(W,v)$ drawn inside the face $C(v)$ of $E(G_1^T)$.

Observe that $D(W,v)$ has a non-intersecting clockwise odd Hamiltonian cycle $H(W,v)$: this is true since $C(v)$ is clockwise odd in any relevant orientation of G^T, and each arc of $D(W,v) - C(v)$ is by definition directed so that both cycles if makes with $C(v)$ are clockwise odd.

Let $G_1^T(W) = G_1^T \cup (H(W,v); v \in V)$. $G_1^T(W)$ is a proper g-graph and each $H(W,v)$ is a face of its planar subgraph. Let $D_0(W)$ denote the restriction of F_2 to the planar subgraph $E_0(G_1^T(W))$ of $G_1^T(W)$. All faces of $D_0(W)$ are clockwise odd and hence $D_0(W)$ is a Pfaffian orientation.

The remaining arcs of $D(W,v) - H(W,v)$ may be drawn inside the area bounded by $H(W,v)$ and they are directed in $D(W,v)$ so that the two cycles they form with $H(W,v)$ are clockwise odd.

Let $W(v)$ be a perfect matching of $H(W,v)$ and let $M(W) = W \cup (W(v); v \in V)$ be a perfect matching of $G_1^T(W)$ (and G^T as well) which contains W.

Claim 1. Let W be a T-join of G and let D be a T-relevant orientation of G^T. Then

$$\sum_{P \in \mathcal{R}(W)} s(D,P) = s(D, M(W)).$$

Proof. Let $P \in \mathcal{R}(W)$. By Theorem 1.8 we have

$$s(D,P) = sgn(D, M(W)\Delta P)s(D, M(W)) = (-1)^n s(D, M(W)),$$

where n equals to the number of clockwise even alternating cycles of P and $M(W)$. Each such alternating cycle belongs to $D(W,v)$ for some vertex v of G. We let $P(W,v) = P \cap D(W,v)$ and we let $\mathcal{M}(W,v)$ denote the set of the perfect matchings of $D(W,v)$.

We have

$$\sum_{P \in \mathcal{R}(W)} s(D,P) =$$

$$s(D, M(W)) \sum_{P \in \mathcal{R}(W)} \prod_{v \in V} sgn(D(W,v), W(v)\Delta P(W,v)) =$$

$$s(D, M(W)) \prod_{v \in V} \sum_{P \in \mathcal{M}(W,v))} sgn(D(W,v), W(v)\Delta P).$$

It suffices to show that $\sum_{P\in\mathcal{M}(W,v)} sgn(D(W,v), W(v)\Delta P) = 1$ for each $v \in V$.

We will prove it by induction on $|V(W,v)|$.

If $|V(W,v)| = 2$ then it is clearly true.

Hence let $|V(W,v)| > 2$ and let $e = xy$ be an edge of $W(v)$.

First observe that $\sum_{P\in\mathcal{M}(W,v),e\in P} sgn(D(W,v), W(v)\Delta P) = 1$ by the induction assumption for $D(W,v) - \{x,y\}$.

The perfect matchings of $D(W,v)$ which do not contain e are paired as follows.

Each perfect matching of $D(W,v)$ not containing e may be obtained from a perfect matching M' containing e and another edge f disjoint with e, by replacing e and f by other two disjoint edges with the same end-vertices. This replacement may be done in two ways, which produces a natural pairing of the perfect matchings of $D(W,v)$ which do not contain e.

Let M_1, M_2 be a pair of perfect matchings of $D(W,v)$ obtained by replacing $e, f \in M'$ by edges e_1, f_1 and e_2, f_2 respectively. Observe that the drawing of f inside face $H(W,v)$ of $G_1^T(W)$ doesnot intersect e, since e is an edge of $H(v)$. Hence the edges of exactly one of the pairs e_1, f_1 and e_2, f_2 intersect when we draw them inside face $H(W,v)$ of $G_1^T(W)$. Let it be e_1, f_1.

In order to simplify the presentation we will write $a \stackrel{2}{=} b$ if $a = b$ modulo 2.

By the definition of sgn we have

$$sgn(D(W,v), W(v)\Delta M_1) + sgn(D(W,v), W(v)\Delta M_2) \stackrel{2}{=} 1 + sgn(D(W,v), M_1\Delta M_2) \stackrel{2}{=} 0$$

since there is exactly one alternating cycle Q of $M_1\Delta M_2$, and it is clockwise even. This may be proved as follows:

Let Q' denote cycle (e_2, e, f_2, f) of $D(W,v)$. Q' is clockwise odd since e_2, f_2 do not intersect, $H(W,v)$ is clockwise odd and each arc of $D(W,v) - H(W,v)$ is by definition directed so that both cycles it makes with $H(W,v)$ are clockwise odd. From the same reason all four cycles of $D(W,v)$, each consisting of e_1 or f_1 and two arcs of Q' are clockwise odd. This implies that Q is clockwise even and thus $sgn(D(W,v), M_1\Delta M_2) = -1$.

Summarising we have $\sum_{P\in M(W,v),e\notin P} sgn(D(W,v), W(v)\Delta P) = 0$.

End of Claim 1.

Let W be a T-join of G. Let $D'_1, ..., D'_{4^g}$ be the relevant orientations of $G_1^T(W)$ containing $D_0(W)$. Let x^W denote the restriction of x^T to the edges of $G_1^T(W)$.

By 1.19, $\mathcal{P}(G_1^T(W), x^W)$ equals the sum over all perfect matchings P of $G_1^T(W)$ of $\sum_{i=1}^{4^g} c'(D'_i)s(D'_i, P)x^W(P)$ where $c'(D'_i) = s(D'_i, M(W))c(r(D'_i))$.

Hence $\sum_{i=1}^{4^g} c'(D'_i)s(D'_i, M(W)) = 1$.

For each D'_i, $i = 1, ..., 4^g$, there is exactly one T-relevant orientation D_i of G^T such that $D'_i \subset D_i$. We have that $s(D'_i, M(W)) = s(D_i, M(W))$. Finally observe that $c'(D'_i) = c'(D^1_i)$: $r(D'_i) = r(D^1_i)$ and $s(D'_i, M(W)) = s(D'_i, M_0(G_1^T)) = s(D^1_i, M_0(G_1^T))$.

Hence we get

Claim 2.

$$\sum_{i=1}^{4^g} c'(D_i^1)s(D_i, M(W)) = 1.$$

End of Claim 2.

In summary, the linear combination $\mathcal{L}_g^T$ of $Pf(A(D))$, where D are the T-relevant orientations of G^T, equals to the sum of $\alpha(W)x(W)$ over the T-joins W of G, and by Claims 1 and 2,

$$\alpha(W) = \sum_{i=1}^{4^g} \sum_{P\in\mathcal{R}(W)} c'(D_i^1))s(D_i, P) = 1$$

. This proves the Theorem.

$\square$

Corollary 2.3 *Let $G = (V, G)$ be a graph of orientable genus g, and let $T \subset V$. Then $\mathcal{T}_T(G, x)$ may be expressed using Theorem 2.2.*

Proof. We proceed in a similar way as in the proof of Corollary 1.20 given in [3].

If G has orientable genus g then it may be embedded on S_g so that each vertex belongs to B_0, but some edges may use several bridges. We construct graph G' so that we replace each edge e which uses k bridges, $k \geq 1$, by a path P_e consisting of new edges $(e_1, ..., e_{2k+1})$. We let $x'_{e_i} = 1$ for each $i > 1$ and $x'_{e_1} = x_e$. Next we add edges so that the outer face of the planar part is a cycle. We let $x'_e = 0$ for each such edge e.

Observe that $\mathcal{T}_T(G, x) = \mathcal{T}_T(G', x')$, G' is a g-graph and each vertex of G' is incident with at most one edge which does not belong to G'_0.

If $T = \emptyset$ then $G_1'^T$ is a proper g-graph since each $C(v)$ has a perfect matching, and we can use Theorem 2.2 to get $\mathcal{T}_T(G', x') = \mathcal{T}_T(G, x)$.

If $T \neq \emptyset$ we need to work more. We construct a graph G'' as follows. Let $|T| = 2k$ be even, otherwise G has no T-join. We connect pairs of vertices of T embedded on B_0 by disjoint lines $L_1, ..., L_k$ on B_0 such that the interior of each L_i, $i = 1, ..., k$, contains no vertex of G'. Let L_i connect vertices x_i, y_i of T.

For each $i \in \{1, ..., k\}$ we do the following construction. Let $z_1, ..., z_m$ be the intersections of L_i with the lines representing the edges of G'_0, ordered along L_i from x_i to y_i. Let f_j be the edge of G'_0 containing z_j, $j = 1, ..., m$ and let z'_j be a vertex of the interior of the line representing f_j which is located on the right hand-side of z_j when staying on L_i and looking towards y_i. Moreover let z'_j be sufficiently near to z_j. We add vertices $z_1, z'_1, ..., z_m, z'_m$ to G', replacing each edge f_j by the corresponding path of an even number of new vertices. As before, if f_j is replaced by path $(f_j^1, ..., f_j^p)$ then we let $x''_{f_j^1} = x'_{f_j}$, $x''_{f_j^l} = 1$ for $l > 1$ and $x''_e = x_e$ for each edge of G'.

Moreover we add edges $x_i z_1, z'_1 z_2, ..., z'_{m-1} z_m, z'_m y_i$ to G'' and we let $x''_e = 0$ for each such edge e.

This finishes the construction of G''.

Let R_i denote the path $x_i z_1, z_1 z'_1, z'_1 z_2, z_2 z'_2, ..., z_m z'_m, z'_m y_i$ of G''.

Observe that $\mathcal{T}_T(G'', x'') = \mathcal{T}_T(G', x')$, the outer face of G''_0 is a cycle and each vertex of G'' is incident with at most one edge which does not belong to G''_0. Moreover $\mathcal{R} = \{R_1, ..., R_k\}$ is a T-join of G''_0. Each path R_i has the property that for each vertex v of R_i, the edges of R_i incident with v are neighbouring in the clockwise order of the edges incident with v. Hence $G_1''^T$ is a proper g-graph and we may use Theorem 2.2 to get $\mathcal{T}_T(G'', x'') = \mathcal{T}_T(G, x)$.

This finishes the proof of the corollary.

$\square$

3 The Multicuts.

We start with a theorem of van der Waerden (see [11], [13], [6]). We use the following standard notation.

$sinh(x) = 1/2(e^x - e^{-x})$, $cosh(x) = 1/2(e^x + e^{-x})$, $th(x) = sinh(x)/cosh(x)$ and $coth(x) = cosh(x)/sinh(x)$.

We let $\delta(x, y)$ equal to 1 if $x = y$ and -1 otherwise.

If $G = (V, E)$ is a graph and $A \subset E$ then we let

$$U_k((V, A)) = \sum_{\sigma \in \{1,...,k\}^V} (\prod_{(ij) \in A} \delta(\sigma_i \sigma_j)).$$

Theorem 3.1 *Let $G = (V, E)$ be a graph and $k > 1$.*

$$e^{\sum_{f \in E} x_f}[k + \sum_{i=2}^{k} i! \binom{k}{i} \mathcal{C}_i(G, (e^{-2x_f} : f \in E))] =$$

$$(\prod_{f \in E} cosh(x_f)) \sum_{A \subseteq E} (U_k((V, A)) \prod_{f \in A} th(x_f)).$$

Proof. We use a simple fact that

$$e^{x\delta(\sigma_i \sigma_j)} = cosh(x) + \delta(\sigma_i \sigma_j) sinh(x).$$

We have:

$$e^{\sum_{f\in E} x_f}[k+\sum_{i=2}^{k} i!\binom{k}{i}\mathcal{C}_i(G,(e^{-2x_f}:f\in E))] =$$

$$\sum_{\sigma\in\{1,\dots,k\}^V}(\prod_{ij\in E} e^{\delta(\sigma_i\sigma_j)x_{ij}}) =$$

$$\sum_{\sigma\in\{1,\dots,k\}^V}(\prod_{ij\in E}(cosh(x_{ij})+\delta(\sigma_i\sigma_j)sinh(x_{ij}))) =$$

$$(\prod_{f\in E} cosh(x_f))\sum_{\sigma\in\{1,\dots,k\}^V}(\prod_{ij\in E}(1+\delta(\sigma_i\sigma_j)tanh(x_{ij}))) =$$

$$(\prod_{f\in E} cosh(x_f))\sum_{\sigma\in\{1,\dots,k\}^V}\sum_{A\subseteq E}(\prod_{ij\in A}\delta(\sigma_i\sigma_j)tanh(x_{ij})) =$$

$$(\prod_{f\in E} cosh(x_f))\sum_{A\subseteq E}(U_k((V,A))\prod_{f\in A} th(x_f)).$$

□

Theorem 3.2 *Let G be a graph and let $\mathcal{C}_2'(G,z)=\mathcal{C}_2(G,z)+1$. Then*

$$2e^{\sum_{f\in E} x_f}\mathcal{C}_2'(G,(e^{-2x_f},f\in E)) =$$

$$(\prod_{f\in E} cosh(x_f))2^{|V|}\mathcal{E}(G,(th(x_f):f\in E)).$$

Proof. We have from Theorem3.1 that

$$2e^{\sum_{f\in E} x_f}\mathcal{C}_2'(G,(e^{-2x_f},f\in E)) =$$

$$(\prod_{f\in E} cosh(x_f))\sum_{A\subseteq E}(U_2((V,A))\prod_{f\in A} tanh(x_f)).$$

Now observe that if $A\subset E$ is a cycle and $\sigma\in\{1,2\}^V$ arbitrary then $\prod_{ij\in A}\delta(\sigma_i\sigma_j)=1$. Hence $U_2((V,A))=2^{|V|}$ when (V,A) is an eulerian subgraph. Moreover, if (V,A) is not an eulerian subgraph, then observe that $U_2((V,A))=0$.

□

Theorem 3.3 *Let $G=(V,E)$ be a graph and let the edges of G be partitioned into a bounded number of classes so that the variables x_f are equal in each class. Then there is a polynomial algorithm which, given $\mathcal{E}(G,x)$ and G as input, produces $\mathcal{C}_2(G,x)$.*

Proof. We have

$$(\prod_{f\in E} cosh(x_f))2^{|V|}\mathcal{E}(G,(tanh(x_f):f\in E)) =$$

$$(\prod_{f\in E}\frac{e^{x_f}+e^{-x_f}}{2}2^{|V|}\mathcal{E}(G,(\frac{e^{x_f}-e^{-x_f}}{e^{x_f}+e^{-x_f}}:f\in E)) =$$

$$(\prod_{f\in E}\frac{e^{2x_f}+1}{2e^{x_f}}2^{|V|}\mathcal{E}(G,(\frac{e^{2x_f}-1}{e^{2x_f}+1}:f\in E)) =$$

$$2\prod_{f\in E} e^{x_f}\mathcal{C}_2'(G,(e^{-2x_f},f\in E)).$$

Hence,

$$\mathcal{C}_2'(G,(e^{-2x_f},f\in E)) = 2^{|V|-|E|-1}\prod_{f\in E} e^{-2x_f}\mathcal{E}^*(e^{2x_f}:f\in E)),$$

where

$$\mathcal{E}^*(e^{2x_f}:f\in E)) = \prod_{f\in E}(e^{2x_f}+1)\mathcal{E}(G,(\frac{e^{2x_f}-1}{e^{2x_f}+1}:f\in E)).$$

It follows that $\mathcal{E}^*$ and, consequently, $\mathcal{C}_2'$ and $\mathcal{C}_2$ may be expressed in polynomial time. □

Next algorithmical result is a consequence of Theorem 3.3, Corollary 2.3 and Theorem 2.2 and Corollary 1.20.

Theorem 3.4 *Let S be an orientable surface and k a fixed positive integer. Let $\mathcal{G}$ be the class of graphs which may be embedded on S and such that the edges are partitioned into at most k classes and the variables x_e are equal in each class. Then $\mathcal{C}_2(G,x)$ and $\mathcal{T}_T(G,x)$ may be determined efficiently for $G\in\mathcal{G}$. In particular it is possible to find efficiently the size of a maximum edge-cut for the graphs of $\mathcal{G}$.*

Remark 3.5 *A possibility to understand more about edge-cuts is to try to solve the problem of finding a maximum weighted edge-cut for the graphs of a fixed surface. This is polynomial for planar graphs using the duality of planar graphs and a polyhedral description of the Chinese Postman Problem.*

A different construction of Fisher (see [2]) also yields an algorithm to compute $\mathcal{C}_2(G,x)$, using Theorem 3.3 and Corollary 1.20.

To split a vertex v of a graph G means to replace v by a path of $deg_G(v)$ vertices so that each vertex of the path is incident with exactly one edge incident with v in G.

Definition 3.6 *Let $G = (V, E)$ be a graph. We denote by $G_s = (V_s, E_s)$ the graph obtained from G by splitting of all the vertices of G. We let $x^s_f = x_f$ for each $f \in G_s \cap G$.*

All vertices of G_s have degree at most three and $\mathcal{C}_2(G, (e^{-2x_f} : f \in E))$ equals the limit of $\mathcal{C}_2(G_s, (e^{-2x_f} : f \in E_s)$ when each x_f, $f \in G_s - G$, goes to infinity. Moreover, if G is embeddable on orientable surface S then so is G_s.

Definition 3.7 *Let H be a graph where each vertex has degree at most three. We denote by H' the graph obtained as follows:*

1. *let v be a vertex of H of degree two and let f_1 and f_2 incident with v. We delete v, add two new vertices v_1 and v_2 and three new edges f'_1, f'_2 and g. f'_i is obtained from f_i by replacing v by v_i, $i = 1, 2$, and $g = v_1v_2$.*

2. *let v be a vertex of H of degree three and let f_1, f_2 and f_3 be the edges incident with v. We delete v, add six new vertices $v_1, \dots, v_6$ and ten new edges f'_1, f'_2, f'_3 and $g_1, \dots, g_7$. f'_i is obtained from f_i by replacing v by v_i, $i = 1, 2, 3$. Moreover, $g_1 = v_1v_2$, $g_2 = v_2v_4$, $g_3 = v_1v_4$, $g_4 = v_4v_5$, $g_= v_5v_3$. $g_6 = v_3v_6$ and $g_7 = v_5v_6$. Hence, g_1, g_2, g_3 and g_5, g_6, g_7 form triangles.*

We let $x'_e = 1$ for each $e \in H' - H$ and $x'_e = x_e$ for each $e \in H' \cap H$.

If H is embeddable on orientable surface S then so is H'.

Proposition 3.8 $\mathcal{P}(H', x') = \mathcal{E}(H, x)$.

Proof. The statement follows from a simple observation that for each eulerian subgraph W of H there exists a unique perfect matching $P(W)$ of H' such that $W \subset P(W)$.

$\square$

In summary, if G is a graph embeddable on an orientable surface S then so is G'_s and $\mathcal{C}_2(G, x)$ may be obtained from $\mathcal{P}(G'_s, x^{s\prime})$ by applying Theorem 3.2 (as indicated above it is necessary to make a limit for each x_f, $f \in E(G'_s) - E(G')$, going to the infinity).

$\mathcal{P}(G'_s, x^{s\prime})$ may be computed using Corollary 1.20.

4 A Duality of Enumeration.

Let $V = \mathbf{F}^n$ be a vector space over a field $\mathbf{F}$. Each subspace of V is called a *linear code of length* n. If linear code C of length n is given as the row space of a $(k,n)-$ matrix A, i.e. $C = \{A^T x; x \in \mathbf{F}^k\}$, and M is the matroid represented by the columns of A, then we let $M = M(\mathbf{F}, C)$ and $C = C(\mathbf{F}, M)$. The *dual linear code* $C^* = C(\mathbf{F}, M)^*$ is the orthogonal complement of C in $\mathbf{F}^n$, i.e. $C^* = \{x \in \mathbf{F}^n; Ax = 0\}$.

The elements of a linear code are called codewords. The weight of a codeword is the number of non-zero entries in it. If C is a linear code of length n then its *weight enumerator* is the polynomial

$$A_C(t) = \sum_{i=0}^{n} A_i t^i,$$

where A_i denote the number of codewords of C of weight i.

The following theorem was proved by MacWilliams ([8]).

Theorem 4.1 *Let C be a linear code of length n and dimension k over $GF[q]$ and let D be its dual code. Then*

$$A_D(t) = q^{-k}[1 + (q-1)t]^n A_C((1-t)(1+(q-1)t)^{-1}).$$

Let $G = (V, E)$ be a graph and let $N(G)$ be the graphic matroid of G. Let $O(G)$ be the oriented incidence matrix of G, i.e. V by E matrix obtained from the incidence matrix of G by replacing exactly one $'1'$ of each column by $'-1'$. The columns of $O(G)$ represent $N(G)$ over an arbitrary field $\mathbf{F}$.

Consider the following example. The set of the characteristic vectors of edge-cuts of a connected graph G (including the empty cut) equals $C(GF[2], N(G))$ and the set of the characteristic vectors of eulerian subgraphs of G equals $C(GF[2], N(G))^*$. Indeed, Theorem 4.1 generalises Theorem 3.2.

Theorem 4.2 *Let $G = (V, E)$ be a graph. Then*

$$A_{C(GF[q],N(G))}(t) = 1 + \sum_{i=2}^{q} (q-1)\ldots(q-i+1)\mathcal{C}_i(G, (t, \ldots, t)).$$

Proof. Let $C = C(GF[q], N(G))$. We have $C = \{O(G)^T x; x \in GF[q]^V\}$ and $A_C(t)$ is the weight enumerator of C. Let us define an equivalence on $GF[q]^V$ by $x \equiv y$ if $O(G)^T x = O(G)^T y$. Observe that each equivalence class consists of q elements since $O(G)^T x = O(G)^T y$ if and only if $x - y$ is a constant vector, i.e. $(x-y)_i = (x-y)_j$

for each $i, j \in \{1, ..., |V|\}$. Let C^+ be the system (i.e. some elements may be repeated several times) defined by $C^+ = (O(G)^T x; x \in GF[q]^V)$. Let $A_{C^+}(t) = \sum_{i=0}^{|E|} A_i^+ t^i$, where A_i^+ equals the number of vectors of C^+ with i non-zero components. Then $A_{C^+}(t) = qA_C(t)$.

If $x = (x_1, ..., x_{|V|}) \in GF[q]^V$ then let l_x be the number of different values of x_i, $i = 1, ..., |V|$ and let $V_1^x, ..., V_{l_x}^x$ be the partition of V into l_x non-empty classes such that the components of x equal in each class.

Let $cut(x)$ be the l_x-cut of G determined by $V_1^x, ..., V_{l_x}^x$, i.e. the subset of E of the edges with the end-vertices in different sets V_i^x, $i = 1, ..., l_x$ and let $Cut(x) = (\{V_1, ..., V_{l_x}\}, cut(x))$.

Each $Cut(x)$ is a l_x-cut of G and the weight of codeword $O(G)^T x$ equals $|cut(x)|$. Let C^{++} be the system defined by $C^{++} = (cut(x); x \in GF[q]^V)$. Let $A_{C^{++}}(t) = \sum_{W \in C^{++}} t^{|W|}$. We have $A_{C^+}(t) = A_{C^{++}}(t)$.

For $i = 1, ..., q$ let $X_i = \{x \in GF[q]^V; l_x = i\}$. Define an equivalence on X_i by $x \equiv y$ if $Cut(x) = Cut(y)$. Observe that each equivalence class consists of $q(q-1)...(q-i+1)$ elements. Hence

$$A_{C^{++}}(t) = q + \sum_{i=2}^{q} q(q-1)...(q-i+1)\mathcal{C}_i(G, (t, ..., t)).$$

This proves the Theorem.

$\square$

We have $C(GF[q], N(G))^* = \{z \in GF[q]^E; O(G)z = 0\}$. Hence the elements of $C(GF[q], N(G))^*$ are *flows* on G with values in $GF[q]$. An element of $C(GF[q], N(G))^*$ is called *nowhere-zero flow* if its weight equals $|E|$. Let $F'(G, q)$ be the subset of $C(GF[q], N(G))^*$ consisting of nowhere-zero flows. $F(G, q) = |F'(G, q)|$ is called *flow polynomial* of G.

Theorems 4.1, 4.2 express a duality between flows and cuts of a graph. It is a duality of the Tutte polynomial.

Tutte polynomial was defined by Tutte ([9]). It may be expressed as a minor modification of the Whitney rank generating function ([15]).

Definition 4.3 *Let $G = (V, E)$ be a graph. For $A \subset E$ let $r(A) = |V| - k(A)$, where $k(A)$ denotes the number of connected components of (V, A). Then let*

$$T(G, x, y) = \sum_{A \subset E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

$T(G, x, y)$ is called Tutte polynomial of graph G.

More generally, Tutte polynomial of a matroid (see [12] for basic notions of matroid theory) is defined as follows.

Definition 4.4 *Let M be a matroid on set E. For $A \subset E$ let $r(A)$ denote the rank of A in M. Then let*

$$T(M, x, y) = \sum_{A \subset E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

$T(M, x, y)$ is called Tutte polynomial of matroid M.

For example if G is a graph and $N(G)$ the graphic matroid of G then $T(G, x, y) = T(N(G), x, y)$.

If M is a matroid and M^* its dual then $r^*(E) - r^*(A) = |A| - r(A)$ and we immediately get that $T(M, x, y) = T(M^*, y, x)$. This expression is called *duality of the Tutte polynomial.*

The following theorem was proved by Green ([4]).

Theorem 4.5 *Let C be a linear code of dimension k and length n over $GF[q]$. Then*

$$A_C(t) = (1-t)^k t^{n-k} T(M(GF[q], C), (1 + (q-1)t)(1-t)^{-1}, t^{-1}).$$

As an immediate corollary we get

Corollary 4.6 *Let M be a matroid represented over $GF[q]$ and let $C(GF[q], M)$ be the linear code of length n and dimension k. If $(x-1)(y-1) = q$ then*

$$T(M, x, y) = y^{k-n}(1 - y^{-1})^k A_{C(GF[q],M)}(y^{-1}).$$

By Corollary 4.6 and Theorem 4.1 we have

Corollary 4.7 *Let $G = (V, E)$ be a graph and let $N(G)$ be the graphic matroid of G. If $(x-1)(y-1) = 2$ then*

$$T(G, x, y) = T(N(G), x, y) = y^{|V|-|E|-1}(1 - y^{-1})^{|V|-1}[1 + \mathcal{C}_2(G, (y^{-1}, ..., y^{-1})].$$

It follows that the Tutte polynomial of a graph of genus g may be expressed along the hyperbola $(x-1)(y-1) = 2$ as a linear combination of 4^g Pfaffians, and hence it may be determined efficiently for the graphs embeddable on an arbitrary fixed orientable surface.

It is natural to ask whether there is an analogy of this statement for binary matroids.

Note that Theorem 4.1 also follows immediately from Theorem 4.5 and the duality of the Tutte polynomial.

Finally we make some remarks on the chromatic polynomial and the flow polynomial of a graph.

Definition 4.8 *Let $G = (V, E)$ be a graph and $x > 1$ an integer. Any function f from V to $\{1, ..., x\}$ is called x-colouring. Let $p_i(G, x)$ be the number of proper x-colorings of vertices of G, in which there are i monochromatic edges. Let*

$$B(G, x, s) = \sum_{i=0}^{|E|} s^i p_i(G, x).$$

$B(G, x, s)$ is called monochrome polynomial. $p_0(G, x)$ is called chromatic polynomial.

Chromatic polynomial was introduced by Birkhoff ([1]) and the following expression was found by Birkhoff ([1]) and Whitney ([15]).

Theorem 4.9 *Let $G = (V, E)$ be a graph. Then $p_0(G, x) = \sum_{H \subset E} (-1)^{|H|} x^{c((V,H))}$, where $c((V, H))$ denotes the number of the connectivity components of graph (V, H).*

The following Theorem was proved by Tutte (see [10]).

Theorem 4.10 *Let $G = (V, E)$ be a graph. Then*

$$p_0(G, k) = (-1)^{|V| - c(G)} k^{c(G)} T(G, 1 - k, 0).$$

Using Theorem 3.1 we obtain another interesting expression for the chromatic polynomial. It is a special case of a Lemma on p. 501 of [14].

Theorem 4.11 *Let $G = (V, E)$ be a graph and $k > 1$. Then*

$$p_0(G, k) = 2^{-|E|} \sum_{H \subset E} (-1)^{|H|} U_k((V, H)).$$

Proof. Let $Q_k(z)$ be the polynomial such that

$$e^{2|E|x}[k + \sum_{i=2}^{k} i! \binom{k}{i} \mathcal{C}_i(G, (e^{-2x}, ..., e^{-2x})) = Q_k(e^{2x}).$$

Observe that $p_0(G, k)$ equals to the constant term of $Q_k(z)$. By Theorem 3.1 we have

$$e^{-|E|x} Q_k(e^{2x}) =$$
$$(e^{2x} + 1)^{|E|} 2^{-|E|} e^{-|E|x} \sum_{H \subseteq E} [U_k((V, H))(e^{2x} - 1)^{|H|} (e^{2x} + 1)^{-|H|}].$$

Hence

$$Q_k(e^{2x}) = 2^{-|E|} \sum_{H \subset E} U_k((V,H))(e^{2x}-1)^{|H|}(e^{2x}+1)^{|E|-|H|}.$$

It follows that the constant term of $Q_k(z)$ equals to

$$2^{-|E|} \sum_{H \subset E} (-1)^{|H|} U_k((V,H)).$$

$\square$

The following proposition is straightforward.

Proposition 4.12 $(-1)^{|H|}U_k((V,H)) = B((V,H),k,-1)$.

Hence we get

Corollary 4.13 *Let* $G=(V,E)$ *be a graph. Then* $p_0(G,k) = 2^{-|E|}\sum_{H\subset E} B((V,H),k,-1)$.

Nowhere-zero flows are studied extensively. The following theorem was proved by Tutte ([10]).

Theorem 4.14 *Let* $G=(V,E)$ *be a graph and let* q *be a power of a prime. Then*

$$F(G,q) = (-1)^{|V|-|E|-c(G)} T(G,0,1-q).$$

We give an interesting expression for $F(G,q)$ which is new as far as we know.

Theorem 4.15 *Let* $G=(V,E)$ *be a connected graph. Then*

$$F(G,q) = q^{-|V|} 2^{-|E|} \sum_{A \subset E} U_q((V,A)) q^{|A|} (q-2)^{|E|-|A|}.$$

Proof. Let $C = C(GF[q], N(G))$ and let $D = C(GF[q], N(G))^*$. By Theorem 4.1 we have

$$A_D(t) = q^{1-|V|}[1+(q-1)t]^{|E|} A_C((1-t)(1+(q-1)t)^{-1}).$$

From Theorem 4.2 and Theorem 3.1 we get for $z>0$

$$qA_C(z^{-1}) = q + \sum_{i=2}^{q} q(q-1)...(q-i+1)\mathcal{C}_i(G,(z^{-1},...,z^{-1})) =$$

$$2^{-|E|} z^{-|E|} \sum_{A \subset E} U_q((V,A))(z-1)^{|A|}(z+1)^{|E|-|A|}.$$

If we let $z = (1+(q-1)t)(1-t)^{-1}$ we get for all $t > 0$, $t \neq 1$

$$A_D(t) = q^{-|V|}2^{-|E|} \sum_{A \subset E} U_q((V,A))(qt)^{|A|}(2+(q-2)t)^{|E|-|A|}.$$

It follows that the leading coefficient of $A_D(t)$, which equals $F(G,q)$, is equal to

$$q^{-|V|}2^{-|E|} \sum_{A \subset E} U_q((V,A))q^{|A|}(q-2)^{|E|-|A|}.$$

□

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